Large Deviations and Queueing Theory

Exam questions—Reading Course 2002—D.J. Wischik

Q 1. The starship Voyager is far away in the Delta quadrant, and is heading for Earth 47000 lightyears away. Every day, it may either be given a hyperspace boost by friendly aliens, bringing it 100 lightyears closer; or it may travel without incident, coming 3 lightyears closer; or it may pause to explore a strange new world, and only come 1 lightyear closer. These events occur independently: the first has probability 1/1000, and the other two are equally likely.

In fact, Voyager returns home in 4700 days, much faster than expected. How many hyperspace boosts did it get on its journey home? State clearly any general results you use.

Q 2. (a) Let X be a geometric random variable with mean $1/p$, and let $X^L = X/L$. Show that X^L satisfies a large deviations principle with good rate function $I(x) = x \log(1-p)^{-1}$.

(b) State the contraction principle.

(c) Let $X_1, \ldots, X_n$ be independent geometric random variables with means $1/p_1, \ldots, 1/p_n$. Find a large deviations principle for $X_1 + \cdots + X_n$.

Q 3. State and prove Varadhan's Integral Theorem.

Q 4. Consider a queue operating in slotted time, with finite buffer B and constant service rate C, receiving x_t units of work in timeslot t. For $t > 0$ let $x(0,t] = x_{-t} + \cdots + x_{-1}$. Let A be the set of input processes for which $x(0,t] \le \lambda t$ eventually, and let $\|x\| = \sup_{t>0} |x(0,t]/t|$. Prove that if $\lambda < C$ then the queue size function is continuous on $(A, \|\cdot\|)$.