Model-comparison Games with Algebraic Rules

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Games in finite model theory

Logic	Corresponding game
First-order logic	Ehrenfeucht-Fraïssé game

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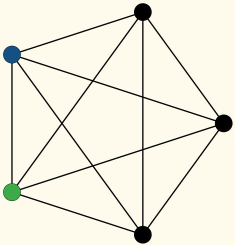

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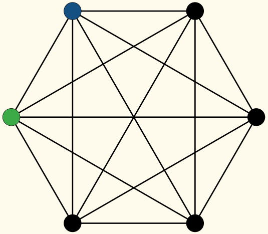

H

Spoiler
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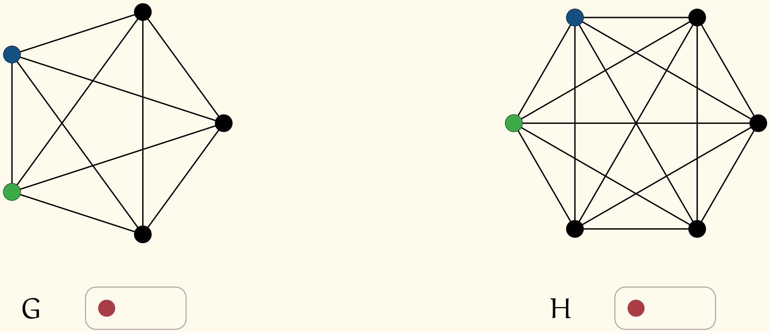

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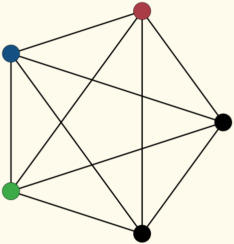

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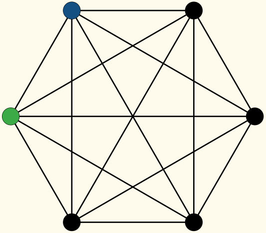

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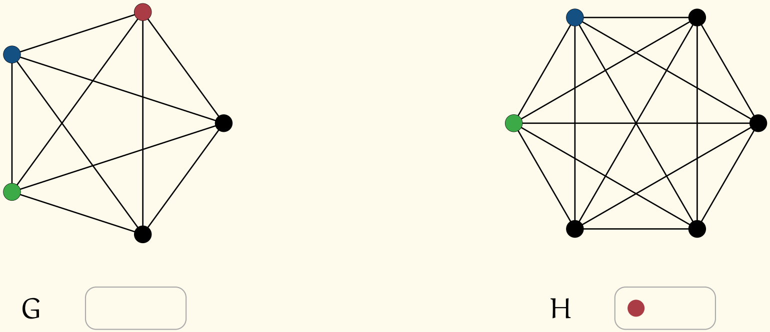

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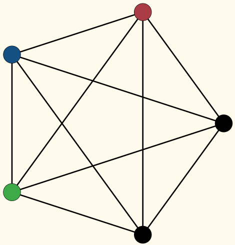

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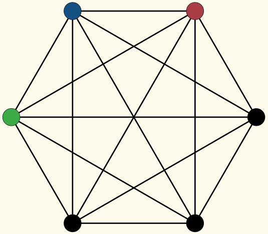

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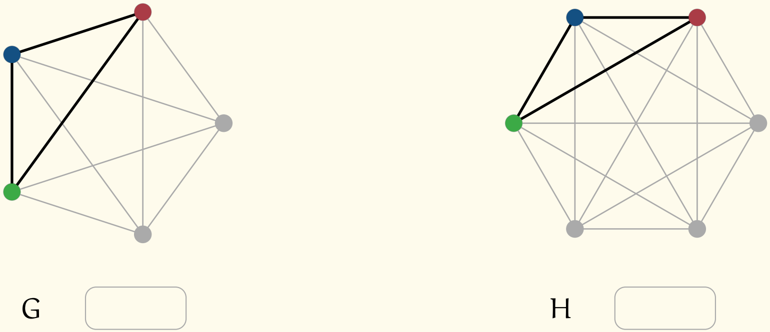

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Pebble mapping: partial isomorphism?

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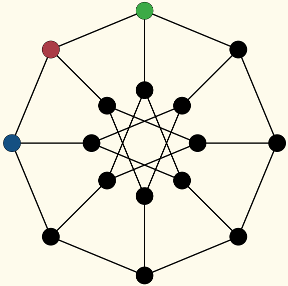

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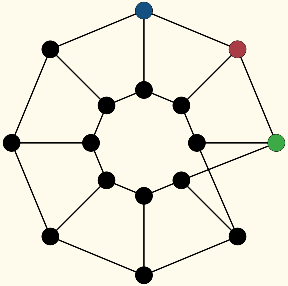

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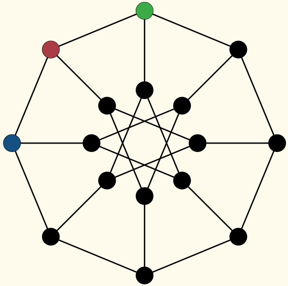

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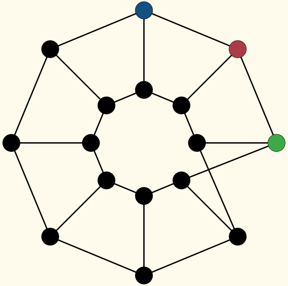

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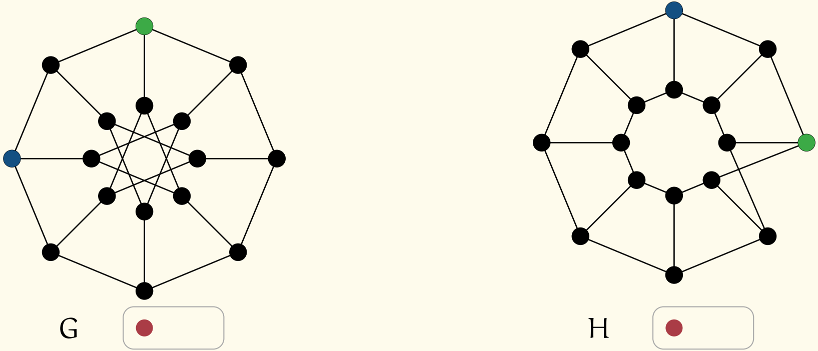

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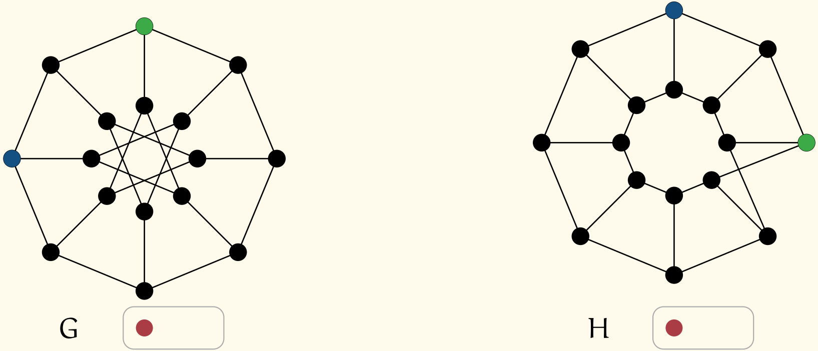

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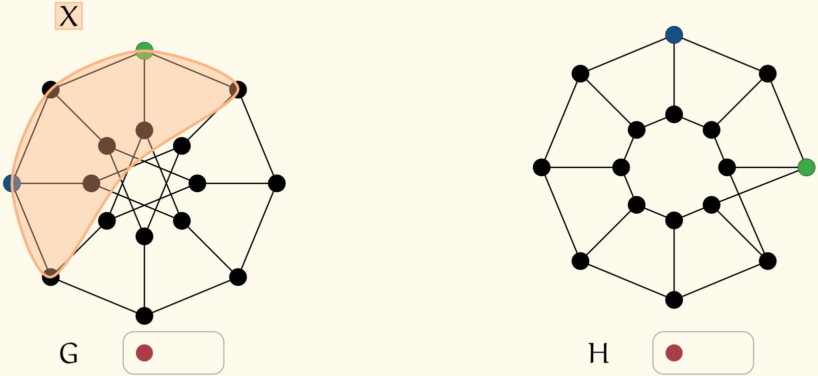

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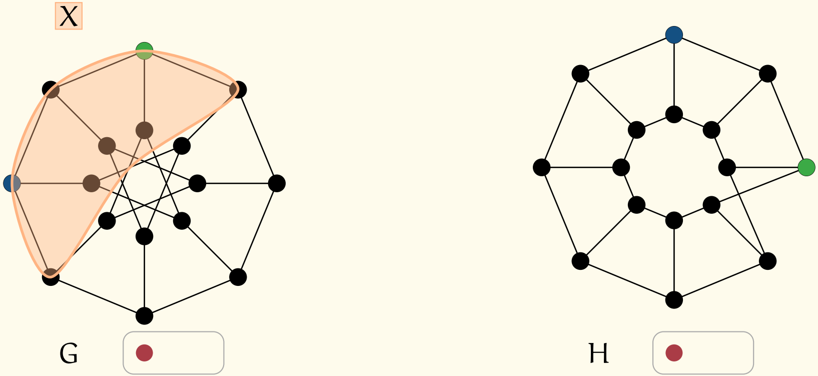

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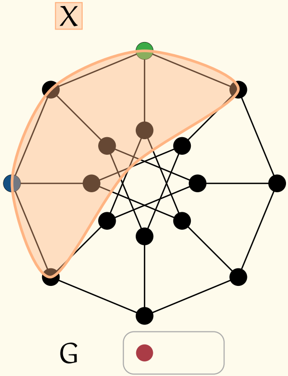

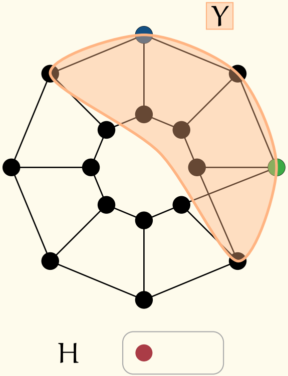

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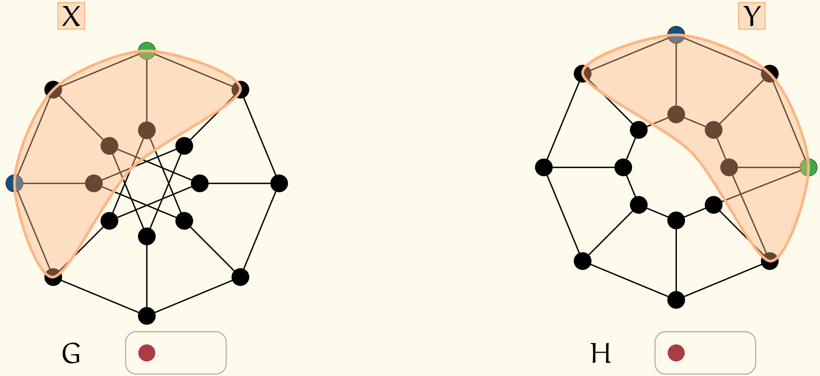

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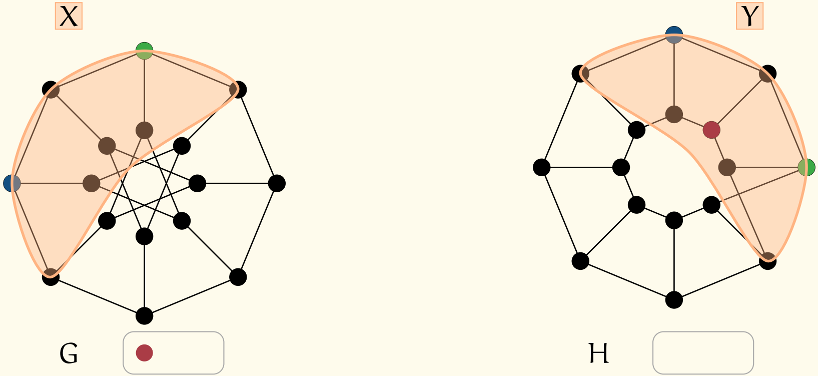

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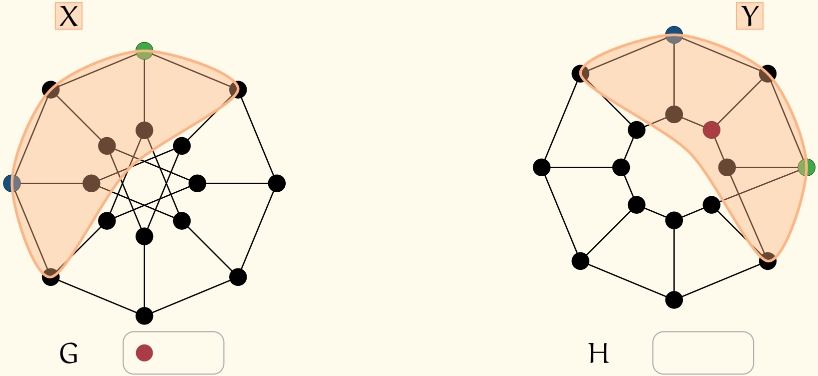

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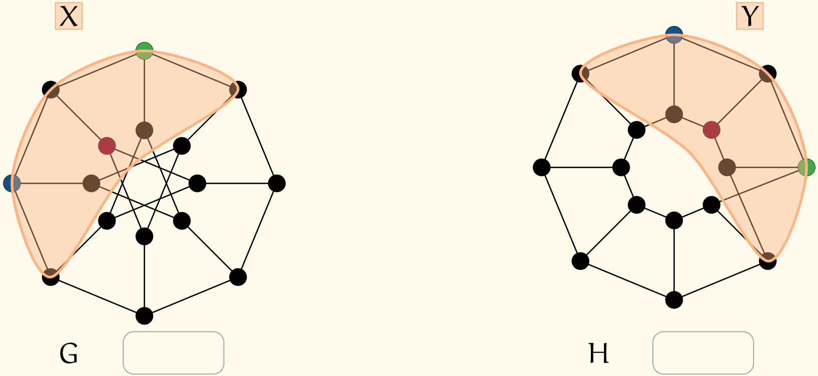

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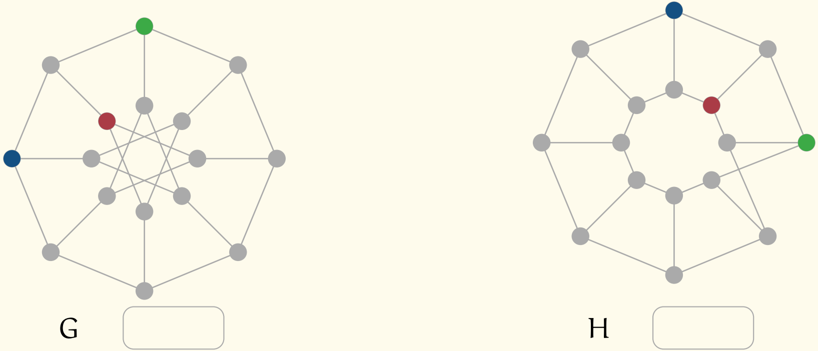

Counting game characterises C^k $\rightsquigarrow$ game method for IFPC

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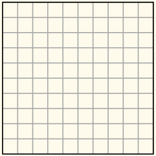

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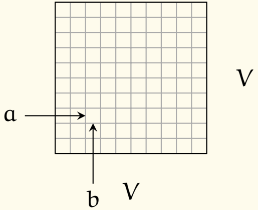

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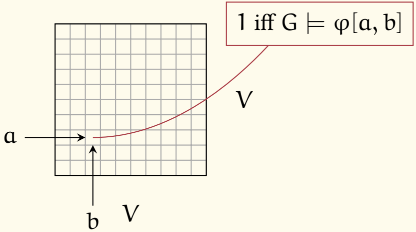

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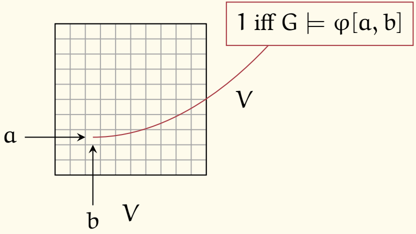

Example

$\varphi(x, y) := Exy \rightsquigarrow M_\varphi^G =$ adjacency matrix of G

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$\rightsquigarrow$ rank of M_φ^G over GF_2

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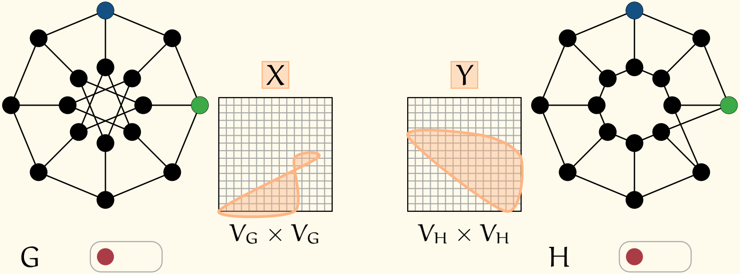

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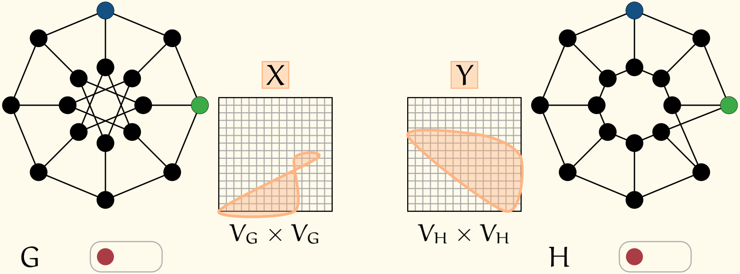

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Problem: rank is not monotone!

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Partition game for L^k - illustrates main idea

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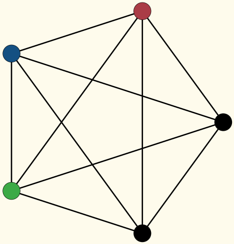

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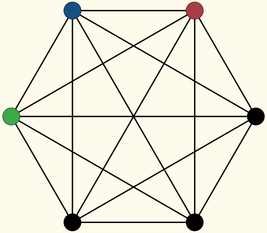

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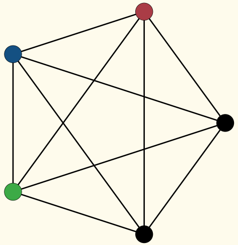

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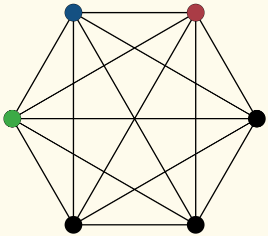

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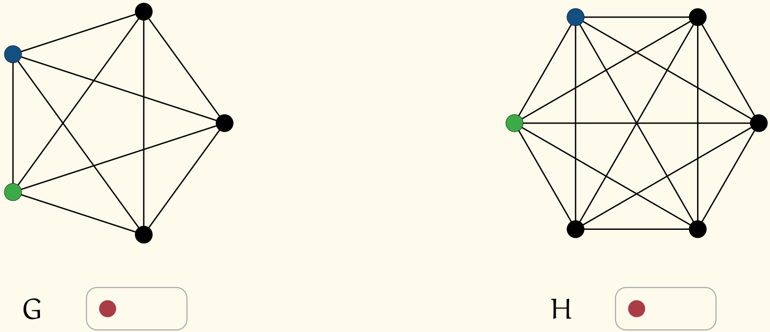

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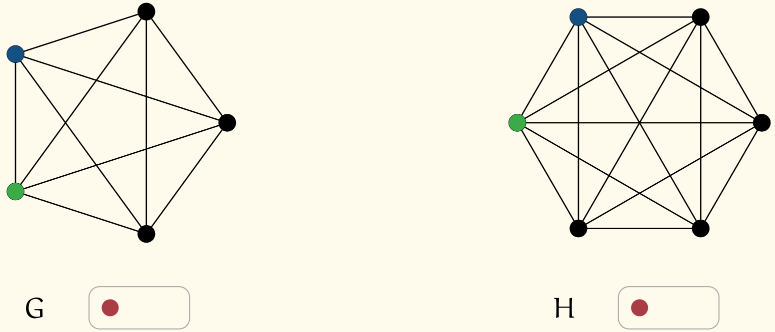

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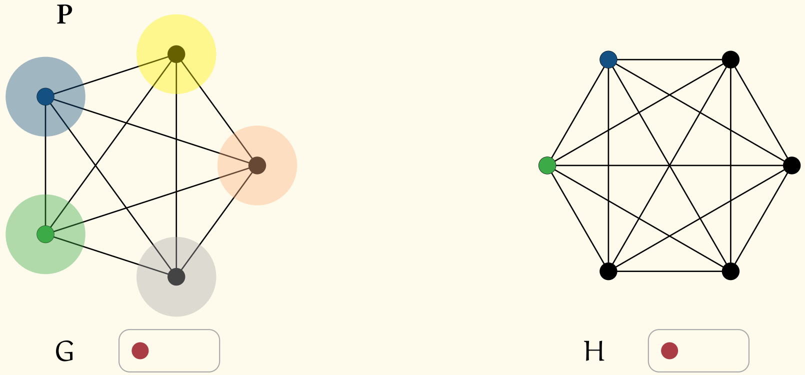

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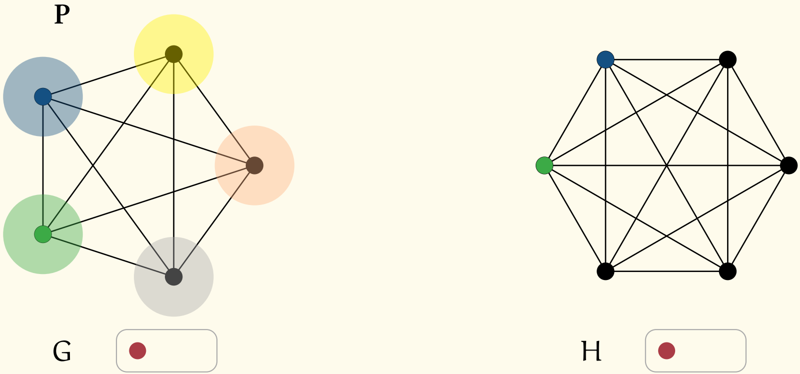

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Duplicator a partition $\mathbf{Q}$ of V_H, and a bijection $f : \mathbf{P} \to \mathbf{Q}$

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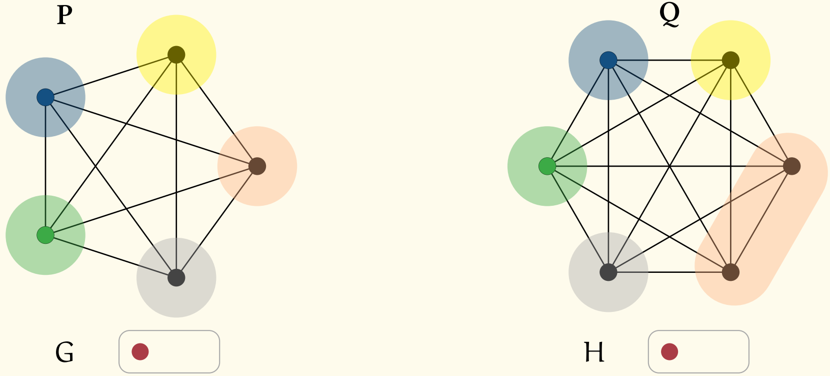

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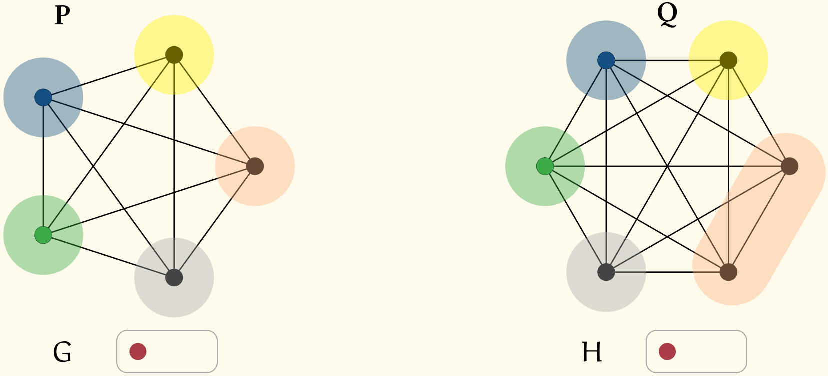

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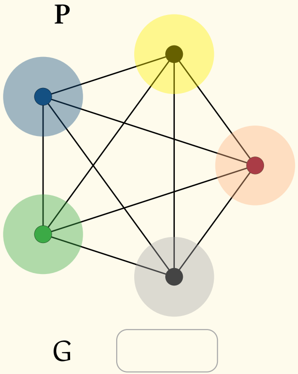

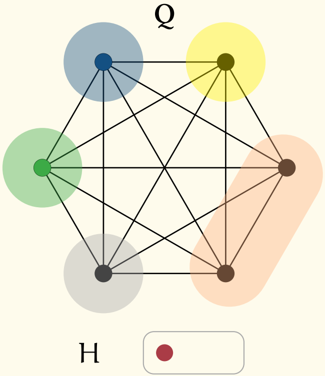

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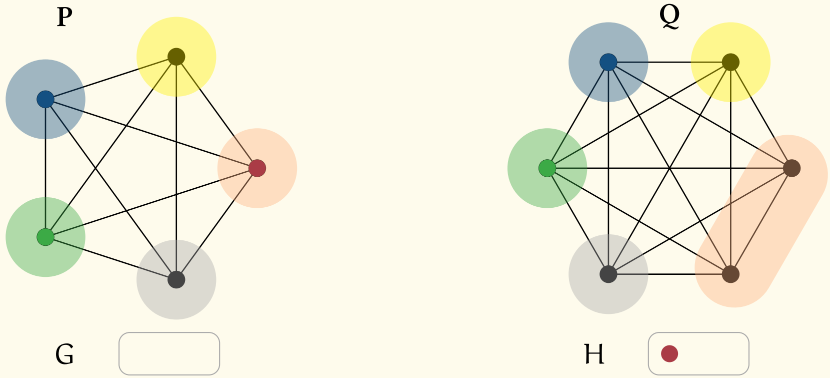
Duplicator

A new pebble-game protocol: partition games

Partition game for L^k - illustrates main idea



Spoiler **and** places red H-pebble on an element of $f(X) \in \mathbf{Q}$

Duplicator

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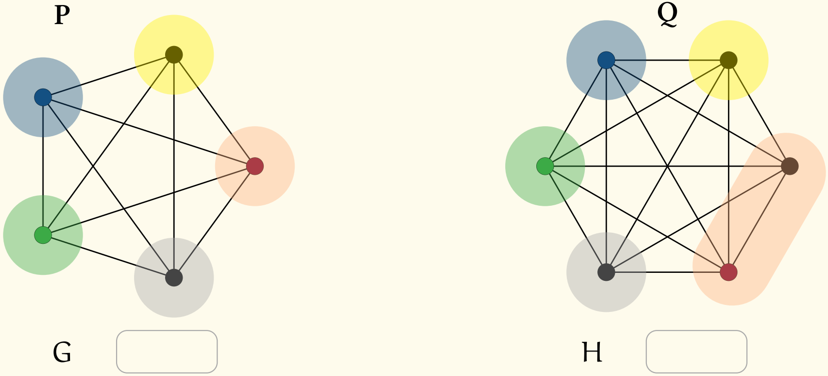
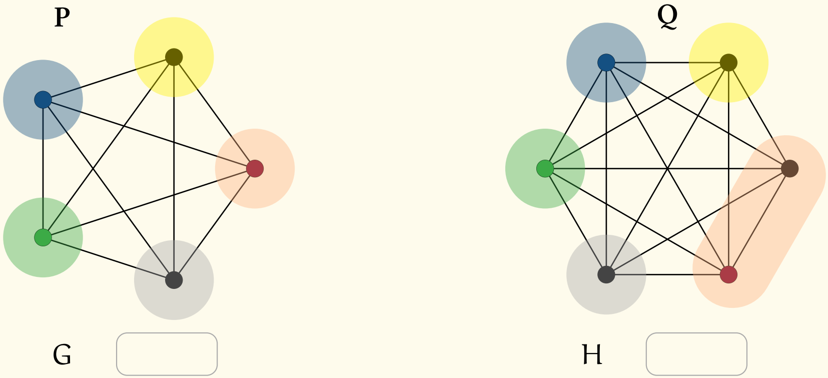


Spoiler **and** places red H-pebble on an element of $f(X) \in \mathbf{Q}$

Duplicator

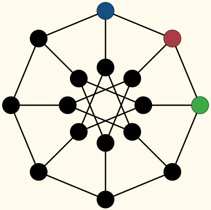
A new pebble-game protocol: partition games

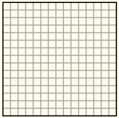
To get a partition game for $C^k \rightsquigarrow$ for all $X \in \mathbf{P}$, $\|X\| = \|f(X)\|$



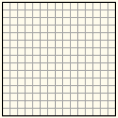
Spoiler
Duplicator

Partition game for rank logics

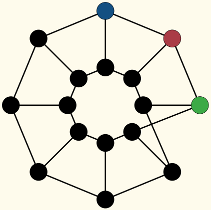




$V_G \times V_G$



$V_H \times V_H$



G

H

Spoiler
Duplicator

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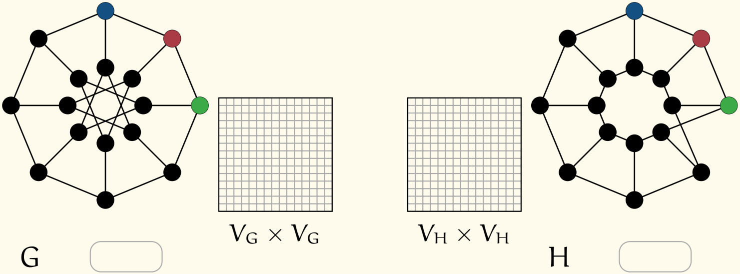
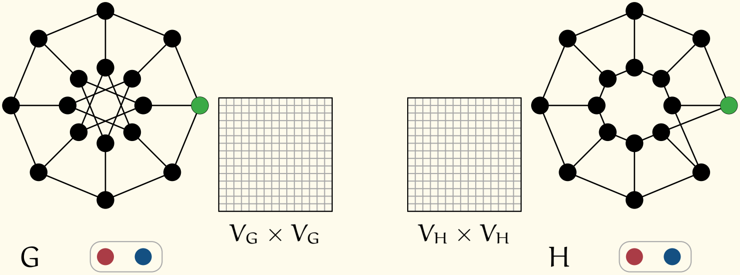


Spoiler removes two pairs of corresponding pebbles

Duplicator

Partition game for rank logics



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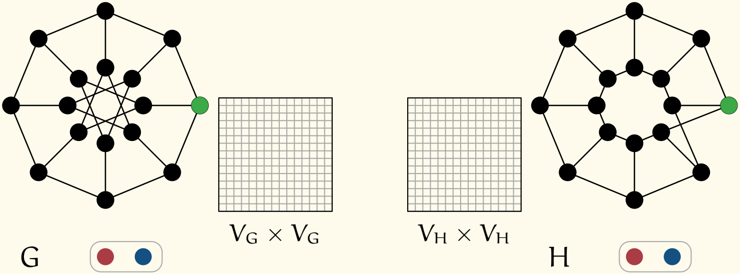
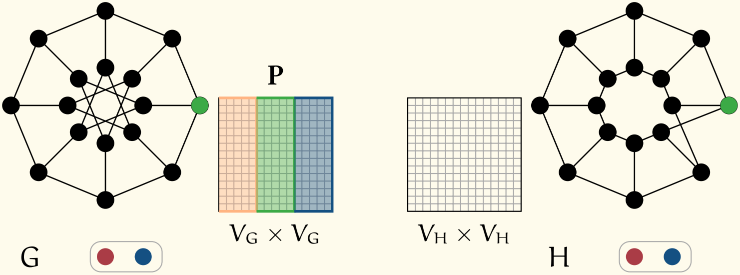


Spoiler

Duplicator gives a partition **P** of $V_G \times V_G$, . . .

Partition game for rank logics



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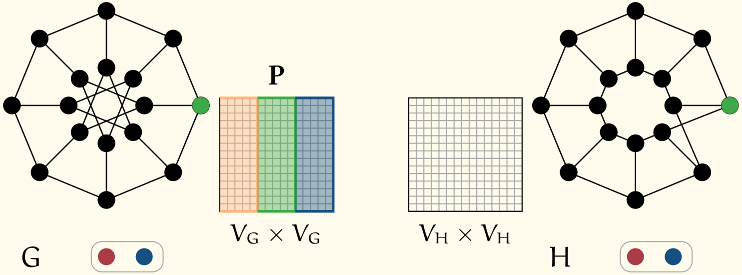
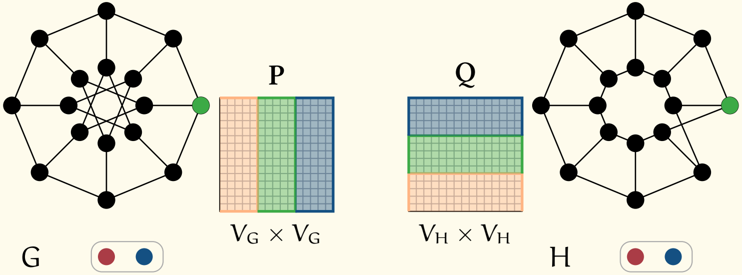


Spoiler

Duplicator   a partition $\mathbf{Q}$ of $V_H \times V_H$, and $f : \mathbf{P} \to \mathbf{Q}$,

Partition game for rank logics



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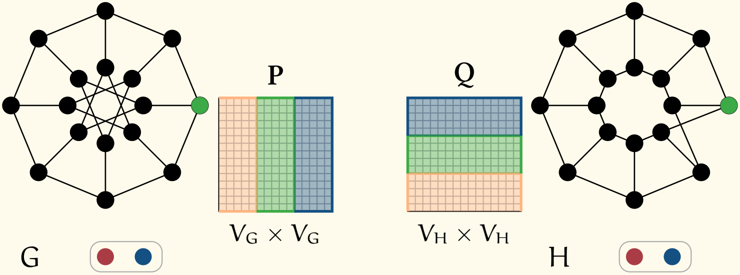

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$$\mathrm{rk}\left(\sum_{M \in X} P\right) = \mathrm{rk}\left(\sum_{M \in X} f(M)\right)$$

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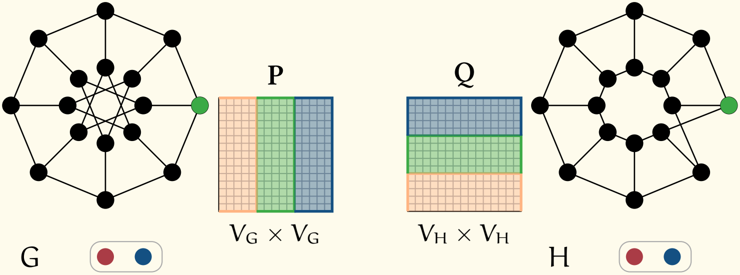

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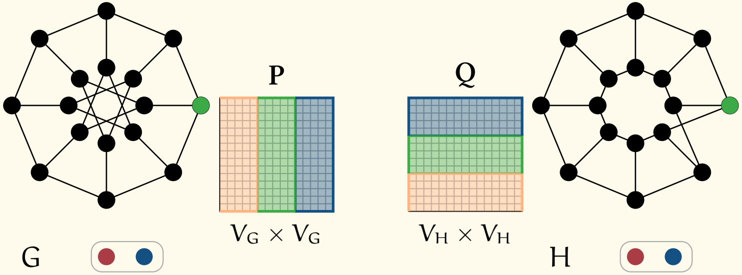
Can we decide in polynomial time who wins the game?

$\rightsquigarrow$ "yes" for standard pebble and cardinality games

Tractable instances of partition games?



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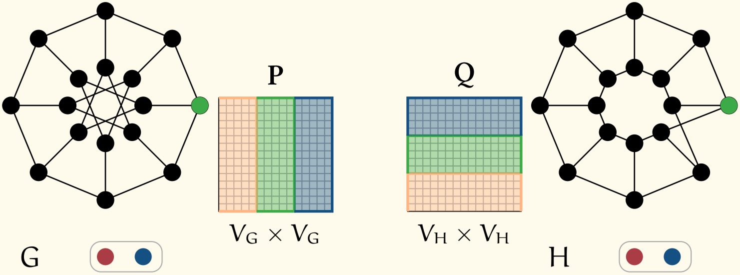

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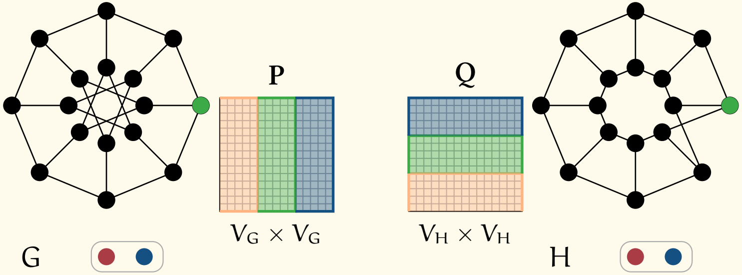
$\sum_{M \in X} P$ and $\sum_{M \in X} f(M)$ are equivalent

Tractable instances of partition games?



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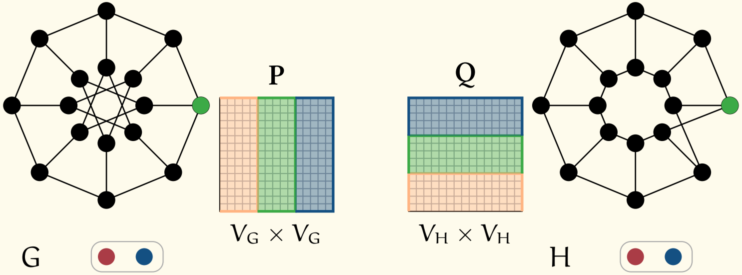
Tractable instances of partition games?



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Duplicator   a partition $\mathbf{Q}$ of $V_H \times V_H$, and $f : \mathbf{P} \to \mathbf{Q}$, such that:

$(M)_{M \in \mathbf{P}}$ and $(f(M))_{M \in \mathbf{P}}$
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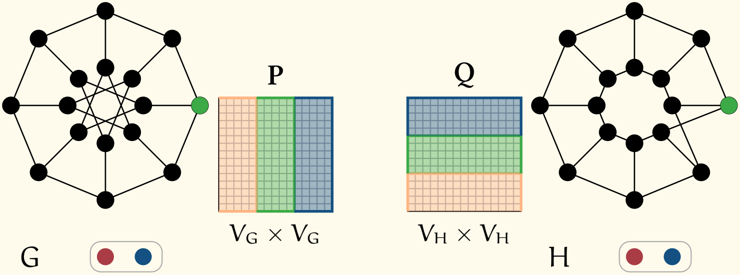

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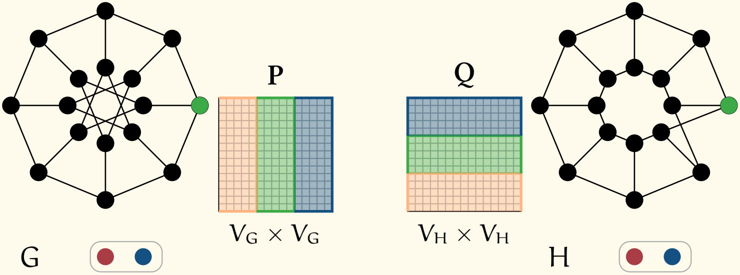
$\rightsquigarrow$ we can decide who wins this game in polynomial time

Tractable instances of partition games?



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⇝ we can decide who wins this game in polynomial time

⇝ Application: family of algorithms for testing graph isomorphism

From logics to games – and back again?

- Does the "simultaneous-similarity game" correspond to a natural logic?
- Duplicator wins the simultaneous-similarity game $\Rightarrow$ Duplicator wins the matrix-rank game. Converse?