Example Sheet 1

These results are required for some of the course material.

Q 1 (Principle of the largest term). Let $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ be sequences in $\mathbb{R}_+$. Prove that

$$\limsup_{n\to\infty} \frac{1}{n}\log(a_n + b_n) \le \limsup_{n\to\infty} \frac{1}{n}\log(a_n) \vee \limsup_{n\to\infty} \frac{1}{n}\log(b_n)$$

and

$$\liminf_{n\to\infty} \frac{1}{n}\log(a_n + b_n) \ge \liminf_{n\to\infty} \frac{1}{n}\log(a_n) \vee \liminf_{n\to\infty} \frac{1}{n}\log(b_n).$$

[Need: elementary limits]

Q 2 (Lower-semicontinuity). Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a lower-semicontinuous function on a Hausdorff space $\mathcal{X}$ (i.e. assume the level sets $\{x : f(x) \le \alpha\}$ are closed for all $\alpha \in \mathbb{R}$.)

i. Let $K \subset \mathcal{X}$ be compact. Show that if $\inf_{x\in K} f(x) < \infty$ then the infimum is attained in K.
ii. Are all convex functions lower-semicontinuous?
iii. Let $g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, and let $g^*(x) = \sup_{\theta\in\mathbb{R}} \theta x - g(\theta)$. Show that g^* is lower-semicontinuous.

[Need: elementary topology]

Q 3 (Useful LDPs). Let $(X_n, n \in \mathbb{N})$ and $(Y_n, n \in \mathbb{N})$ satisfy large deviations principles in Hausdorff spaces $\mathcal{X}$ and $\mathcal{Y}$ with good rate functions I and J.

i. Suppose that (for each n) X_n and Y_n are independent, and that $\mathcal{X}$ and $\mathcal{Y}$ are separable (i.e. that they have countable bases of open sets.) Show that (X_n, Y_n) satisfies a large deviations principle in $\mathcal{X} \times \mathcal{Y}$ with good rate function $(x, y) \mapsto I(x) + J(y)$.
ii. Suppose $\mathcal{X} = \mathcal{Y}$, and let

$$Z_n = \begin{cases} X_n & \text{if } B_n = 0 \\ Y_n & \text{if } B_n = 1 \end{cases}$$

where $B_n \sim \mathrm{Bin}(1, p)$, and is independent of X_n and Y_n. Show that Z_n satisfies an LDP in $\mathcal{X}$ with rate function $z \mapsto I(z) \wedge J(z)$.

[Need: abstract large deviations]

Q 4 (Restricting an LDP). Let $(X_n,\ n \in \mathbb{N})$ be a sequence of random variables taking values in $\mathcal{X}$. Let $\mathcal{E}$ be a measurable subset of $\mathcal{X}$ such that $\mathbb{P}(X_n \in \mathcal{E}) = 1$ for all n. Equip $\mathcal{E}$ with the topology induced by $\mathcal{X}$, and suppose $\mathcal{E}$ is closed. Prove the following.

i. If $(X_n,\ n \in \mathbb{N})$ satisfies an LDP in $\mathcal{E}$ with rate function I, then it satisfies an LDP in $\mathcal{X}$ with rate function

$$I'(x) = \begin{cases} I(x) & \text{if } x \in \mathcal{E} \\ \infty & \text{otherwise.} \end{cases}$$

ii. If $(X_n,\ n \in \mathbb{N})$ satisfies an LDP in $\mathcal{X}$ with good rate function I then it satisfies an LDP in $\mathcal{E}$ with the same rate function I.

[Need: abstract large deviations]

Q 5 (Restricted contraction principle). Suppose that X_n satisfies a large deviations principle in some Hausdorff space $\mathcal{X}$ with good rate function I, and let $f : \mathcal{X} \to \mathcal{Y}$ be a map to another Hausdorff space $\mathcal{Y}$. Suppose there exists an open neighbourhood $\mathcal{E}$ of the effective domain of I, such that f is continuous on $\bar{\mathcal{E}}$. Show that $f(X_n)$ satisfies a large deviations principle in $\mathcal{Y}$ with good rate function $J(y) = \inf_{x:f(x)=y} I(x)$. [Need: contraction principle]

Q 6 (Moderate Deviations). Let X be a real-valued random variable, with log moment generating function $\Lambda(\theta) = \log \mathbb{E}e^{\theta X}$ finite in a neighbourhood of the origin. Let X_n be the average of n independent copies of X. Show that for any $\beta \in (0,1)$,

$$\frac{1}{n^\beta} \log \mathbb{P}\big(n^{(1-\beta)/2}(X_n - \mu) \in B\big) \approx - \inf_{x \in B} \tfrac{1}{2}x^2/\sigma^2$$

where $\mu = \mathbb{E}X$ and $\sigma^2 = \operatorname{Var} X > 0$, and the approximation means that the appropriate large deviations upper and lower bounds apply. Interpret this result, in light of Cramér's Theorem and the Central Limit Theorem. [Need: Cramér]

Example Sheet 2

These questions test your understanding of the course material.

Q 7 (Definition of queue size). State Lindley's recursion, for a queue with constant service rate C and infinite buffer, fed by a random arrival process A. Let $R_0^{-T}(r)$ be the queue size at time 0, subject to the boundary condition that the queue size at time $-T$ is r. Show that

$$R_0^{-T}(r) = \max_{0 \le s \le T} \Big[A(-s,0] - Cs \Big] \vee (r + A(-T,0] - CT).$$

Deduce that, if $A(-t,0]/t \to \mu$ almost surely as $t \to \infty$ for some $\mu < C$, then almost surely

$$\lim_{T \to \infty} R_0^{-T}(r) = \sup_{t \ge 0} A(-t,0] - Ct \quad \text{for all } r.$$

This shows that we could just as well take any value for the 'queue size at time $-\infty$'—it makes no difference to the queue size at time 0. [Need: Lindley's recursion]

Q 8 (Rate functions). Calculate the cumulant generating function, and its convex conjugate, for each of the following.

i. $X \sim$ Bernoulli(p),
ii. $X \sim$ Binomial(n, p),
iii. $X \sim$ Poisson(λ),
iv. $X \sim$ Exponential(λ),
v. $X \sim$ Geometric(ρ),
vi. $X \sim$ Normal(μ, σ^2),
vii. $X \sim$ Cauchy, with density $f(x) = \pi^{-1}(1 + x^2)^{-1}$, $x \in \mathbb{R}$.

[Need: Cramér]

Q 9 (Extended LDP for simple queue).

i. Let A be a random stationary arrival process, and define

$$\Lambda_t(\theta) = \frac{1}{t} \log \mathbb{E} e^{\theta A(-t,0]}.$$

Suppose that the limit

$$\Lambda(\theta) = \lim_{t \to \infty} \Lambda_t(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$ for each $\theta \in \mathbb{R}$, and that it is essentially smooth, finite in a neighbourhood of $\theta = 0$, and lower-semicontinuous. State a large deviations principle for $L^{-1}A(-L,0]$.

ii. Consider a queue fed by A. Suppose the queue has infinite buffer, and constant service rate $C > \mathbb{E}X_1$. Let Q be the queue size at time 0. State and prove a large deviations principle for $L^{-1}Q$.

[Need: Cramér, LDP for a simple queue]

Q 10 (Example arrival processes). In the setting of Question 9, verify the conditions and find the rate function for queue size, for the following arrival processes.

i. $(A_t, t \in \mathbb{Z})$ is a two-state Markov chain, representing a traffic source which produces an amount of work h in each timestep while in the on state, and no work while in the off state, and which flips from on to off with probability p, and from off to on with probability q.

ii. $(A_t, t \in \mathbb{Z})$ is a stationary autoregressive process of degree 1, that is, $A_t = \mu + X_t$ where

$$X_t = \alpha X_{t-1} + (1 - \alpha^2)\varepsilon_t$$

where $|\alpha| < 1$ and the ε_t are independent normal random variables with mean 0 and variance σ^2. Hint: The marginal distribution of X_t is $\mathrm{N}(0, \sigma^2)$.

[Need: Question 9]

Q 11. Let $(X^N/N,\ N \in \mathbb{N})$ satisfy a large deviations principle in $\mathbb{R}$ with convex rate function I. Let α be a positive real number. Show that $(X^{\lfloor \alpha N \rfloor}/N,\ N \in \mathbb{N})$ satisfies a large deviations principle in $\mathbb{R}$ with rate function $J(x) = \alpha I(x/\alpha)$. Hint: recall the proof of Cramér's theorem. [Need: abstract large deviations]

Q 12 (Empirical distributions). A discrete-time Markov chain (X_t) on the states $\{1,2,3,4\}$ moves according to the transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-p & p & 0 & 0 \\ 1-q & 0 & q & 0 \\ 0 & r & 1-r & 0 \end{pmatrix}$$

and $X_0 = 4$. Given that the empirical distribution of $X_1, \dots, X_n$ on $\{1,2,3\}$ satisfies a large deviations principle as $n \to \infty$, write down (without proof) what you expect its rate function to be. For what choices of p, q and r is the rate function good? convex? [Need: abstract large deviations, Sanov's theorem]

Q 13. Let $A_1, A_2, \dots$ be normal random variables with mean μ and variance σ^2. Let B be an exponential random variable with mean $1/\lambda$. Let C be a normal random variable with mean ν and variance ρ^2. Let all of these random variables be independent.

i. State, without proof, a large deviations principle for $L^{-1}B$.
ii. Find a large deviations principle for $L^{-1}(A_1 + \cdots + A_L)$.
iii. Find a large deviations principle for $L^{-1}(B + A_1 + \cdots + A_L)$.
iv. Find a large deviations principle for $L^{-1}(C + A_1 + \cdots + A_L)$.
v. Comment on your results.

State clearly any general results to which you appeal. [Need: Cramér, abstract large deviations]

Q 14 (Linear geodesics). A Brownian bridge is a Brownian motion over the interval $[0,1]$ conditioned to be 0 at the right endpoint. An easy way to construct a Brownian bridge is to take a standard Brownian motion $B(t)$ and set $X(t) = B(t) - tB(1)$. Then X is a Brownian bridge. Its vertical span is

$$R = \sup_{t \in [0,1]} X(t) - \inf_{t \in [0,1]} X(t).$$

Find an LDP for $R/\sqrt{N}$. What is the most likely path to lead to a large value of R? [Need: Schilder]

Q 15 (Underflow in queues fed by many flows).

i. Let q be the queue size function for a queue with infinite buffer size and finite service rate C. Let $\mathcal{D}_\mu$ be the space of discrete-time arrival processes with mean rate $\mu < C$. Let

$$B = \{a \in \mathcal{D}_\mu : q(a) > 0\}.$$

Show that

$$\bar{B} = \bigcup_{t>0} \{a \in \mathcal{D}_\mu : a(-t,0] \geq Ct\}.$$

ii. Suppose that A^L is the average of L independent copies of some stationary random arrival process, with mean rate μ, and that it satisfies the conditions of the many-flows sample path LDP. Show that

$$\limsup_{L\to\infty} \frac{1}{L} \log \mathbb{P}(B) \leq \inf_{t>0} \sup_{\theta \in \mathbb{R}} \theta C t - \Lambda_t(\theta) \leq \sup_{\theta\in\mathbb{R}} \theta C - \Lambda_1(\theta) < 0$$

where $\Lambda_t(\theta) = \log \mathbb{E} e^{\theta A(-t,0]}$.

This shows that it is rare for the queue to be non-empty. [Need: many flows limit]

Q 16 (Duality of convex conjugate). Let X be a real-valued random variable, and let $\Lambda(\theta) = \log \mathbb{E} e^{\theta X}$. Suppose that Λ is finite in a neighbourhood of the origin. Show that $(\Lambda^*)^* = \Lambda$. [Need: Varadhan's Theorem. Although this can be proved directly, for any convex lower-semicontinuous function Λ.]