The logic of Weihrauch degrees

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Based on joint work with Vasco Brattka, Kojiro Higuchi and Paulo Oliva.
A very short overview

- Weihrauch reducibility compares multivalued functions between represented spaces.
- The induced degrees have a rich algebraic structure.
- Many mathematical theorems can be interpreted as multivalued functions, with the associated Weihrauch degrees measuring the computational content of the theorem.
- The algebraic operations have logic-like meanings regarding such theorems.
- Many concrete theorems have been classified via Weihrauch reducibility; and this classification is reminiscent of reverse mathematics and Brouwerian counterexamples.
- Various techniques have been developed to prove separation results.
Represented spaces and computability

Definition
A represented space $X$ is a pair $(X, \delta_X)$ where $X$ is a set and $\delta_X : \subseteq \mathbb{N}^\mathbb{N} \rightarrow X$ a surjective partial function.

Definition
$F : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ is a realizer of $f : \subseteq X \Rightarrow Y$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \delta_X^{-1}(\text{dom}(f))$. Abbreviate: $F \vdash f$.

\[
\begin{array}{ccc}
\mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\
\downarrow{\delta_X} & & \downarrow{\delta_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

Definition
$f : \subseteq X \Rightarrow Y$ is called computable (continuous), iff it has a computable (continuous) realizer.
Weihrauch reducibility

Definition (GHERARDI & MARCONE 2009)
Say that $f \leq_W g$, if there are computable functions $H, K : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $G \vdash g$ implies $K\langle\text{id}, GH\rangle \vdash f$.

Figure: The black box model for Weihrauch reducibility
Weihrauch reducibility, equivalent definition

Proposition
Let $f : \subseteq X \Rightarrow Y$, $g : \subseteq U \Rightarrow V$. Then $f \leq_W g$, iff there are computable multivalued partial functions $h : \subseteq X \Rightarrow Y$, $k : \subseteq X \times V \Rightarrow Y$ such that $k \circ (\text{id}_X \times (g \circ h)) \circ \Delta_X \leq f$.

This is a special case of a general construction explored in:

A. Pauly.
Many-one reductions between search problems.

From theorems to multivalued functions

Identify the theorem:

\[ \forall x \in X \ \exists y \in Y \ (D(x) \Rightarrow P(x, y)) \]

with the multivalued function \( F : \subseteq X \Rightarrow Y \) with

\[ \text{dom}(F) = \{ x \in X \mid D(x) \} \] and \( y \in F(x) \) iff \( P(x, y) \).
Non-uniqueness

Observation (BRATTKA)

The Baire Category Theorem has 4 different incarnations as a multivalued function:

1. \( \forall (A_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{A}(X)) \ \exists x \in X \)

   \( ((\forall i \in \mathbb{N} \ A_i \text{ is nowhere dense}) \rightarrow x \notin \bigcup_{i \in \mathbb{N}} A_i) \)

2. \( \forall (A_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{A}(X)) \ \exists n \in \mathbb{N} \)

   \( (\bigcup_{i \in \mathbb{N}} A_i = X \rightarrow A_n \text{ is somewhere dense}) \)

3. \( \forall (A_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{V}(X)) \ \exists x \in X \)

   \( ((\forall i \in \mathbb{N} \ A_i \text{ is nowhere dense}) \rightarrow x \notin \bigcup_{i \in \mathbb{N}} A_i) \)

4. \( \forall (A_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{V}(X)) \ \exists n \in \mathbb{N} \)

   \( (\bigcup_{i \in \mathbb{N}} A_i = X \rightarrow A_n \text{ is somewhere dense}) \)
Theorem (P. 2010, Brattka & Gherardi 2012) 

\((\mathcal{W}, \oplus, +, 0, \emptyset)\) is a distributive bounded lattice.

- For \(f_i : X_i \Rightarrow Y_i\), let \(f_1 + f_2 : X_1 + X_2 \Rightarrow Y_1 + Y_2\) be defined via \((f_1 + f_2)(i, x) = (i, f_i(x))\).

- and let \(f_1 \oplus f_2 : X_1 \times X_2 \Rightarrow Y_1 + Y_2\) be defined via 
  \((f_1 \oplus f_2)(x, y) = (\{1\} \times f_1(x)) \cup (\{2\} \times f_2(y))\).

- The representatives of 0 are the multivalued functions with the no-where defined function as realizer.

- The representatives of \(\emptyset\) are the multivalued functions without realizer.
Theorem (HIGUCHI & P. 2013)
No non-trivial countable suprema exist in $\mathcal{W}$. Only some non-trivial countable infima exist in $\mathcal{W}$.

Corollary
$\mathcal{W}$ is not isomorphic to its dual – no classical logic.
The want for an implication

Question

*Can we internalize the meta-implication* $f \leq_W g$ *to some operator* $g \rightarrow f$?

Proposition (HIGUCHI & P. 2013)

There are $f$, $g$ such that $\{ h \mid f \leq_W g + h \}$ has no minimal element.

This means we cannot define $g \rightarrow f$ by just requiring $f \leq_W (g \rightarrow f) + g$ – so it’s not intuitionistic logic either.
Theorem (P. 2010, Brattka & Gherardi 2012)

Both $*$ and $\hat{}$ are closure operators on $\mathcal{W}$.

$0^* = 1^* = \hat{1} = 1$, $\hat{0} = 0$

- Let $\hat{1}$ be the degree of $\mathrm{id} : \{0\} \to \{0\}$.
- Let $f^0 = (\mathrm{id} : \{0\} \to \{0\})$, $f^{n+1} = f^n \times f$ and $f^* = \bigsqcup_{n \in \mathbb{N}} f^n$.
- $\hat{X} = X \times X \times \ldots$, then $\hat{f} : \subseteq \hat{X} \Rightarrow \hat{Y}$ is defined via $\hat{f}(x_1, x_2, \ldots) = (f(x_1), f(x_2), \ldots)$.
Theorem ((BRATTKA), HIGUCHI & P. 2013)

$(\mathbb{W}, \leq_W, +, \times, 0, 1, *)$ is a commutative Kleene algebra.

i.e.:

1. $+$ is the supremum for $\leq_W$
2. $0$ is the neutral element for $+$ and $1$ is the neutral element for $\times$
3. $\times$ is associative, commutative and distributes over $+$
4. $1 + a \times a^* \leq_W a^*$
5. $a \times b \leq_W b$ implies $a^* \times b \leq_W b$

$\times$ is just the usual product
So is it intuitionistic linear logic?

Now we have the additive **and** $+$ and the multiplicative **and** $\times$, and two (!) bang operators $\ast$, $\hat{}$. Does this make for a model of intuitionistic logic?

**Proposition (BRATTKA, OLIVA, P.)**

$\times$ **does not distribute over** $\oplus$, i.e.

$$(a \times b) \oplus (a \times c) \leq_W a \times (b \oplus c).$$

**Example**

$$C_N \times C\{0,1\}_N \not\leq_W (C_N + C\{0,1\}_N) \times (C_N \oplus C\{0,1\}_N)$$

**Corollary**

*There are $f, g$ such that $\{h \mid f \leq_W g \times h\}$ has no minimum.*
Theorem (BRATTKA, OLIVA & P.)
\[ a \star b = \max_{\leq W} \{ f \circ g \mid f \leq_W a \land g \leq_W b \} \]

Theorem (BRATTKA, OLIVA & P.)
\[ a \rightarrow b = \min_{\leq W} \{ c \mid b \leq_W a \star c \} \]

- Let \( f : \subseteq U \Rightarrow V, g : \subseteq X \Rightarrow Y \), then define
\[ g^t_Z : \subseteq X \times \mathcal{M}(Y, Z) \Rightarrow Z \text{ via } g^t_Z(x, k) = k(g(x)) \]

- \( f \star g := (\text{id}_Y \times f) \circ g^t_{Y \times U} \)

- Define \( g \rightarrow f : \subseteq U \Rightarrow (X \times \mathcal{M}(Y, V)) \) via
\[ (x, H) \in (g \rightarrow f)(u) \text{ iff } H(g(x)) \subseteq f(u). \]
Non-commutativity

Observation
★ is not commutative.

Proposition (BRATTKA, OLIVA & P.)

There are $f, g$ such that $\{ h \mid f \leq_W g \times h \}$ and $\{ h \mid f \leq_W h \star g \}$ have no minimal elements.

So generally, the implication only work via $f \leq_W g \star (g \to f)$, whereas $f \not\leq_W (g \to f) \star g$. 
Overview

We have a set \( \mathcal{W} \) partially ordered by \( \leq_W \), with unary operations \(*\) and \(\wedge\), binary operations \(+, \oplus, \times, \star, \rightarrow\) and constants \(0, 1, \emptyset\).

- \(0\) is trivially true, \(1\) is true, \(\emptyset\) is false
- \(+, \times, \star\) are all conjunctions, with \(\star\) being a non-commutative one
- \(\oplus\) is the disjunction
- \(\rightarrow\) the implication regarding \(\star\) (only working on the right)
- \(*\) and \(\wedge\) are exponentials (as in linear logic)
K. Higuchi & A. Pauly.
The degree structure of Weihrauch reducibility.
Logical Methods of Computer Science, 2013.