The Unreasonable Effectiveness of (High School) Mathematics

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Some sums...

(elementary school!)

\[2 + 2 = 4\]
\[1 + 0 = 1\]
\[0 + 3 = 3\]
\[0 + 0 = 0\]
... with variables

(high school)

\[ x + 0 = x \]

\[ 0 + x = x \]
... with three numbers

\[3 + 4 + 2 = 9\]
\[6 + 1 + 4 = 11\]

How did you do it?
Reducing “pairs”

\[
(6 + 1) + 4 \quad 6 + (1 + 4)
\]

\[
= 7 + 4 \quad = 6 + 5
\]

\[
= 11 \quad = 11
\]
Some axioms of +

0 does nothing (with respect to +)

\[ x + 0 = x \]

\[ 0 + x = x \]

grouping into pairs doesn’t change result

\[ (x + y) + z = x + (y + z) \]
Some more sums...

\[ 2 \times 2 = 4 \]
\[ 3 \times 4 = 12 \]
\[ 1 \times 3 = 3 \]
\[ 4 \times 1 = 4 \]
... with variables

\[ x \times 1 = x \]

\[ 1 \times x = x \]
Reducing “pairs”

\[(2 \times 3) \times 4\]  
\[= 6 \times 4\]  
\[= 24\]

\[2 \times (3 \times 4)\]  
\[= 2 \times 12\]  
\[= 24\]
Some axioms of $\times$

1 does nothing (with respect to $\times$)

$$x \times 1 = x$$

$$1 \times x = x$$

grouping into pairs doesn’t change result

$$(x \times y) \times z = x \times (y \times z)$$
Axioms of $\times$ and $+$

$$x \times 1 = x$$

$$1 \times x = x$$

$$(x \times y) \times z = x \times (y \times z)$$

$$\left( x \times y = y \times x \right)$$

$$x + 0 = x$$

$$0 + x = x$$

$$(x + y) + z = x + (y + z)$$

$$x + y = y + x$$
Common structure...

monoids

Not the Dr. Who aliens with one eye.....
Monoids

• A collection of things $X$
  e.g. (whole) numbers

• An operation $\oplus$ that turns two $X$s into one $X$
  e.g. $\pm$ or $\times$

• A special $X$, call it $n$, that does “nothing” with $\oplus$
  e.g. $0$ (for $\pm$) or $1$ (for $\times$)

\[
\begin{align*}
  x \oplus n & = x & \{\text{right unit}\} \\
  n \oplus x & = x & \{\text{left unit}\} \\
  (x \oplus y) \oplus z & = x \oplus (y \oplus z) & \{\text{associativity}\}
\end{align*}
\]
Unreasonably effective

- Monoids are a simple (abstract) concept (cf. counting*)
- Extremely ubiquitous
  - Maths
  - Physics
  - Topology
- Computer Science
  - Logic
  - Linguistics (semantics)
  - Everyday phenomena

* The Unreasonable Effectiveness of Mathematics (R.W. HAMMING, 1980)
Not *trivially* effective

- Plenty of things are not *monoids*
  
  
  \[ 3 - 0 = 3 \]  
  \[ 0 - 3 = -3 \]  
  \[ (2 - 3) - 4 = -5 \]  
  \[ 2 - (3 - 4) = 3 \]

- Interesting to study the things that are not
New example: Paint mixing!
Paint-mixing

- Collection of “things” $\times$
  Acrylic paints

- Operation that turns two $\times$s into one $\times$
  $\oplus = \text{mixing of paints}$

- Special $\times$ that does “nothing” with $\oplus$
  $n = \text{“Extender” base}$
Monoids of “containment”

- \( \odot \) means take two containers where one is inside the other, and flatten into one:
Monoids of “containment”

- “Nothing” (trivial) container, \( n = x \)

\[ \begin{array}{c}
\circled{+} \quad \boxed{x} \\
\circled{+} \quad \boxed{x}
\end{array} = \boxed{x} \]
Dreams as containers...

(idea originally due to Dan Piponi)
("Containment monoids" usually called monads)
Dreams as container monoid...

• Anything can be put into a (trivial) dream
Dreams as container monoid...

- A dream inside a dream is just a dream (collapse)
THE DREAM IS COLLAPSING
Computations (functions)

A \xrightarrow{f} B \xrightarrow{\oplus} C

B \xrightarrow{g} C

= \hspace{1cm}

A \xrightarrow{f \oplus g} C
Computations (functions)

\[ \begin{array}{c}
\text{A} \quad f \quad \text{B} \\
\hline
\end{array} \quad \oplus \quad \begin{array}{c}
x \\
\hline
\end{array} \quad \text{n} \quad \begin{array}{c}
x \\
\hline
\end{array} \\
\quad = \quad \begin{array}{c}
\text{A} \\
\hline
\end{array} \quad f \quad \begin{array}{c}
\text{B} \\
\hline
\end{array}
\]
My work....

\[ \text{A} \xrightarrow{f} \text{B} \oplus \text{B} \xrightarrow{g} \text{C} = \text{A} \xrightarrow{f \oplus g} \text{C} \]

- Special kinds of monoids for \( \triangle \) and \( \star \)
Importance of monoidality

• Underlying equational theory

\[ \ldots + x + 5 + (-5) + y + \ldots. \]

Additional property

\[ \downarrow \]

\[ \ldots + x + 0 + y + \ldots. \]

Monoidality

\[ \downarrow \]

\[ \ldots + x + y + \ldots. \]
<table>
<thead>
<tr>
<th>Expression</th>
<th>Equation</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x ⊕ n$</td>
<td>$= x$</td>
<td>right unit</td>
</tr>
<tr>
<td>$n ⊕ x$</td>
<td>$= x$</td>
<td>left unit</td>
</tr>
<tr>
<td>$(x ⊕ y) ⊕ z$</td>
<td>$= x ⊕ (y ⊕ z)$</td>
<td>associativity</td>
</tr>
</tbody>
</table>

- Monoids pervasive (many more examples)
- Unreasonably effective but very simple
- My work: more complex models of computation with underlying monoid properties:

Thanks! [dorchard.co.uk](http://dorchard.co.uk)
Back-up slides
Computations (functions)

\[
\begin{align*}
\text{A} &\quad \boxed{f} \quad \text{B} \quad \bigcirc \quad \text{B} \quad \boxed{g} \quad \text{C} \quad \bigcirc \quad \text{C} \quad \boxed{h} \quad \text{D} \\
\text{A} &\quad \boxed{(f \oplus g) \oplus h} \quad \text{C} \\
\end{align*}
\]
Computations (functions)

\[ A \xrightarrow{f} B \oplus \xrightarrow{g} C \oplus \xrightarrow{h} D \]

\[ = \]

\[ A \xrightarrow{f} B \oplus \xrightarrow{g \oplus h} C \]

\[ = \]

\[ A \xrightarrow{f \oplus (g \oplus h)} C \]
Non-deterministic Computations

• Previously, output single result

• Non-deterministic: output many possible results
Non-deterministic Computations

\[ \begin{array}{c}
A \\
\xrightarrow{f} \\
\oplus \\
\xrightarrow{g} \\
\end{array} \quad \begin{array}{c}
B^* \\
\oplus \\
C^* \\
\end{array} \]
Non-deterministic Computations

\[ f \oplus g^* = f \oplus g^* \]
Non-deterministic Computations

- Need to design the \( * \) operation and a “nothing” computation \( id^* \) to satisfy monoid axioms e.g.: