$\boxed{Q}$ Suppose that X_n satisfies an LDP in some regular space $\mathcal{X}$, good rate func. I

Y_n ——— $\mathcal{Y}$ ——— J

and that X_n and Y_n are independent.

Show that (X_n, Y_n) satisfies an LDP in $\mathcal{X} \times \mathcal{Y}$, good rate func.

$$K(x,y) = I(x) + J(y).$$

$\boxed{A}$. Note first the topology on $\mathcal{X} \times \mathcal{Y}$.

If σ and τ are bases for $\mathcal{X}$ and $\mathcal{Y}$, then $\{O \times P : O \in \sigma, P \in \tau\}$ is a basis for $\mathcal{X} \times \mathcal{Y}$.

Thus open sets are of the form $\bigcup_n O_n \times P_n$ $\quad O_n$ open in $\mathcal{X}$, P_n open in $\mathcal{Y}$

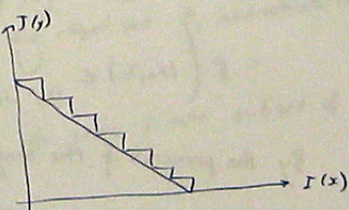
closed ——— $\bigcap_n \left((C_n \times \mathcal{Y}) \cup (\mathcal{X} \times D_n) \right)$ $\quad C_n$ closed in $\mathcal{X}$, D_n closed in $\mathcal{Y}$.

Goodness of K. Clearly $K(x,y) \geq 0$.

A typical level set is

$$\{(x,y) : K(x,y) \leq \alpha\} = \{(x,y) : I(x) + J(y) \leq \alpha\}$$

$$= \bigcap_{n \in \mathbb{N}} \bigcup_{m \leq n+1} \left\{ (x,y) : I(x) \leq \tfrac{m}{n}\alpha,\ J(y) \leq \tfrac{n+1-m}{n}\alpha \right\}$$



$$= \bigcap_{n \in \mathbb{N}} \bigcup_{m \leq n+1} \left\{ x : I(x) \leq \tfrac{m}{n}\alpha \right\} \times \left\{ y : J(y) \leq \tfrac{n+1-m}{n}\alpha \right\}$$

$$= \bigcap_{n \in \mathbb{N}} \bigcup_{m \leq n+1} \text{closed} \times \text{closed} = \bigcap_{n \in \mathbb{N}} \text{closed} = \text{closed}.$$

Furthermore, this level set is a subset of the compact set

$$\{x : I(x) \leq \alpha\} \times \{y : J(y) \leq \alpha\}.$$

So this level set is compact.

Lower Bound

Let B be open in $\mathcal{X}\times\mathcal{Y}$. Then $B = \bigcup_n O_n \times P_n$.

Pick $(x,y)\in B$. Then $(x,y)\in O\times P$ for some O open in $\mathcal{X}$, P open in $\mathcal{Y}$, $O\times P\subset B$.

$$\mathbb{P}\big((X_n,Y_n)\in B\big) \geq \mathbb{P}\big((X_n,Y_n)\in O\times P\big) = \mathbb{P}(X_n\in O)\,\mathbb{P}(Y_n\in P).$$

$$\Rightarrow \liminf \tfrac{1}{n}\log \mathbb{P}\big((X_n,Y_n)\in B\big)$$

$$\geq \liminf \tfrac{1}{n}\Big[\log \mathbb{P}(X_n\in O) + \log \mathbb{P}(Y_n\in P)\Big]$$

$$\geq \liminf \tfrac{1}{n}\log \mathbb{P}(X_n\in O) + \liminf \tfrac{1}{n}\log \mathbb{P}(Y_n\in P)$$

$$\geq -\inf_{x'\in O} I(x') - \inf_{y'\in P} J(y')$$

$$\geq -\big(I(x)+J(y)\big) = -K(x,y).$$

Upper Bd for Cylinder sets

Consider first a simple closed set of the form $B = \bigcap_{n\leq N} (C_n\times\mathcal{Y})\cup(\mathcal{X}\times D_n)$.

For such a set,

$$\mathbb{P}\big((X_n,Y_n)\in B\big) = \mathbb{P}\Bigg((X_n,Y_n)\in \bigcup_{\substack{(i_1,\dots,i_N)\\ \in\{0,1\}^N}} \bigcap_{n\leq N} \begin{cases} i_n=0: & C_n\times\mathcal{Y}\\ i_n=1: & \mathcal{X}\times D_n\end{cases}\Bigg).$$

$$= \mathbb{P}\Bigg((X_n,Y_n)\in \bigcup_{(i_1,\dots,i_N)} \Big(\bigcap_{n:i_n=0} C_n\Big)\times\Big(\bigcap_{n:i_n=1} D_n\Big)\Bigg)$$

By the principle of the largest term,

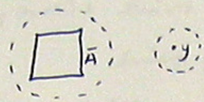
$$\limsup \tfrac{1}{n}\log\mathbb{P}\big((X_n,Y_n)\in B\big) \leq \max_{i_1,\dots,i_N} \limsup\tfrac{1}{n}\log\mathbb{P}\Bigg((X_n,Y_n)\in \Big(\bigcap_{n:i_n=0} C_n\Big)\times\Big(\bigcap_{n:i_n=1} D_n\Big)\Bigg)$$

$$\leq -\min_{i_1,\dots,i_N}\Bigg(\inf_{x\in\bigcap_{i_n=0} C_n} I(x) + \inf_{y\in\bigcap_{i_n=1} D_n} J(y)\Bigg).$$

$$= -\inf_{(x,y)\in B} I(x)+J(y).$$

Upper bound for general closed sets

Recall: a space is regular if for every closed set $\bar{A}$ and every point y there exist disjoint open neighbourhoods $\bar{A} \subset B_{\bar{A}}$ and $y \in B_y$.



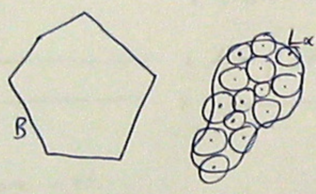
It can be shown that if $\mathcal{X}$ and $\mathcal{Y}$ are regular then $\mathcal{X} \times \mathcal{Y}$ is regular also. [simple exercise].

Let $B \subset \mathcal{X} \times \mathcal{Y}$ be closed.

Let $\alpha = \inf_{z \in B} K(z)$. If $\alpha = 0$, upper bd is trivial.

Otherwise, for any $\varepsilon > 0$, consider the level set $L_{\alpha-\varepsilon}$.

Clearly, $B \cap L_{\alpha-\varepsilon} = \emptyset$. By goodness, $L_{\alpha-\varepsilon}$ is compact.



For every ~~B_z~~ $z \in L_{\alpha-\varepsilon}$, by regularity, we can pick an open ball B_z which does not intersect the closed set B, and which is of the form $O \times P$.

By compactness of $L_{\alpha-\varepsilon}$,

$$L_{\alpha-\varepsilon} \subseteq \bigcup_{n \in N} B_{z_n} \quad \text{for some finite subset of the } z \in L_{\alpha-\varepsilon}.$$

Now, by construction of the B_z,

$$B \subseteq \Big(\bigcup_{n \in N} B_{z_n} \Big)^c = \bigcap_{n \in N} C_{z_n} \quad \text{where } C_{z_n} = B_{z_n}^c \text{ is closed.}$$

So $\limsup \frac{1}{n} \log \mathbb{P}\big((X_n, Y_n) \in B \big) \le \limsup \frac{1}{n} \log \mathbb{P}\big((X_n, Y_n) \in \bigcap_{n \in N} C_{z_n} \big)$

$$\le - \inf_{z \in \bigcap_{n \in N} C_{z_n}} K(z) \quad \text{by upper bd for cylinder sets}$$

But for any such z, by choice of the B_z, $K(z) > \alpha - \varepsilon$.

Hence $\limsup \frac{1}{n} \log \mathbb{P}\big((X_n, Y_n) \in B \big) \le -(\alpha - \varepsilon)$. Since ε arbitrary, the LD upper bound holds.

$\boxed{1}$ (i). • $\limsup \frac{1}{n}\log(a_n+b_n) \le \underbrace{\limsup \frac{1}{n}\log a_n}_{= A} \vee \underbrace{\limsup \frac{1}{n}\log b_n}_{= B}$? wlog $A \ge B$.

$\forall \varepsilon > 0\ \exists n_0$ s.t. $n \ge n_0 \Rightarrow a_n \le e^{n(A+\varepsilon)}$ and $b_n \le e^{n(B+\varepsilon)}$.

$$\Rightarrow \limsup \tfrac{1}{n}\log(a_n+b_n) \le \limsup \tfrac{1}{n}\log\left[e^{n(A+\varepsilon)} + e^{n(B+\varepsilon)}\right]$$

$$= \limsup \tfrac{1}{n}\log e^{n(A+\varepsilon)}\left[1 + e^{-n(A-B)}\right]$$

$$= A+\varepsilon.$$

But $\varepsilon > 0$ arbitrary, so $\limsup \frac{1}{n}\log(a_n+b_n) \le A$.

• $\liminf \frac{1}{n}\log(a_n+b_n) \ge \overset{\liminf \frac{1}{n}\log a_n}{\overset{=}{A}} \vee \overset{\liminf \frac{1}{n}\log b_n}{\overset{=}{B}}$? wlog $A \ge B$

$\ge \liminf \frac{1}{n}\log a_n$ since $b_n \ge 0$

$= A$.

(ii). • $\lim \frac{1}{n}\log \mathbb{P}(A_n \cup B_n) \le \limsup \frac{1}{n}\log(a_n+b_n)$ $a_n = \log \mathbb{P}(A_n)$ $b_n = \log \mathbb{P}(B_n)$

$\le (-a) \vee (-b) = -(a \wedge b)$.

——— $\ge \liminf \frac{1}{n}\log a_n = -a$

——— $\ge$ ——— $b_n = -b$.

agree.

so $\lim = -(a \wedge b)$

• suppose $a > b$.

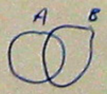
$\frac{1}{n}\log \mathbb{P}(A_n \mid A_n \cup B_n) = \frac{1}{n}\log \mathbb{P}(A_n) - \frac{1}{n}\log \mathbb{P}(A_n \cup B_n) \longrightarrow -a - (-(a \wedge b)) = -(a-b) < 0$.

So $\mathbb{P}(A_n \mid A_n \cup B_n) \longrightarrow 0$.

Suppose $a < b$.

By above, $\mathbb{P}(B_n \mid A_n \cup B_n) \longrightarrow 0$.

But $\mathbb{P}(A_n \mid A_n \cup B_n) \ge 1 - \mathbb{P}(B_n \mid A_n \cup B_n) \longrightarrow 1$.



$\boxed{2}$ (i) $K \subset X$ compact, $I = \inf_{x \in K} f(x) < \infty$.

Pick $x_n \in K$, $I \le f(x_n) < I + \frac{1}{n}$.

By compactness, (x_n) has cgt subsequence $(x_{k(n)})$,

$$x_{k(n)} \in K, \quad x_{k(n)} \longrightarrow \underset{\substack{\in K \\ \text{compact} \Rightarrow \text{closed}}}{x^*}, \quad f(x_{k(n)}) \longrightarrow I.$$

What is $f(x^*)$?

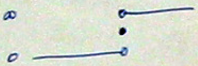
$L_{I+\varepsilon} = \{x : f(x) \le I + \varepsilon\}$ is closed, and $x_k \in L_{I+\varepsilon}$ for suff. large k,

so the limit $x^* \in L_{I+\varepsilon}$. ie $f(x^*) \le I + \varepsilon$

This is true for all $\varepsilon > 0$; so $f(x^*) \le I$.

Since $x^* \in K$, $f(x^*) \ge I$.

Thus $f(x^*) = I$.

(ii). No. 

But any convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

PROOF. Suppose not. Let $x_n \to x$, $f(x_n) \not\to f(x)$.

So either $\limsup f(x_n) > f(x)$ or $\liminf f(x_n) < f(x)$.

Suppose $x_n \uparrow x$. (The case $x_n \downarrow x$ is similar, and these two cases are sufficient.)

(1) $x_n = (1-\lambda_n)x_1 + \lambda_n x$ for some $\lambda_n \longrightarrow 1$.

By convexity, $f[(1-\lambda_n)x_1 + \lambda_n x] \le (1-\lambda_n) f(x_1) + \lambda_n f(x) \longrightarrow f(x)$.

Thus $\limsup f(x_n) \le f(x)$. ※

(2). By convexity, $f(y) \ge f(x) + m(y-x)$ [a supporting line, and $m < \infty$ since $f < \infty$].

$f(x_n) \ge f(x) + m(x_n - x)$

$\liminf f(x_n) \ge f(x)$. ※

(iii) Consider a level set $\{x : g^*(x) \le \alpha\} = \{x : \sup_{\theta \in \mathbb{R}} \theta x - g(\theta) \le \alpha\}$

$= \{x : \theta x \le \alpha + g(\theta) \text{ for all } \theta \in \mathbb{R}\}$.

If $0 > \alpha + g(0)$, this set is empty, hence closed.

If $0 \le \alpha + g(0)$: $= \{x : x \le \frac{\alpha + g(\theta)}{\theta} \ \forall \theta > 0, \ x \ge \frac{\alpha + g(\theta)}{\theta} \ \forall \theta < 0\}$

$= \{x : \sup_{\theta < 0} \frac{\alpha + g(\theta)}{\theta} \le x \le \inf_{\theta > 0} \frac{\alpha + g(\theta)}{\theta}\}$

which is a closed interval, hence closed.

$\boxed{2}$ (iv). • Suppose f is lower-semicontinuous, ie level sets closed.

Let $I = \liminf f(x_n)$, so that $f(x_n) \le I+\varepsilon$ for ∞ many n, any $\varepsilon>0$.

Since $x_n \to x$, $x \in L_{I+\varepsilon}$ — level set $L_{I+\varepsilon}$ is closed, and these $x_{k(n)}$ all lie in $L_{I+\varepsilon}$, so limit pt does too.

So $f(x) \le I+\varepsilon$.

This is true for all $\varepsilon>0$; thus $f(x) \le I$.

metric: closed $\Leftrightarrow$ has all limit points.

• Suppose $x_n \to x \Rightarrow \liminf f(x_n) \ge f(x)$.

Is $L_I = \{x : f(x) \le I\}$ closed? Pick $x_n \in L_I$, $x_n \to x$. Is $x \in L_I$?

Certainly $f(x) \le \liminf f(x_n) \le I$. So $x \in L_I$.

$\boxed{3}$ • $\limsup \frac{1}{n} \log \mathbb{P}(Z_n \in B)$, B closed

$$= \limsup \frac{1}{n} \log \left[\mathbb{P}(Z_n \in B, B_n = 0) + \mathbb{P}(Z_n \in B, B_n = 1) \right]$$

$$= \limsup \frac{1}{n} \log \left[\mathbb{P}(X_n \in B, B_n = 0) + \mathbb{P}(Y_n \in B, B_n = 1) \right]$$

$$= \limsup \frac{1}{n} \log \left[(1-p)\,\mathbb{P}(X_n \in B) + p\,\mathbb{P}(Y_n \in B) \right]$$

$$\le \left(\limsup \frac{1}{n} \log (1-p)\mathbb{P}(X_n \in B) \right) \vee \left(\limsup \frac{1}{n} \log p\,\mathbb{P}(Y_n \in B) \right)$$

$$= \limsup \frac{1}{n} \log \mathbb{P}(X_n \in B) \quad \vee \quad \limsup \frac{1}{n} \log \mathbb{P}(Y_n \in B) \qquad \text{if } 0<p<1.$$

$$= \left(-\inf_{x \in B} I(x) \right) \vee \left(-\inf_{y \in B} J(y) \right) = -\inf_{x \in B} I(x) \wedge J(y).$$

• $\liminf \frac{1}{n} \log \mathbb{P}(Z_n \in B)$, B open

$$\ge \liminf \frac{1}{n} \log \mathbb{P}(X_n \in B, B_n = 0) \ge -\inf_{x \in B} I(x).$$

Also $\ge -\inf_{x \in B} J(x)$.

Hence $\ge -\inf_{x \in B} I(x) \wedge J(x)$.

Or: if the spaces are separable, rate fncs are good:

B_n satisfies LDP. $\frac{1}{n}\log \mathbb{P}(B_n \in B) = 0$

$B \subset \{0,1\}$.

So (X_n, Y_n, B_n) satisfies LDP, grf $K(x, y, b) = I(x) + J(y) + 0$

$\in \mathcal{X} \; \in \mathcal{X} \; \in \{0,1\}$.

So Z_n, a cts function of (X_n, Y_n, B_n), satisfies LDP,

$$\text{grf} \quad L(z) = \inf_{\substack{x,y,b:\\ b=0 \,\&\, x=z\\ \text{or } b=1 \,\&\, y=z}} I(x) + J(y)$$

$$= \min \begin{cases} b=0: & \inf_y I(z) + J(y) \\ b=1: & \inf_x I(x) + J(z) \end{cases}$$

$$= \min \begin{cases} b=0: & I(z) \\ b=1: & J(z) \end{cases} \qquad \text{since } \inf_{x \in \mathcal{X}} I(x) = 0, \text{ by LD upper bd for } \mathcal{X}.$$

$$= I(z) \wedge J(z).$$

$\boxed{4}$:

- $\limsup \frac{1}{n} \log \mathbb{P}(X_n \in B)$ in $\mathcal{E}$ for B closed in $\mathcal{E}$

$\Rightarrow$ B^c open in $\mathcal{E}$ $\Rightarrow$ $B^c \cap \mathcal{E}$ open in $\mathcal{X}$.

$\Rightarrow$ $B^c \cap \mathcal{E} = A^c$, for some A closed in $\mathcal{X}$

$= A^c \cap \mathcal{E}$ since $B^c \cap \mathcal{E} \subset \mathcal{E}$, ~~and closed~~

$\Rightarrow$ $B \cup \mathcal{E}^c = A \cup \mathcal{E}^c = (A \cap \mathcal{E}) \cup \mathcal{E}^c$

$\Rightarrow$ $B = A \cap \mathcal{E}$, A closed in $\mathcal{X}$.

$\leq \limsup \frac{1}{n} \log \mathbb{P}(X_n \in A \cap \mathcal{E})$ in $\mathcal{X}$

$$= -\inf_{x \in A \cap \mathcal{E}} I(x) = -\inf_{x \in B} I(x).$$

- $\liminf \frac{1}{n} \log \mathbb{P}(X_n \in B)$ in $\mathcal{E}$ for B open in $\mathcal{E}$

$\Rightarrow$ $B = A \cap \mathcal{E}$, A open in $\mathcal{X}$

$= \liminf \frac{1}{n} \log \mathbb{P}(X_n \in A \cap \mathcal{E})$

$= \liminf \frac{1}{n} \log \mathbb{P}(X_n \in A)$ since $\mathbb{P}(\mathcal{E}) = 1$

$$\geq -\inf_{x \in A} I(x) \geq -\inf_{x \in A \cap \mathcal{E}} I(x) = -\inf_{x \in B} I(x).$$

$$\inf_{x \in A} I(x) \leq \inf_{x \in A \cap \mathcal{E}} I(x)$$

larger   smaller

$\boxed{5}$. (i). $\limsup \frac{1}{n^{2(1-G)}} \log P(X_n \in C)$? If $\inf_{x \in C} I'(x) = 0$, trivial. So assume $\inf_{x \in C} I'(x) = \infty$.

$$= \limsup n^{2(G-H)} \cdot \underbrace{\frac{1}{n^{2(1-H)}} \log P(X_n \in C)}.$$

$$\limsup \frac{1}{n^{2(1-H)}} \log P(X_n \in C) \le - \inf_{x \in C} I(x).$$

What is this inf? Since C is good, the inf is attained.

If $\inf_{x \in C} I(x) = 0$, $\inf_{x \in C} I'(x) = 0$ by nontriviality (iii) — discard this case.

Else $\inf_{x \in C} I(x) > 0 \Rightarrow \inf_{x \in C} I'(x) = \infty$. Let $\alpha = \inf_{x \in C} I(x) > 0$.

$\le -\alpha$

So $\frac{1}{n^{2(1-H)}} \log \mathbb{P}(X_n \in C) \le -\alpha + \varepsilon$ for n suff. large. (where $\varepsilon < \alpha$).

So $n^{2(G-H)} \cdot \frac{1}{n^{2(1-H)}} \log \mathbb{P}(X_n \in C) \le -n^{2(G-H)}(\alpha - \varepsilon) \longrightarrow -\infty$.

Thus $\limsup \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in C) \le -\inf_{x \in C} I'(x)$.

(ii). • $\limsup \frac{1}{n} \log \mathbb{P}(X_n \notin D) \le -\inf_{x \in D^c} I(x) < 0$ (by goodness, closure of D $\to$ inf attained).

So $\mathbb{P}(X_n \notin D) \to 0$.

• $\liminf \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in E)$?

If E contains μ, $\inf_{x \in E} I'(x) = 0$;

and $\mathbb{P}(X_n \in E) \to 1$ so $\log \mathbb{P}(X_n \in E) \to 0$ so $\liminf \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in E) \ge 0$.

If E does not contain μ, $\inf_{x \in E} I'(x) = \infty$, so bd is trivial.

(iii) So we have LDP at speed $n^{2(1-G)}$, rate function I'.

Clearly a good rate function (non-negative, compact level sets.)

(iv). wlog $H < G$.
By LDP for product spaces, (X_n, Y_n) satisfies LDP at speed $n^{2(1-G)}$, grf $\underbrace{I'(x) + J(y)}$. non-trivial too!
So $X_n + Y_n$ satisfies LDP at same speed, grf $K(z) = \inf_{x,y : x+y=z} I'(x) + J(y) = J(z - \mu)$
which is non-trivial. $\square$

$\boxed{7}$ (i). $B \sim \mathrm{Exp}(\lambda)$.

So $\frac{B}{L}$ satisfies LDP in $\mathbb{R}_+$, grf $J(x) = \lambda x$.

(ii) By Cramér:

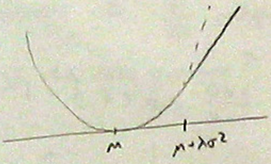
$\frac{A_1 + \cdots + A_L}{L}$ satisfies LDP in $\mathbb{R}$, grf

$$J(x) = \sup_\theta \theta x - \log \mathbb{E} e^{\theta A_1}$$
$$= \sup_\theta \theta x - (\theta\mu + \tfrac{1}{2}\theta^2\sigma^2)$$
$$= \frac{1}{2\sigma^2}(x-\mu)^2.$$

(iii). By contraction; Product Spaces:

$\frac{B}{L} + \frac{A_1 + \cdots + A_L}{L}$ satisfies LDP, grf

$$K(x) = \inf_{\substack{b \geq 0,\, a: \\ b+a=x}} \lambda b + \frac{1}{2\sigma^2}(a-\mu)^2 = \begin{cases} x \geq \mu + \lambda\sigma^2: & \lambda(x-\mu) - \frac{\lambda^2\sigma^2}{2} \\ & \text{at } b = x-\mu-\lambda\sigma^2,\ a = \mu+\lambda\sigma^2 \\ x \leq \mu+\lambda\sigma^2: & \frac{(x-\mu)^2}{2\sigma^2} \\ & \text{at } b=0,\ a=x. \end{cases}$$



Exp only kicks in at a certain pt.

NOTE: G-E won't work.

$$\frac{1}{L}\log \mathbb{E} e^{\theta(B + A_1 + \cdots + A_L)} = \theta\mu + \tfrac{1}{2}\theta^2\sigma^2 + \frac{1}{L}\cdot\frac{\lambda}{\lambda-\theta}$$

$$\to \begin{cases} \theta\mu + \frac{1}{2}\theta^2\sigma^2 & \text{if } \theta < \lambda \\ \infty & \theta \geq \lambda \end{cases}$$ Not steep!

(iv). By Gärtner-Ellis.

$$\frac{1}{L}\log \mathbb{E} e^{\theta(C + A_1 + \cdots + A_L)} \to \theta\mu + \tfrac{1}{2}\theta^2\sigma^2.$$

So $\frac{C}{L} + \frac{A_1 + \cdots + A_L}{L}$ satisfies same LDP as $\frac{A_1 + \cdots + A_L}{L}$.

OR by product & contraction. Try $C \sim N(0,1)$ first — result for $C \sim N(\nu, \rho^2)$ follows by contraction.

$\frac{C}{L}$ satisfies LDP at speed L^2. $\mathbb{P}(\frac{C}{L} \approx x) \approx \mathbb{P}(N(0,1) \approx Lx) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}L^2x^2}$.

Or use Cramér: $\frac{C}{\sqrt{L}} \sim \frac{1}{L}\left(N(0,1) + \cdots + N(0,1)\right)$

Via Hwistiness, satisfies LDP at speed L,

grf $J'(x) = \begin{cases} 0 & x = 0 \\ \infty & \text{elsw.} \end{cases}$

$\boxed{8}$. (i) ~~Wha~~ We need only model the header queue.

This has Poisson arrivals of rate λ,

and service times are Exp with mean $1/c$.

So queue size evolves like $M_\lambda / M_c / 1$.

Stationary queue size dist is

$$\mathbb{P}(Q \geq q) = \left(\frac{\lambda}{c}\right)^q.$$

(ii). Let A = amount of work arriving in a typical timeslot.

$\longrightarrow$ made up from $N \sim \text{Poisson}(\lambda\delta)$ packets. of size $X_1 \cdots X_N \sim \text{Exp}(1)$

$$\mathbb{E}\, e^{\theta A} = \mathbb{E}\, e^{\theta(X_1 + \cdots + X_N)}$$

$$= \mathbb{E}\left[\, \mathbb{E}\, e^{\theta(X_1 + \cdots + X_N)} \mid N \,\right]$$

$$= \mathbb{E}\left[\, (\mathbb{E}\, e^{\theta X_1})^N \,\right] = \mathbb{E}\left[\, \left(\frac{1}{1-\theta}\right)^N \,\right] = e^{\lambda\delta\left(\frac{1}{1-\theta} - 1\right)} = e^{\lambda\delta\theta/1-\theta}.$$

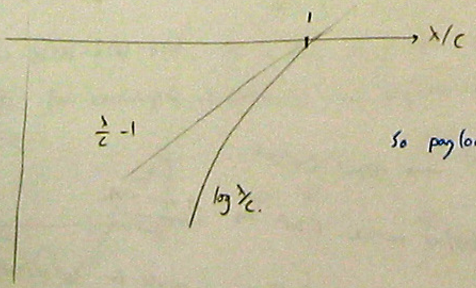
By theorem,

$$\frac{1}{r} \log \mathbb{P}(R_\delta > r) \simeq -\sup\left\{\theta > 0 : \frac{\lambda\delta\theta}{1-\theta} < \theta c \delta\right\} = 1 - \frac{\lambda}{c}.$$

NOTE: doesn't depend on δ! Good!

(iii) Header-queue: $\mathbb{P}(Q \geq q) = \left(\frac{\lambda}{c}\right)^q = e^{q \log \lambda/c}$

Payload-queue: $\mathbb{P}(R_\delta > r) \approx e^{r\cdot(\lambda/c - 1)}$.



So payload-queue more likely to overflow.

This is reasonable — there are more ways for the payload queue to overflow

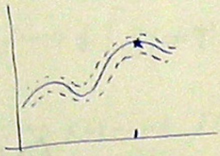
e.g. average # packets which are all v. large.

$\boxed{9}$ Note first:

* a continuous function on a compact set attains its supremum.

  [let $f(x_k) \uparrow$ supremum.
  Then (x_k) has cvgt subsequence, $(x_j) \to x$
  Clearly $f(x_j) \to$ supremum $= f(x)$.

* The function $x \mapsto \sup_{0 \le t \le 1} x(t)$ is cts w.r.t. sup-norm on $\mathcal{C}[0,1]$.

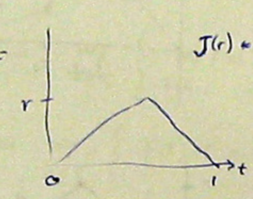


So $b \mapsto R(b) = \sup_{t \in [0,1]} x(t) - \inf_{t \in [0,1]} x(t)$, where $x(t) = b(t) - b(1)$,

is continuous. So $\frac{R}{\sqrt{N}}$ satisfies LDP.

What is the rate function? $J(r) = \inf_{b : R(b) = r} I(b)$. $I(b) = \begin{cases} \frac{1}{2}\int_0^1 \dot{b}_t^2 \, dt & \text{if } b \text{ is abs. cts} \\ \infty & \text{else.} \end{cases}$

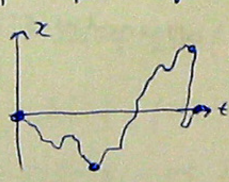
$\boxed{\le}$. Consider $\hat{b}(t) =$



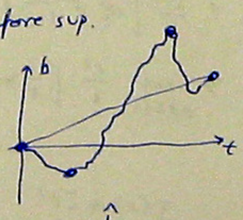
$J(r) \le I(\hat{b}) = \frac{1}{2}\left[\frac{1}{2}\left(\frac{r}{1/2}\right)^2 + \frac{1}{2}\left(\frac{r}{1/2}\right)^2 \right] = 2r^2.$

$\boxed{\ge}$. Suppose $R(b) = r$. We will argue $I(b) \ge 2r^2$ . . .
Either $I(b) = \infty$ (in which case we are done) or $I(b) < \infty$; in which case . . .

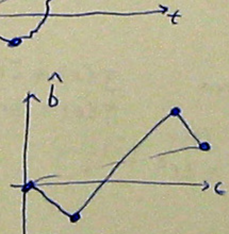
Look at $x(t) = b(t) - b(1)$. It is cts, so it attains its inf and sup.
Suppose for example it attains inf before sup.



so b is



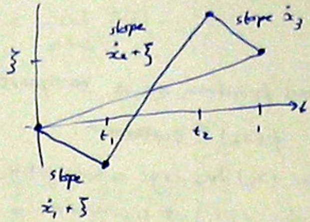
Now consider the straightened path $\hat{b}$ =



By convexity of I,

$I(b) \ge I(\hat{b})$.

The rate function of $\hat{b}$ is



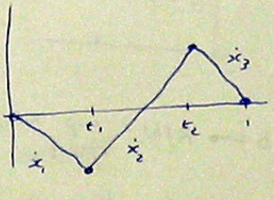
where $t_1 \dot{x}_1 + (t_2 - t_1)\dot{x}_2 + (1-t_2)\dot{x}_3 = 0$

$$I(\hat{b}) = t_1 (\dot{x}_1 + \xi)^2 + (t_2 - t_1)(\dot{x}_2 + \xi)^2 + (1-t_2)(\dot{x}_3 + \xi)^2$$

$$= \xi^2 + 2\xi \cancel{\left(t_1 \dot{x}_1 + (t_2-t_1)\dot{x}_2 + (1-t_2)\dot{x}_3\right)} + \text{const.}$$

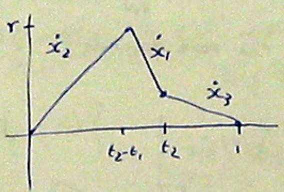
$\geq$ const; min achieved at $\xi = 0$.

Thus $I(\hat{b}) \geq I(\hat{\hat{b}})$, $\hat{\hat{b}}$ is



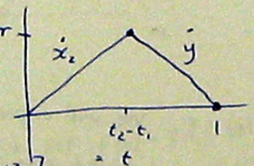
By just shuffling the pieces,

$I(\hat{\hat{b}}) \geq I(\hat{\hat{\hat{b}}})$, $\hat{\hat{\hat{b}}}$ is



By convexity,

$I(\hat{\hat{\hat{b}}}) \geq I(\hat{b}^{(*)})$, $\hat{b}^{(*)}$ is



where $t = t_2 - t_1$
$\dot{x}_2 = r/t$
$\dot{y} = -r/1-t$.

$$I(\hat{b}^{(*)}) = \frac{1}{2}\left[t\left(\frac{r}{t}\right)^2 + (1-t)\left(-\frac{r}{1-t}\right)^2 \right]$$

$$= \frac{r^2}{2}\left[\frac{1}{t} + \frac{1}{1-t}\right].$$

$$\geq \inf_{0 \leq t \leq 1} \frac{r^2}{2}\left[\frac{1}{t} + \frac{1}{1-t}\right] = 2r^2, \text{ attained at } t = \tfrac{1}{2}.$$

We've shown: $J(r) \leq 2r^2$
$J(r) \geq 2r^2$.

Thus $J(r) = 2r^2$.

$\boxed{10}$. (i) $\limsup \frac{1}{n} \log \mathbb{P}(X_n \in C_i) \le - \inf_{x \in C_i} I(x)$.

$$\limsup \frac{1}{n} \log \mathbb{E} e^{n f(X_n)}$$

$$\le \limsup \frac{1}{n} \log \Big[\sum_i e^{n \cdot \sup_{x \in C_i} f(x)} \cdot \mathbb{P}(X_n \in C_i) \Big]$$

$$\le \max_i \limsup \frac{1}{n} \log e^{n \cdot \sup_{x \in C_i} f(x)} \mathbb{P}(X_n \in C_i) \qquad \text{by PLT}$$

$$= \max_i \limsup \Big[\sup_{x \in C_i} f(x) + \frac{1}{n} \log \mathbb{P}(X_n \in C_i) \Big]$$

$$\le \max_i \sup_{x \in C_i} f(x) - \inf_{x \in C_i} I(x).$$

(ii). $\limsup \frac{1}{n} \log \mathbb{E} e^{n f(X_n)} \le \max_i \sup_{x \in C_i} f(x) - \inf_{x \in C_i} I(x)$

$\cancel{\le \max_i \sup_{x \in C_i} f(x) \quad \sup_{x \in C_i} f(x) - I(\hat{x})}$ ~~where $\hat{x}$ attains inf in C_i, so $f(\hat{x}) \ge \sup_{x \in C_i} f(x) - \varepsilon$.~~

$$\le \max_i \sup_{x \in C_i} \Big\{ \sup_{z \in C_i} f(z) - I(x) \Big\}$$

$\sup_{z \in C_i} f(z) \le f(x) + \varepsilon$ for any $x \in C_i$

$$\le \max_i \sup_{x \in C_i} \Big\{ f(x) + \varepsilon - I(x) \Big\}$$

$$= \sup_{x \in \mathcal{X}} f(x) - I(x) + \varepsilon.$$

But $\varepsilon > 0$ arbitrary, so $\limsup \le \sup_{x \in \mathcal{X}} f(x) - I(x)$.

(iii). $\liminf \frac{1}{n} \log \mathbb{E} e^{n f(X_n)} \ge \liminf \frac{1}{n} \log \mathbb{E}(e^{n f(X_n)}; X_n \in B)$

$$\ge \liminf \frac{1}{n} \log \big(e^{n(f(\hat{x}) - \varepsilon)} \mathbb{P}(X_n \in B) \big)$$

$$= \liminf \Big[f(\hat{x}) - \varepsilon + \frac{1}{n} \log \mathbb{P}(X_n \in B) \Big]$$

$$\ge f(\hat{x}) - \varepsilon - \inf_{x \in B} I(x)$$

$\ge f(\hat{x}) - \varepsilon - I(\hat{x})$. Since ε arb., $\ge f(\hat{x}) - I(\hat{x})$.

Since $\hat{x}$ arb., $\ge - \inf_{x \in C} f(x) - I(x)$.

(iv). $\liminf = \limsup$.

$\boxed{Q15}$. INTUITION.

$$\mathbb{P}\left(\frac{X^{\lfloor\alpha N\rfloor}}{N} \approx x\right) \approx \mathbb{P}\left(\frac{X^{\alpha N}}{\alpha N} \approx \frac{x}{\alpha}\right) \approx e^{-\alpha N \cdot I(x/\alpha)}$$

so $\frac{X^{\lfloor\alpha N\rfloor}}{N}$ satisfies LDP, rate func. $J(x) = \alpha I(x/\alpha)$.

PROOF.

1.LD Upper bound for semi-infinite intervals $[x, \infty)$.

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\frac{X^{\lfloor\alpha N\rfloor}}{N} \geq x\right) = \limsup \frac{\lfloor\alpha N\rfloor}{N} \cdot \frac{1}{\lfloor\alpha N\rfloor} \log \mathbb{P}\left(\frac{X^{\lfloor\alpha N\rfloor}}{\lfloor\alpha N\rfloor} \geq \frac{N}{\lfloor\alpha N\rfloor} x\right)$$

$$\leq \limsup \frac{\lfloor\alpha N\rfloor}{N} \cdot \frac{1}{\lfloor\alpha N\rfloor} \log \mathbb{P}\left(\frac{X^{\lfloor\alpha N\rfloor}}{\lfloor\alpha N\rfloor} \geq \frac{x}{\alpha} - \varepsilon\right) \quad \text{for any } \varepsilon > 0$$

$$\leq -\alpha \cdot \inf_{y \geq \frac{x}{\alpha} - \varepsilon} I(y).$$

This is true for any $\varepsilon > 0$, so

$$\limsup \frac{1}{N} \log \mathbb{P}\left(\frac{X^{\lfloor\alpha N\rfloor}}{N} \geq x\right) \leq -\alpha \lim_{\varepsilon \downarrow 0} \inf_{y \geq \frac{x}{\alpha} - \varepsilon} I(y) \leq -\alpha \inf_{y \geq \frac{x}{\alpha}} I(y).$$

For the last step, see Q2 (iv).

Specifically, suppose f is lower-semicontinuous and

$$\lim_{\varepsilon \downarrow 0} \inf_{y \geq z - \varepsilon} f(y) < \inf_{y \geq z} f(y).$$

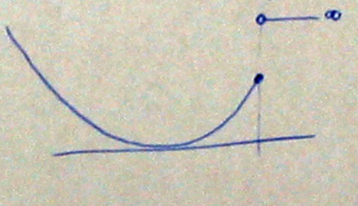
Then, for ~~infinitely many~~ arbitrarily small $\varepsilon > 0$, $\exists\, y_\varepsilon$ with $z - \varepsilon \leq y_\varepsilon \leq z$

and $f(y_\varepsilon) < \inf_{y \geq z} f(z) - \delta$, for some $\delta > 0$.

By Q2(iv), $\liminf_{\varepsilon \downarrow 0} f(y_\varepsilon) \geq f(\lim_{\varepsilon \downarrow 0} y_\varepsilon) = f(z)$. #.

Thus $\lim_{\varepsilon \downarrow 0} \inf_{y \geq z - \varepsilon} f(y) \geq \inf_{y \geq z} f(y)$.

To make this less abstract · convince yourself that I must look something like



2. LD Upper bound for semi-infinite intervals $(-\infty, x]$.

Proof similar to above.

3. LD Upper bound for general closed sets.

Proof exactly as for Cramér's theorem.

4. LD lower bound.

Pick any $\hat{x} \in B$, for some given open set B. Let $(\hat{x}-\delta, \hat{x}+\delta) \subset B$.

$$\liminf \frac{1}{N} \log \mathbb{P}\left(\frac{X^{\lfloor \alpha N \rfloor}}{N} \in B \right)$$

$$\geq \liminf \frac{\lfloor \alpha N \rfloor}{N} \cdot \frac{1}{\lfloor \alpha N \rfloor} \log \mathbb{P}\left(\frac{X^{\lfloor \alpha N \rfloor}}{\lfloor \alpha N \rfloor} \in \frac{N}{\lfloor \alpha N \rfloor} (\hat{x}-\delta, \hat{x}+\delta) \right)$$

$$\geq \liminf \frac{\lfloor \alpha N \rfloor}{N} \cdot \frac{1}{\lfloor \alpha N \rfloor} \log \mathbb{P}\left(\frac{X^{\lfloor \alpha N \rfloor}}{\lfloor \alpha N \rfloor} \in \left(\frac{\hat{x}}{\alpha} - \delta', \frac{\hat{x}}{\alpha} + \delta' \right) \right) \text{ for suff. small } \delta'$$

$$\geq -\alpha \cdot \inf_{\frac{\hat{x}}{\alpha} - \delta' < y < \frac{\hat{x}}{\alpha} + \delta'} I(y) \geq -\alpha\, I\left(\frac{\hat{x}}{\alpha}\right).$$

Take the $\inf_{\hat{x} \in B}$ to give the result.

NOTE. The upper bound would be much simpler if we had assumed that I was everywhere finite (and hence, by convexity, continuous). That would make this question exam-standard.

As it is, it is too technical to be exam-standard.