Example sheet 1—course material

Q 1 (Principle of the largest term).

i. Let $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ be sequences in $\mathbb{R}_+$. Prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log(a_n + b_n) \leq \limsup_{n\to\infty} \frac{1}{n} \log(a_n) \vee \limsup_{n\to\infty} \frac{1}{n} \log(b_n)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log(a_n + b_n) \geq \liminf_{n\to\infty} \frac{1}{n} \log(a_n) \vee \liminf_{n\to\infty} \frac{1}{n} \log(b_n).$$

ii. Let $(A_n,\ n \in \mathbb{N})$ and $(B_n,\ n \in \mathbb{N})$ be sequences of events. Suppose that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(A_n) = -a \quad \text{and that} \quad \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(B_n) = -b.$$

Deduce that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(A_n \cup B_n) = -(a \wedge b)$$

and that

$$\lim_{n\to\infty} \mathbb{P}(A_n | A_n \cup B_n) = \begin{cases} 1 & \text{if } a < b \\ 0 & \text{if } a > b \end{cases}$$

[Need: elementary limits]

Q 2 (Lower-semicontinuity). Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a lower-semicontinuous function on a Hausdorff space $\mathcal{X}$ (i.e. assume the level sets $\{x : f(x) \leq \alpha\}$ are closed for all $\alpha \in \mathbb{R}$.)

i. Let $K \subset \mathcal{X}$ be compact. Show that if $\inf_{x\in K} f(x) < \infty$ then the infimum is attained in K.
ii. Are all convex functions lower-semicontinuous?
iii. Let $g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, and let $g^*(x) = \sup_{\theta\in\mathbb{R}} \theta x - g(\theta)$. Show that g^* is lower-semicontinuous.
iv. Suppose $\mathcal{X}$ is a metric space. Show that f is lower-semicontinuous if and only if whenever $x_n \to x$ then $\liminf_{n\to\infty} f(x_n) \geq f(x)$.

[Need: elementary topology]

Q 3 (LDP for alternatives). Let $(X_n, n \in \mathbb{N})$ and $(Y_n, n \in \mathbb{N})$ satisfy large deviations principles with good rate functions I and J. Let

$$Z_n = \begin{cases} X_n & \text{if } B_n = 0 \\ Y_n & \text{if } B_n = 1 \end{cases}$$

where $B_n \sim \mathrm{Bin}(1, p)$, and is independent of X_n and Y_n. Show that Z_n satisfies an LDP with rate function $z \mapsto I(z) \wedge J(z)$. [Need: abstract large deviations]

Q 4 (Restricting an LDP). Let $(X_n,\ n \in \mathbb{N})$ be a sequence of random variables taking values in $\mathcal{X}$. Let $\mathcal{E}$ be a measurable subset of $\mathcal{X}$ such that $\mathbb{P}(X_n \in \mathcal{E}) = 1$ for all n. Equip $\mathcal{E}$ with the topology induced by $\mathcal{X}$, and suppose $\mathcal{E}$ is closed. Suppose that $(X_n,\ n \in \mathbb{N})$ satisfies an LDP in $\mathcal{X}$ with good rate function I. Prove that it satisfies an LDP in $\mathcal{E}$ with the same rate function I. [Need: abstract large deviations]

Q 5 (Hurstiness). A sequence of random variables $(X_n,\ n \in \mathbb{N})$ taking values in a metric space $\mathcal{X}$ is said to have *Hurstiness* $H \in (0, 1)$ if the following three conditions are satisfied:

i. $(X_n,\ n \in \mathbb{N})$ satisfies a large deviations principle with good rate function I at speed $n^{2(1-H)}$ (defined below);
ii. there is some $\hat{x} \in \mathcal{X}$ such that $0 < I(\hat{x}) < \infty$;
iii. there is some $\mu \in \mathcal{X}$ such that $I(x) = 0$ only if $x = \mu$.

Suppose $(X_n,\ n \in \mathbb{N})$ has Hurstiness H. Let $G > H$, $G \in (0,1)$, and define

$$I'(x) = \begin{cases} 0 & \text{if } I(x) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

i. Prove that for any closed set C

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in C) \leq - \inf_{x \in C} I'(x).$$

ii. Show that if D is an open set containing μ then

$$\mathbb{P}(X_n \notin D) \to 0.$$

Hence (or otherwise) show that for any open set E

$$\liminf_{n\to\infty} \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in E) \geq - \inf_{x \in E} I'(x).$$

iii. Deduce that $(X_n,\ n \in \mathbb{N})$ satisfies an LDP at speed $n^{2(1-G)}$ with good rate function I'.
iv. Suppose that $(X_n,\ n \in \mathbb{N})$ has Hurstiness H, that $(Y_n,\ n \in \mathbb{N})$ has Hurstiness G, that X_n is independent of Y_n, and that both take values in $\mathbb{R}$. Show that $(X_n + Y_n,\ n \in \mathbb{N})$ has Hurstiness equal to the greater of H and G.

[Need: abstract LDP]

Example sheet 2—useful exercises

Q 6 (Rate functions). Calculate the cumulant generating function, and its convex conjugate, for each of the following.
i. $X \sim$ Bernoulli(p),
ii. $X \sim$ Binomial(n, p),
iii. $X \sim$ Poisson(λ),
iv. $X \sim$ Exponential(λ),
v. $X \sim$ Geometric(ρ),
vi. $X \sim$ Normal(μ, σ^2),
vii. $X \sim$ Cauchy, with density $f(x) = \pi^{-1}(1 + x^2)^{-1}$, $x \in \mathbb{R}$.
[Need: Cramér]

Q 7. Let $A_1, A_2, \ldots$ be normal random variables with mean μ and variance σ^2. Let B be an exponential random variable with mean $1/\lambda$. Let C be a normal random variable with mean ν and variance ρ^2. Let all of these random variables be independent.
i. State, without proof, a large deviations principle for $L^{-1}B$.
ii. Find a large deviations principle for $L^{-1}(A_1 + \cdots + A_L)$.
iii. Find a large deviations principle for $L^{-1}(B + A_1 + \cdots + A_L)$.
iv. Find a large deviations principle for $L^{-1}(C + A_1 + \cdots + A_L)$.
v. Comment on your results.
State clearly any general results to which you appeal. [Need: Cramér, abstract large deviations]

Q 8. Packets arrive at an Internet router as a Poisson process of rate λ packets per second. Each packet has a payload; payload sizes are independent of each other and of the arrival process, and have an exponential distribution with mean 1 kilobyte.
The router maintains two parallel queues, a 'payload queue' and a 'header queue'. When a packet arrives, the payload is stored in the former, and a packet header is stored in the latter. Packets are served in the order they arrive. The payload is served at constant rate C kilobytes per second, and when the entire payload of a packet has been served then that packet's header is removed from the header queue.
Both queues have finite space. The payload queue has space for 1000 kilobytes; the header queue has space for 1000 headers. As a queueing theorist, you are called in to advise on which queue is more likely to overflow.
i. Let Q be the number of packet headers in the header queue. With reference to an $M/M/1$ queue (or otherwise), estimate the probability that $Q \geq q$. (For modelling purposes, you can treat both queues as having infinite space.)
ii. The payload queue may be modelled by a discrete-time queue, with timeslots of length δ, in which the number of packets arriving in each timeslot is a Poisson random variable with mean $\delta\lambda$, and the service offered in that timeslot is $C\delta$. Let R_δ be the amount of work in this discrete-time queue. Estimate the probability that $R_\delta \geq r$. (Again, for modelling purposes, you can treat both queues as having infinite space.)
iii. Which queue is more likely to overflow? Give an intuitive explanation for your answer.
[Need: LDP for simple queue]

Q 9 (Linear geodesics). A Brownian bridge is a Brownian motion over the interval $[0, 1]$ conditioned to be 0 at the right endpoint. An easy way to construct a Brownian bridge is to take a standard Brownian motion $B(t)$ and set $X(t) = B(t) - tB(1)$. Then X is a Brownian bridge. Its vertical span is

$$R = \sup_{t\in[0,1]} X(t) - \inf_{t\in[0,1]} X(t).$$

Find an LDP for $R/\sqrt{N}$. What is the most likely path to lead to a large value of R? [Need: Schilder]

Q 10 (Varadhan's lemma). Let $(X_n,\ n \in \mathbb{N})$ satisfy a large deviations principle in some space $\mathcal{X}$ with good rate function I. Let f be a bounded continuous function $f : \mathcal{X} \to \mathbb{R}$.

i. Let $C_1, \ldots, C_m$ be closed subsets of $\mathcal{X}$ with $\bigcup_i C_i = \mathcal{X}$. Prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \le \max_{1\le i\le m} \Big\{ \sup_{x\in C_i} f(x) - \inf_{x\in C_i} I(x) \Big\}.$$

ii. Let $f(\mathcal{X})$ be contained in the interval $[a, b]$. Pick any $\varepsilon > 0$ and define the closed intervals

$$D_i = \big[a + (i-1)\varepsilon, a + i\varepsilon\big], \quad i = 1, \ldots, \lceil (b-a)/\varepsilon \rceil.$$

Let $C_i = f^{-1}(D_i)$. Using your answer to part (i), or otherwise, prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \le \sup_{x\in\mathcal{X}} f(x) - I(x) + \varepsilon.$$

iii. Pick any $\hat{x} \in \mathcal{X}$ and any $\varepsilon > 0$. Define the open interval

$$D = \big(f(\hat{x}) - \varepsilon, f(\hat{x}) + \varepsilon\big).$$

Let $B = f^{-1}(D)$. Using this set, or otherwise, prove that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \ge f(\hat{x}) - I(\hat{x}) - \varepsilon.$$

iv. Deduce that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} = \sup_{x\in\mathcal{X}} f(x) - I(x).$$

[Need: abstract LDP]

Example sheet 3—further exercises

Q 11 (Moderate Deviations). Let X be a real-valued random variable, with log moment generating function $\Lambda(\theta) = \log \mathbb{E}e^{\theta X}$ finite in a neighbourhood of the origin. Let X_n be the average of n independent copies of X. Show that for any $\beta \in (0,1)$,

$$\frac{1}{n^\beta} \log \mathbb{P}\big(n^{(1-\beta)/2}(X_n - \mu) \in B\big) \approx -\inf_{x \in B} \tfrac{1}{2}x^2/\sigma^2$$

where $\mu = \mathbb{E}X$ and $\sigma^2 = \operatorname{Var} X > 0$, and the approximation means that the appropriate large deviations upper and lower bounds apply. Interpret this result, in light of Cramér's Theorem and the Central Limit Theorem. [Need: Cramér]

Q 12 (Lindley's construction). State Lindley's recursion, for a queue with constant service rate C and infinite buffer, fed by a random arrival process A. Let $R_0^{-T}(r)$ be the queue size at time 0, subject to the boundary condition that the queue size at time $-T$ is r. Show that

$$R_0^{-T}(r) = \max_{0 \le s \le T} \Big[A(-s,0] - Cs\Big] \vee (r + A(-T,0] - CT).$$

Deduce that, if $A(-t,0]/t \to \mu$ almost surely as $t \to \infty$ for some $\mu < C$, then almost surely

$$\lim_{T\to\infty} R_0^{-T}(r) = \sup_{t \ge 0} A(-t,0] - Ct \quad \text{for all } r.$$

This shows that we could just as well take any value for the 'queue size at time $-\infty$'—it makes no difference to the queue size at time 0. [Need: Lindley's recursion]

Q 13 (Extended LDP for simple queue).

i. Let A be a random stationary arrival process, and define

$$\Lambda_t(\theta) = \frac{1}{t} \log \mathbb{E}e^{\theta A(-t,0]}.$$

Suppose that the limit

$$\Lambda(\theta) = \lim_{t\to\infty} \Lambda_t(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$ for each $\theta \in \mathbb{R}$, and that it is essentially smooth, finite in a neighbourhood of $\theta = 0$, and lower-semicontinuous. State a large deviations principle for $L^{-1}A(-L,0]$.

ii. Consider a queue fed by A. Suppose the queue has infinite buffer, and constant service rate $C > \mathbb{E}X_1$. Let Q be the queue size at time 0. State and prove a large deviations principle for $L^{-1}Q$.

[Need: Cramér, LDP for a simple queue]

Q 14 (Example arrival processes). In the setting of Question 13, verify the conditions and find the rate function for queue size, for the following arrival processes.

i. $(A_t, t \in \mathbb{Z})$ is a two-state Markov chain, representing a traffic source which produces an amount of work h in each timestep while in the on state, and no work while in the off state, and which flips from on to off with probability p, and from off to on with probability q.

ii. $(A_t, t \in \mathbb{Z})$ is a stationary autoregressive process of degree 1, that is, $A_t = \mu + X_t$ where

$$X_t = \alpha X_{t-1} + (1-\alpha^2)\varepsilon_t$$

where $|\alpha| < 1$ and the ε_t are independent normal random variables with mean 0 and variance σ^2. Hint: The marginal distribution of X_t is $\mathrm{N}(0,\sigma^2)$.

[Need: Question 13]

Q 15. Let $(X^N/N,\ N \in \mathbb{N})$ satisfy a large deviations principle in $\mathbb{R}$ with convex rate function I. Let α be a positive real number. Show that $(X^{\lfloor \alpha N \rfloor}/N,\ N \in \mathbb{N})$ satisfies a large deviations principle in $\mathbb{R}$ with rate function $J(x) = \alpha I(x/\alpha)$. Hint: recall the proof of Cramér's theorem. [Need: abstract large deviations]