Large Deviations and Queueing Theory

Example Sheet—Lent 2001—D.J. Wischik

Q 1 (Laplace's Principle). Prove that if a_i^L, $1 \le i \le k$, $L \ge 1$, are such that

$$\lim_{L\to\infty} \frac{1}{L} \log a_i^L = \alpha_i,$$

then

$$\lim_{L\to\infty} \frac{1}{L} \log\Big(\sum_{i=1}^k a_i^L\Big) = \max_{1\le i\le k} \alpha_i.$$

Q 2. Let $X^L = L^{-1} \sum_{i=1}^{N_L} Y_i$, where $Y_1, Y_2, \dots$ are IID with distribution $\mathrm{N}(0,1)$, and N_L is Poisson with mean L. Show that the X^L satisfy a large deviations principle, and calculate the rate function. Is the rate function good? Is it convex?

Q 3. (a) State the Gärtner-Ellis theorem.

(b) Let $W_0, W_1, \dots$ be IID exponential random variables with mean λ^{-1}. Let $|a| < 1$, and consider the process defined by $X_0 = 0; X_{n+1} = aX_n + W_n$, $n = 0, 1, \dots$. Define $Y^L = L^{-1}(X_1 + \dots + X_L)$. Show that the sequence Y^L satisfies a large deviations principle with rate

$$I(y) = \lambda y(1-\alpha) - 1 - \log\big(\lambda y(1-\alpha)\big).$$

Q 4. (a) State and prove the Contraction Principle.

(b) Let $\bar{Q}$ be the queue size function for a queue with service rate C and buffer B. Define the space $(\mathcal{X}_\mu, \|\cdot\|)$, and show that $\bar{Q}$ is continuous on it, for $\mu < C$.

Q 5. Let $\mathbf{X}$ be a stationary auto-regressive random process, given by

$$\begin{aligned} X_0 &\sim \mathrm{N}(\lambda, \sigma^2) \\ X_{t+1} &= \lambda + a(X_t - \lambda) + \sqrt{1-a^2}\varepsilon_t \end{aligned}$$

where $0 < a < 1$ and the random variables ε_t are independent and normally distributed with mean 0 and variance σ^2. Consider a queue fed by the average of L independent copies of $\mathbf{X}$, served at rate $C > \lambda$, with infinite buffer. Let Q be the queue size at time 0. Explain carefully why a path most likely to lead to $Q \ge b$ has the form

$$\hat{x}_i = \lambda + \hat{\theta}\sigma^2\Big(\frac{1-a^i}{1-a} + \frac{1-a^{\hat{t}-i+1}}{1-a} - 1\Big), \quad -\hat{t} \le i < 0,$$

for suitable $\hat{\theta}$ and $\hat{t}$.

Q 6. What is the Hurst parameter of the Bellcore traffic data? (`http://ita.ee.lbl.gov/html/contrib/BC.html`)

Q 7. I have thrown six million fair dice, and my total score is 25 million. How many sixes have I thrown?

Q 8. Let $\mathbf{X}^L$ be a sequence of traffic processes, satisfying a large deviations principle in $(\mathcal{X}_\mu, \|\cdot\|)$ with good rate function

$$I(\mathbf{x}) = \sum_{s=1}^{\infty} \Lambda^*(x_s),$$

where Λ^* is a convex function, with $\Lambda^*(\lambda) = 0$ for some $0 < \lambda < \mu$.

(a) Give an example of a such a sequence.

(b) Let $D(\mathbf{X}^L)$ be the output process from a queue fed by $\mathbf{X}^L$, served at constant rate $C > \mu$, with an infinite buffer. Show that $D(\mathbf{X}^L)$ satisfies a large deviations principle, and find its rate function. Explain why the output process is smoother than the input process. (You may assume that D is continous.)

Q 9. Let X^L be the average of L independent copies of a real-valued random variable X, whose moment generating function is finite in a neighbourhood of the origin. State and prove a large deviations principle for

$$L^{(1-\beta)/2}(X^L - \mu),$$

where $\mu = \mathbb{E}X$ and $0 < \beta < 1$.

Q 10. (a) Let $\mathbf{X}$ be a random stationary traffic process, and define

$$\Lambda_t(\theta) = \frac{1}{t} \log \mathbb{E} e^{\theta X(0,t]}.$$

Suppose that the limit

$$\Lambda(\theta) = \lim_{t \to \infty} \Lambda_t(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$ for each $\theta \in \mathbb{R}$, and that it is essentially smooth, finite in a neighbourhood of $\theta = 0$, and lower-semicontinuous. State a large deviations principle for $L^{-1}X(0, L]$.

(b) Consider a queue fed by $\mathbf{X}$. Suppose the queue has infinite buffer, and constant service rate $C > \mathbb{E}X_1$. Let Q be the queue size at time 0. State and prove a large deviations principle for $L^{-1}Q$.

Q 11. A discrete-time Markov chain (X_t) on the states $\{1, 2, 3, 4\}$ moves according to the transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-p & p & 0 & 0 \\ 1-q & 0 & q & 0 \\ 0 & r & 1-r & 0 \end{pmatrix}$$

and $X_0 = 4$. Given that the empirical distribution of $X_1, \ldots, X_n$ on $\{1, 2, 3\}$ satisfies a large deviations principle as $n \to \infty$, write down (without proof) what you expect its rate function to be. For what choices of p, q and r is the rate function good? convex?