Tripos Theory in Retrospect

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The origin of the notion of 'tripos' (in 1979)

- Higgs' construction of the topos of sheaves on a complete Heyting algebra as H-valued sets.
- Hyland's effective topos, based on Kleene's '1945' recursive realizability.

Definitions look very similar, but result in toposes with *very* different properties.

Qu. *Is there a common generalisation*?

Ans. Tripos! (=topos representing indexed partially ordered set)

Outline of the talk

The 'tripos-to-topos' construction: the category of PERs of a first order hyperdoctrine, $\mathbf{C}[\mathcal{P}]$.

When is $\mathbf{C}[\mathcal{P}]$ a topos? A comprehension property of $(\mathbf{C}, \mathcal{P})$ and its 'Skolemisation'.

Example of a non-tripos for which $\mathbf{C}[\mathcal{P}]$ is nevertheless a topos.

Conclusion. Was the lack of generality in the definition of 'tripos' fortuitous?

Since I’m only interested in provability rather than proofs, I’ll restrict attention to indexed posets, rather than indexed categories.

Definition: first order hyperdoctrine $\mathcal{P}$ over a category $\mathbf{C}$ with finite products [Lawvere,1969]

Entailment structure: $\mathcal{P} : \mathbf{C}^{op} \longrightarrow \mathbf{Poset}$.

Propositional structure: $\mathcal{P}$ factors through
$\mathbf{Heyt} \longrightarrow \mathbf{Poset}$.

Equality: $=_X \in \mathcal{P}(X \times X)$ (each $\mathbf{C}$-object X) satisfying

$$\top \leq \mathcal{P}(\Delta_X)(A) \qquad \text{if and only if} \qquad =_X \leq A$$

for all $A \in \mathcal{P}(X \times X)$.

Quantifiers: Left and right adjoints $(\exists X)_\Gamma, (\forall X)_\Gamma$ to
$\mathcal{P}(\pi) : \mathcal{P}(\Gamma) \longrightarrow \mathcal{P}(\Gamma \times X)$ for each product projection
$\pi : \Gamma \times X \longrightarrow \Gamma$ in $\mathbf{C}$, natural in Γ.

N.B. the *structure* needed to specify a fist order hyperdoctrine over $\mathbf{C}$ is just $\mathcal{P} : \mathbf{C}^{op} \longrightarrow \mathbf{Poset}$. All the other things mentioned on the slide are *properties*, in as much as they are uniquely determined (being instances of adjoints).

Hyperdoctrine semantics of first order intuitionistic logic (FOIL)

syntax		semantics
sort X	$\mapsto$	$\mathbf{C}$-object $[\![X]\!]$
context $\Gamma = [x : X, \ldots]$	$\mapsto$	product of $\mathbf{C}$-objects $[\![\Gamma]\!] = [\![X]\!] \times \cdots$
term $t : X[\Gamma]$	$\mapsto$	$\mathbf{C}$-morphism $[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![X]\!]$
formula $A[\Gamma]$	$\mapsto$	$\mathcal{P}$-predicate $[\![A]\!] \in \mathcal{P}([\![\Gamma]\!])$

hence

sentence A	$\mapsto$	$[\![A]\!] \in \mathcal{P}(1)$

The interpretation of FOIL in a hyperdoctrine proceeds by induction on the structure of expressions, starting from a given *structure* for a first order *signature* assigning objects, morphisms and predicates to the sorts, function symbols and relation symbols.

Properties of the semantics

Soundness: If $\mathrm{FOIL} \vdash A$, then $\mathcal{P} \models A$, i.e. $[\![A]\!] = \top$ in $\mathcal{P}(1)$.

Completeness: $\ldots$ (not very informative).

Internal language of $(\mathbf{C}, \mathcal{P})$: names the $\mathbf{C}$-objects, $\mathbf{C}$-morphisms, and $\mathcal{P}$-predicates.

- Useful for describing constructions on $\mathbf{C}$-morphisms and $\mathcal{P}$-predicates via first order terms and formulas.
- Useful for stating properties of $\mathcal{P}$ via satisfaction of first order sentences.

A good example of the use of the internal language of $(\mathbf{C}, \mathcal{P})$ is to give the definition and properties of the associated category of PERs . . .

The category of PERs, $\mathbf{C}[\mathcal{P}]$

Objects (X, E) are given by $\mathbf{C}$-object X plus $\mathcal{P}$-predicate $E \in \mathcal{P}(X \times X)$, such that $\mathcal{P}$ satisfies

'E is symmetric and transitive.'

Morphisms $(X_1, E_2) \rightarrow (X_2, E_2)$ are $\mathcal{P}$-predicates $F \in \mathcal{P}(X_1 \times X_2)$, such that $\mathcal{P}$ satisfies

'F respects the PERs E_1, E_2, is single-valued and total.'

Identity on (X, E) is E.

Composition of $F : (X_1, E_1) \rightarrow (X_2, E_2)$ and $G : (X_2, E_2) \rightarrow (X_3, E_3)$ is

$$[\![\exists x_2 : X_2 \, . \, F(x_1, x_2) \,\&\, G(x_2, x_3)]\!].$$

Hyland's realizability topos on a partial combinatory algebra (PCA) $\mathbb{A}$

is $\mathbf{Set}[\mathcal{P}_{\mathbb{A}}]$ for $\mathcal{P}_{\mathbb{A}}$ the first order hyperdoctrine given by

$$\mathcal{P}_{\mathbb{A}}(-) \stackrel{\text{def}}{=} \mathbf{Set}(-, Sub(\mathbb{A}))/(\leq \cap \geq)$$

where $\Phi \leq \Phi' \in \mathbf{Set}(X, Sub(\mathbb{A}))$ iff there is some $a' \in \mathbb{A}$ satisfying

$$x \in X \;\&\; a \in \Phi(x) \;\Rightarrow\; a' \cdot a \text{ defined and in } \Phi'(x).$$

PCA = set $\mathbb{A}$ + partial binary operation $(-) \cdot (-)$ + elements $k, s \in \mathbb{A}$ satisfying . . . .

E.g. effective topos is case $\mathbb{A} = \mathbb{N}$, with
$n \cdot m \stackrel{\text{def}}{=}$ value at m of nth partial recursive function.

Category of sheaves on a complete Heyting algebra H as H-valued sets à la Higgs

is $\mathbf{Set}[\mathcal{P}_H]$ for $\mathcal{P}_H$ the first order hyperdoctrine given by

$$\mathcal{P}_H(-) \stackrel{\mathrm{def}}{=} \mathbf{Set}(-, H)$$

where $\Phi \leq \Phi' \in \mathcal{P}_H(X)$ iff for all $x \in X$, $\Phi(x) \leq \Phi'(x)$ in H.

When is $\mathbf{C}[\mathcal{P}]$ a topos?

In general, $\mathbf{C}[\mathcal{P}]$ is a logos:

finite limits, pb-stable images and dual images of subobjects along morphisms, and pb-stable finite joins of subobjects.

For a topos, need powerobjects:

for each object (X, E), an object $P(X, E)$ and a subobject $\in \rightarrowtail (X, E) \times P(X, E)$, universal for subobjects of the form $\cdot \rightarrowtail (X, E) \times \ ?$.

In fact $\mathbf{C}[\mathcal{P}]$ also has quotients for all equivalence relations. Indeed, it is the universal solution to the problem of realizing $\mathcal{P}$-predicates as subobjects in such a logos over $\mathbf{C}$: see Theorem 3.6 in the paper accompanying this talk.

Theorem [AMP, 1982, unpublished]

For $\mathcal{P}$ a first order hyperdoctrine over a category $\mathbf{C}$ with finite products, $\mathbf{C}[\mathcal{P}]$ is a topos if and only if $(\mathbf{C}, \mathcal{P})$ satisfies the following form of comprehension.

(CA) For all $\mathbf{C}$-objects X, there is a $\mathbf{C}$ object PX and a $\mathcal{P}$-predicate $In_X \in \mathcal{P}(X \times PX)$ such that for any $\mathbf{C}$-object Γ and $\mathcal{P}$-predicate $R \in \mathcal{P}(X \times \Gamma)$, $\mathcal{P}$ satisfies the following sentence of its internal language.

$$\forall i : \Gamma \,.\, \exists s : PX \,.\, \forall x : X \,.\, In_X(x, s) \Leftrightarrow R(x, i).$$

'Skolemised' CA

Can satisfy

$$\forall i : \Gamma \,.\, \exists s : PX \,.\, \forall x : X \,.\, In_X(x, s) \Leftrightarrow R(x, i).$$

by requiring existence of a $\mathbf{C}$-morphism $\{R\} : \Gamma \longrightarrow PX$ picking out the required $s = \{R\}(i)$ for each $i : \Gamma$:

$$\forall i : \Gamma \,.\, \forall x : X \,.\, In_X(x, \{R\}(i)) \Leftrightarrow R(x, i)$$

Definition. $\mathcal{P}$ is a $\mathbf{C}$-tripos if for each $\mathbf{C}$ object X there is a $\mathbf{C}$-object PX and a $\mathcal{P}$-predicate $In_X \in \mathcal{P}(X \times PX)$ such that for any $R \in \mathcal{P}(X \times \Gamma)$ there is a $\mathbf{C}$-morphism $\{R\} : \Gamma \longrightarrow PX$ with

$$R = \mathcal{P}(id_X \times \{R\})(In_X).$$

Examples of triposes

Realizability hyperdoctrine, $\mathcal{P}_{\mathbb{A}}$: can take $PX = Sub(\mathbb{A})^X$. Given $R \in \mathcal{P}_{\mathbb{A}}(X \times \Gamma) = \mathbf{Set}(X \times \Gamma, Sub(\mathbb{A}))/(\leq \cap \geq)$, represented by $\Phi \in \mathbf{Set}(X \times \Gamma, Sub(\mathbb{A}))$ say, can take

$$\{R\} : \Gamma \longrightarrow Sub(\mathbb{A})^X$$

to be $\lambda i \in \Gamma \,.\, \lambda x \in X \,.\, \Phi(x, i)$.

Hyperdoctrine of a cHa, $\mathcal{P}_H$: can take $PX = H^X$. Given $R \in \mathcal{P}_H(X \times \Gamma) = \mathbf{Set}(X \times \Gamma, H)$, can take

$$\{R\} : \Gamma \longrightarrow H^X$$

to be $\lambda i \in \Gamma \,.\, \lambda x \in X \,.\, R(x, i)$.

Is every first order hyperdoctrine satisfying (CA) actually a tripos? The next example shows this is not the case.

A non-tripos satisfying CA

Any Heyting algebra (not necessarily complete, or finite) determines a first order hyperdoctrine

$$\mathcal{P} : \mathbf{Fin}^{op} \hookrightarrow \mathbf{Set}^{op} \xrightarrow{\mathbf{Set}(-,H)} \mathbf{Poset}$$

over the category $\mathbf{Fin}$ of finite sets and functions.

Claim: *If H is Boolean, then $\mathcal{P}$ satisfies (CA)* (and hence $\mathbf{Fin}[\mathcal{P}]$ is a topos).

But *if H is infinite, $\mathcal{P}$ is not a $\mathbf{Fin}$-tripos*. (If it were, we'd have

$$\begin{array}{ccc} P1 \cong \mathbf{Fin}(1, P1) & \twoheadrightarrow \mathcal{P}(1 \times 1) & \cong H \\ \uparrow & f \mapsto \mathcal{P}(id_1 \times f)(In_1) & \uparrow \\ \text{finite,} & \text{epi (split by } R \mapsto \{R\}\text{),} & \text{infinite} \end{array}$$

contradiction.)

Proof that $(\mathbf{Fin}, \mathcal{P}_B)$ satisfies (CA) when B Boolean

For each finite set X, define PX to be the finite set $\{\bot, \top\}^X$ and let $In_X \in \mathcal{P}_B(X \times PX) = B^{X \times PX}$ be $\lambda(x, f) \in X \times PX \,.\, f(x)$. Then for any $R \in \mathcal{P}_B(X \times \Gamma)$

$$
\begin{aligned}
\top &= \textstyle\bigwedge_{i\in\Gamma} \bigwedge_{x\in X} \top \\
&= \textstyle\bigwedge_{i\in\Gamma} \bigwedge_{x\in X} R(x,i) \vee \neg R(x,i) \\
&= \textstyle\bigwedge_{i\in\Gamma} \bigwedge_{x\in X} \bigvee_{b\in\{\bot,\top\}} b \Leftrightarrow R(x,i) \\
&= \textstyle\bigwedge_{i\in\Gamma} \bigvee_{s\in\{\bot,\top\}^X} \bigwedge_{x\in X} s(x) \Leftrightarrow R(x,i)
\end{aligned}
$$

the last step using the fact that B is a distributive lattice. Hence the formula

$$\forall i : \Gamma \,.\, \exists s : PX \,.\, \forall x : X \,.\, In_X(x, s) \Leftrightarrow R(x, i)$$

of the internal language of $\mathcal{P}$ is satisfied.

An open problem in topos theory

Note: for any first order hyperdoctrine $(\mathbf{C}, \mathcal{P})$, one always has $Sub_{\mathbf{C}[\mathcal{P}]}(1) \cong \mathcal{P}(1)$.

So the topos $\mathbf{Fin}[\mathcal{P}_B]$ has its Heyting algebra of subobjects of 1 isomorphic to the (arbitrary) Boolean algebra B.

For any given Heyting algebra H, is there a topos $\mathbf{E}$ with $Sub_{\mathbf{E}}(1) \cong H$?

Probably not, when H is free on countably many generators.

The topos $\mathbf{Fin}[\mathcal{P}_B]$ was already known (e.g. to Mike Barr and Peter Freyd) in other, equivalent disguises. For example $\mathbf{Fin}[\mathcal{P}_B]$ is equivalent to a colimit in the bicategory of toposes and logical morphisms of a filtered diagram whose nodes are toposes of sheaves on the finite Boolean subalgebras of B.

The answer to the open question is probably ‘no’: I conjecture that the free Heyting algebra on countably many generators cannot occur as the Heyting algebra of truth values of any topos. See (Pitts, 1992) for more on this.

Summary

Higgs' H-valued sets and Hyland's realizability toposes are both instances of the $(\mathbf{C}, \mathcal{P}) \mapsto \mathbf{C}[\mathcal{P}]$ construction, and both are toposes. In general, $\mathbf{C}[\mathcal{P}]$ is a topos iff

$\boxed{\mathcal{P} \text{ satisfies (CA)}}$ $\leftarrow$ **should this be the definition of 'tripos'?**

More permissive definition potentially (actually?) useful for constructing specific examples.

But many examples are made by constructing new triposes from old using rather rich theory of triposes (geometric morphisms, iteration, etc). In not clear what remains of this if we just use (CA).

For example, one of the most useful results in (Pitts, 1981) concerns the question of *iteration*. To explain . . .

A difference between ‘hyperdoctrine+CA’ and ‘tripos’

$Prf \in \mathcal{P}(Prop)$ is a generic predicate for hyperdoctrine $\mathcal{P}$ over a category $\mathbf{C}$ if for any $A \in \mathcal{P}(\Gamma)$ there is a $\mathbf{C}$-morphism $\ulcorner A \urcorner : \Gamma \longrightarrow Prop$ with $A = \mathcal{P}(\ulcorner A \urcorner)(Prf)$.

Theorem. If $\mathcal{P}$ is a tripos, then it has a generic predicate. Conversely, if $\mathbf{C}$ is cartesian closed and $\mathcal{P}$ has a generic predicate, then it is a tripos.

So, the hyperdoctrine $\mathcal{P}_B$ over $\mathbf{Fin}$ does not have a generic predicate when B is infinite.

N.B. In the presence of a generic predicate we can always standardise $\mathcal{P}$-predicates to be functions: $\mathcal{P}(-) \simeq \mathbf{C}(-, Prop)$ (provided we use indexed *preorders*, rather than posets).

A bit of Tripos Theory—Iteration [Pitts, 1981]

Suppose $\mathcal{P}$ is a tripos over $\mathbf{C}$ and $\mathcal{R}$ is a tripos over $\mathbf{C}[\mathcal{P}]$.

Then $\mathbf{C}[\mathcal{P}][\mathcal{R}]$ is of the form $\mathbf{C}[\mathcal{S}]$ for a $\mathbf{C}$-tripos $\mathcal{S}$ provided quantifiers in $\mathcal{R}$ are *fibrewise*, i.e. induced by morphisms

$$\bigwedge_{\mathcal{R}}, \bigvee_{\mathcal{R}} : \Omega^{Prop} \longrightarrow Prop$$

(where $Prop$ is the generic predicate of the tripos $\mathcal{R}$ and Ω the subobject classifier of the topos $\mathbf{C}[\mathcal{P}]$).

The above result is very useful in practice, because fibrewise quantification seems very common for triposes.

Concept doesn't even make sense for hyperdoctrines+CA, since they don't necessarily have generic predicates.