

Equational Systems and Free Constructions

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Contributions of the paper

- ① General abstract **Definition** of Equational System.
- ② Development of the **Theory** of Equational Systems.
- ③ **Applications** of Equational Systems.

Overview: Definition of Equational System

Equational Systems are a **framework** for **defining models** of systems.

- **Models of Logics**

what we want to reason about.

- **Models of Computational Calculi**

semantic domains where meanings of programs are defined.

e.g. λ -calculus, π -calculus, ...

- **Models of Data Types**

semantic domains where data types and type constructors are interpreted.

⋮

Overview: Theory of Equational Systems

$$\begin{array}{ccc} & \text{Model}(\mathbb{S}) & \\ & \uparrow & \\ F^{\downarrow} & \dashv & U \\ & \searrow & \downarrow \\ & \mathcal{D} & \end{array}$$

1 Construction of free models

Theoretically, the models of \mathbb{S} can be **represented by a monad**.
Practically, it gives interesting models:

- it may give **syntactic models**. (initial algebra semantics)
- it may give **fully abstract models**.
- \vdots

2 $\text{Model}(\mathbb{S})$ is cocomplete.

Models can be **combined** to form new ones (e.g. by coproducts or pushouts) in a **compositional** fashion.

Overview: Applications of Equational System

- Algebraic Theories
 - First-order equational logic
 - Specification, correctness and implementation of abstract data types [ADJ Group '78]
- Enriched Algebraic Theories [Kelly & Power '93]
 - Algebraic treatment of computational effects [Plotkin & Power '03, '04]
- Equational Systems
 - Σ -monoids [Fiore, Plotkin & Turi '99]
 - π -algebras [Stark '05]
 - **Nominal equational logic** [Clouston & Pitts '07]

Motivation for definition: I. Signatures

Signatures as Endofunctors

• Algebraic Theory

$$\Sigma_{\text{Num}} = \{ \text{zero} : 0, \text{succ} : 1, \text{plus} : 2 \}$$

- Σ_{Num} -algebra
- $D \in \mathbf{Set}$
 - $\llbracket \text{zero} \rrbracket : D^0 \rightarrow D$
 - $\llbracket \text{succ} \rrbracket : D^1 \rightarrow D$
 - $\llbracket \text{plus} \rrbracket : D^2 \rightarrow D$

• Equational System

$$\Sigma_{\text{Num}}(X) = X^0 + X^1 + X^2 \text{ on } \mathbf{Set}$$

- Σ_{Num} -algebra
- $D \in \mathbf{Set}$
 - $s : \Sigma_{\text{Num}} D \rightarrow D$
: $D^0 + D^1 + D^2 \rightarrow D$

Motivation for definition: II. Equations

Equations as parallel pairs of Functors

$$\{x, y\} \vdash \text{plus}(\text{succ}(x), y) = \text{succ}(\text{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ \llbracket \text{zero} \rrbracket \downarrow & \llbracket \text{succ} \rrbracket \downarrow & \llbracket \text{plus} \rrbracket \downarrow \\ D & D & D \end{array} \mapsto \forall \rho : \{x, y\} \rightarrow D$$
$$\llbracket \text{plus}(\text{succ}(x), y) \rrbracket_{\rho} = \llbracket \text{succ}(\text{plus}(x, y)) \rrbracket_{\rho} \in D$$

Equational System

$$\boxed{\Sigma_{\text{Num-Alg}}} \quad \Longrightarrow \quad \boxed{(-)^2\text{-Alg}}$$
$$\begin{array}{ccc} 1 + D + D^2 & \mapsto & \begin{array}{c} D^2 \\ \swarrow \quad \searrow \\ \llbracket \text{succ} \rrbracket \times \text{id} \quad \llbracket \text{plus} \rrbracket \\ D^2 = D \\ \swarrow \quad \searrow \\ \llbracket \text{plus} \rrbracket \quad \llbracket \text{succ} \rrbracket \\ D \end{array} \\ \llbracket \llbracket \text{zero} \rrbracket, \llbracket \text{succ} \rrbracket, \llbracket \text{plus} \rrbracket \rrbracket \downarrow & & \end{array}$$

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Definition of Equational System

$$\begin{array}{ccc} \Sigma\text{-Alg} & \begin{array}{c} \xrightarrow{L} \\ \xrightarrow{R} \end{array} & \Gamma\text{-Alg} \\ U_\Sigma \downarrow & \text{=} & \swarrow U_\Gamma \\ \mathcal{D} & & \end{array}$$

- **Equational System \mathbb{T}**

$$(\mathcal{D} \triangleright \Sigma \vdash L = R : \Gamma)$$

- **\mathbb{T} -Algebra**

$$(D, s : \Sigma D \rightarrow D)$$

such that

$$L(D, s) = R(D, s)$$

Definition of Equational System

$$\begin{array}{ccccc} \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} & \xrightleftharpoons[L]{L} & \Gamma\text{-Alg} \\ & \searrow^{U_{\mathbb{T}}} & \downarrow^{U_{\Sigma}} & \swarrow_{U_{\Gamma}} & \\ & & \mathcal{D} & & \end{array}$$

$=$

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such that

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Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} \triangleright \Sigma \vdash L = R : \Gamma)$ an Equational System,

$$\begin{array}{ccc} & \xleftarrow{K_{\mathbb{T}}} & \\ & \perp & \\ \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\ & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\ & & \mathcal{D} \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

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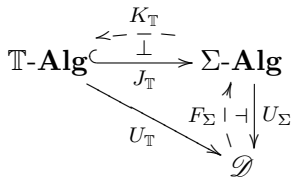
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 & \overset{K_{\mathbb{T}}}{\longleftarrow} & \\
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 \downarrow s & \searrow s_1 & \\
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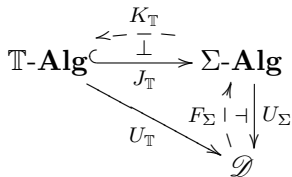
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 X & \xrightarrow{e_1} & X_1 & \xrightarrow{e_2} & X_2 & \xrightarrow{e_3} & X_3 \cdots X_{\omega} \\
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 X & \xrightarrow{e_1} & X_1 & \xrightarrow{e_2} & X_2 & \xrightarrow{e_3} & X_3 \cdots X_{\omega} \cdots \\
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 \end{array}$$

Theorem: Properties of Categories of Algebras

For $\mathbb{T} = (\mathcal{D} \triangleright \Sigma \vdash L = R : \Gamma)$ an Equational System,

$$\begin{array}{ccc} & \mathbb{T}\text{-Alg} & \\ F_{\mathbb{T}} \uparrow & \dashv & \downarrow U_{\mathbb{T}} \\ & \mathcal{D} & \\ & \downarrow T_{\mathbb{T}} = U_{\mathbb{T}} F_{\mathbb{T}} & \end{array}$$

\mathcal{D} is cocomplete.
 Σ, Γ preserve ω -colimits.

- 1 $\mathbb{T}\text{-Alg}$ is cocomplete.
- 2 $\mathbb{T}\text{-Alg}$ is monadic over \mathcal{D} .
- 3 $T_{\mathbb{T}}$ preserves colimits of ω -chains.

An Application: Nominal Equational Logic (Clouston & Pitts '07)

Term = a, b, c, \dots | $\lambda[a]$ **Term** | **Term**@**Term**

- A NEL-theory \mathbb{T}_λ for λ -calculus

$\langle - \rangle : \mathbb{A} \rightarrow \mathbf{tm}$

$\lambda : \mathbb{A} \times \mathbf{tm} \rightarrow \mathbf{tm}$

$-@- : \mathbf{tm} \times \mathbf{tm} \rightarrow \mathbf{tm}$

$a : \mathbb{A}, x : \mathbf{tm} \vdash a \# \lambda(a, x) : \mathbf{tm} \quad (\alpha)$

$a : \mathbb{A}, a \# x : \mathbf{tm} \vdash \lambda(a, x@ \langle a \rangle) = x : \mathbf{tm} \quad (\eta)$

$a : \mathbb{A}, a \# x : \mathbf{tm}, y : \mathbf{tm} \vdash \lambda(a, x)@y = x : \mathbf{tm} \quad (\beta-1)$

\vdots

- An Equational System for \mathbb{T}_λ

$(\mathbf{Nom} \triangleright \Sigma_\lambda \vdash L_\lambda = R_\lambda : \Gamma_\lambda)$

$\Sigma_\lambda D = \mathbb{A} + \mathbb{A} \times D + D \times D$

$\Gamma_\lambda D = \mathbb{A} \times D + \mathbb{A} \otimes D + (\mathbb{A} \otimes D) \times D + \dots$

Concluding Remarks

Conclusion

- As an advantage, Equational Systems provide a general, abstract and **practical** theory for the specification and free construction of equational models suitable for **modern applications**.
- As a drawback, but in favour of generality, Equational Systems do not have associated syntactic (Lawvere) theories in general.

Further work

- Enriched Equational Systems
- Equational Cosystems
- Inequational Systems – Rewriting
 - Modularity of Confluence
 - Confluence of Orthogonal Systems
- Conditional (In)equational System