On Proofs of Equality as Paths

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joint work with Ian Orton

Logic & Semantics Seminar 2016-10-07

(Martin-Löf) Type Theory

is formulated in terms of

"judgements"

$a : A$	a has type A
$a = b : A$	a & b are equal and of type A

Also: A is a type and A & B are equal types

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Also : ~~A is a type~~ and ~~A & B are equal types~~

Here : $A : \mathcal{U}$ $A = B : \mathcal{U}$ ← a universe

(Martin-Löf) Type Theory

is formulated in terms of

hypothetical judgements

$$x : A,\ y : B(x) \vdash a(x,y) : C(x,y)$$

$$x : A,\ y : B(x) \vdash a(x,y) = b(x,y) : C(x,y)$$

dependent types!

Identity types

$$\boxed{x : A,\ y : A \vdash \mathrm{Id}_A\ x\ y \ : \mathcal{U}}$$

type of proofs that x equals y

Identity introduction

If $a : A$, then there is a proof

$$\boxed{\mathsf{refl} : \mathrm{Id}_A\, a\, a}$$

(reflexivity of equality)

Identity elimination

If $a : A$ and $x : A,\ p : \mathrm{Id}_A\, a\, x \vdash B(x,p) : \mathcal{U}$,

Identity elimination

If $a : A$ and $x : A, p : \mathrm{Id}_A\, a\, x \vdash B(x,p) : \mathcal{U}$,

given any $x : A$ & $p : \mathrm{Id}_A\, a\, x$,

to construct an element of $B(x,p)$,

Identity elimination

If $a : A$ and $x : A, p : \mathrm{Id}_A\, a\, x \vdash B(x,p) : \mathcal{U}$,
given any $x : A$ & $p : \mathrm{Id}_A\, a\, x$,
to construct an element of $B(x,p)$,
it suffices to give some $b : B(a, \mathrm{refl})$

$$\boxed{J_{a,B} : B(a,\mathrm{refl}) \to (x : A)(p : \mathrm{Id}_A\, a\, x) \to B(x,p)}$$

Identity elimination & computation

If $a : A$ and $x : A, p : \mathrm{Id}_A\, a\, x \vdash B(x,p) : \mathcal{U}$,
given any $x : A$ & $p : \mathrm{Id}_A\, a\, x$,
to construct an element of $B(x,p)$,
it suffices to give some $b : B(a, \mathrm{refl})$

$$\boxed{J_{a,B} : B(a,\mathrm{refl}) \to (x : A)(\ \ : \mathrm{Id}_A\, a\, x) \to B(x,p)}$$

$$J_{a,B}\ b\ a\ \mathrm{refl} = b \ :\ B(a, \mathrm{refl})$$

"Extensional" MLTT

$$\frac{p : \mathrm{Id}_A\, a\, b}{a = b : A}$$

$$\frac{p : \mathrm{Id}_A\, a\, a}{p = \mathrm{refl} : \mathrm{Id}_A\, a\, a}$$

"Extensional" MLTT

$$\frac{p : \mathrm{Id}_A\, a\, b}{a = b : A}$$

$$\frac{p : \mathrm{Id}_A\, a\, a}{p = \mathrm{refl} : \mathrm{Id}_A\, a\, a}$$

Provability of judgements is undecidable

Higher identity proofs

in "intensional" MLTT

$$A$$

$$\mathrm{Id}_A\, a\, a'$$

$$\mathrm{Id}_{\mathrm{Id}_A a a'}\, p\, p'$$

$$\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A a a'} p p'}\, u\, u'$$

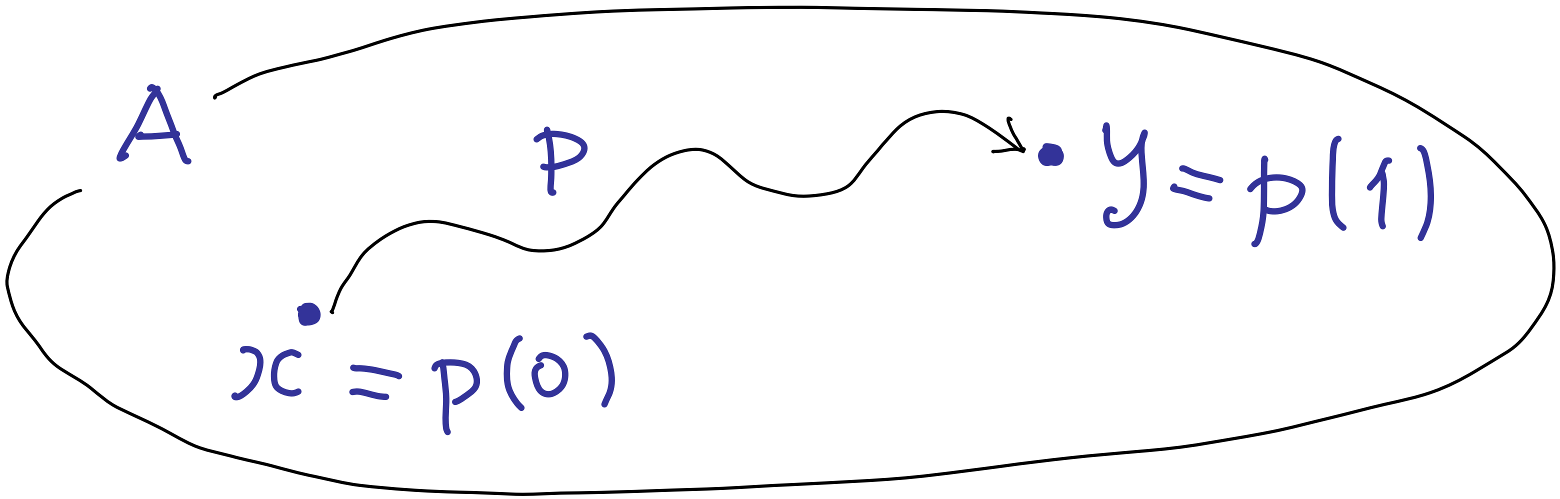
$$\vdots$$

OMG!

Homotopical view of Equality

$\mathrm{Id}_A\ x\ y$	$[\ x, y : A\ ]$
~~type of proofs that x equals y~~	

type of [abstract] paths from x to y in A



Homotopical view of Equality

If "path" means "function $I \to A$", what does an interval I (in a topos, say) have to satisfy to get a model of identity types?

Rest of the talk explores this question (cf. Michael Warren's 2006 PhD thesis)

Homotopical view of Equality

If "path" means "function $I \to A$", what does an interval I (in a topos, say) have to satisfy to get a model of identity types?

Rest of the talk explores this question from a new angle...

Propositional identity types

Replace

judgemental computation rule

$$J_{a,B}\, b\, a\, \mathrm{refl} = b : B(a, \mathrm{refl})$$

Propositional identity types

Replace

judgemental computation rule

$$J_{a,B}\, b\; a\; \mathsf{refl} = b : B(a, \mathsf{refl})$$

by weaker propositional version

$$\boxed{H_{a,B}\, b : \mathrm{Id}_{B(a,\mathsf{refl})}\,(J_{a,B}\, b\; a\; \mathsf{refl})\; b}$$

Propositional identity types

- for <u>extensional</u> TT, makes no difference
- Coquand-Danielsson: makes no difference in practice (?)

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Propositional identity types

- for extensional TT, makes no difference
- Coquand-Danielsson : makes no difference in practice (?)
- VanDenBerg (Apr. 2016): category theoretic semantics
- Swan : prop. id. type $\rightsquigarrow$ judg. id. type in a model of cubical type theory

What about in general ?

Coquand's axioms for propositional identity types

$\mathrm{refl} \quad : \quad x \simeq x$

$\mathrm{contr} \quad : \quad (x, \mathrm{refl}) \simeq (y, p)$

$\_\bullet\_ \quad : \quad x \simeq y \to B\,x \to B\,y$

$\mathrm{refl}\bullet \quad : \quad \mathrm{refl} \bullet b \simeq b$

from now on we write $x \simeq y$
for $\mathrm{Id}_A\, x\, y$ (A implicit)

Coquand's axioms for propositional identity types

$p : x \simeq y$

$\mathsf{refl} : x \simeq x$

$\mathsf{contr} : (x, \mathsf{refl}) \simeq (y, p)$

$\_\bullet\_ : x \simeq y \to B\,x \to B\,y$

family of types

$x : A \vdash B(x) : \mathcal{U}$

$\mathsf{refl}\bullet : \mathsf{refl} \bullet b \simeq b$

Given
$$\boxed{\begin{tikzcd} 1 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & I \end{tikzcd}}$$
interval with end-points

in a topos $\mathcal{E}$

for each $A \in \mathcal{E}$ we get

$$\boxed{x \simeq y \overset{\text{def}}{=} \{p : A^I \mid p0 = x \wedge p1 = y\} \qquad (x, y : A)}$$

What's needed for this $\simeq$ to satisfy Coquand's axioms?

Coquand's axioms for propositional identity types

refl : $x \simeq x$

contr : $(x, \mathrm{refl}) \simeq (y, p)$

$\_ \bullet \_$: $x \simeq y \to Bx \to By$

refl• : $\mathrm{refl} \bullet b \simeq b$

Coquand's axioms for propositional identity types

$\mathsf{refl} \ : \ x \simeq x$

$\mathsf{refl} \stackrel{\text{def}}{=} \lambda i.\, x$ constant function

Coquand's axioms for propositional identity types

✓ $\mathsf{refl} : x \simeq x$

? $\mathsf{contr} : (x, \mathsf{refl}) \simeq (y, p)$

$\lambda i.\, x$

$p : x \simeq y$

Coquand's axioms for propositional identity types

$$\mathrm{contr} : \left(x, \overset{\lambda i.\, x}{\mathrm{refl}}\right) \simeq \left(y, \overset{p : x \simeq y}{p}\right)$$

$$\mathrm{contr} \overset{\mathrm{def}}{=} \lambda i : I.\ (p\, i,\ ?\, i)$$

$$\boxed{\begin{array}{l} ? : I \to (I \to A) \\ ?\, 0 = \lambda j.\, x \\ ?\, 1 = p \end{array}}$$

Coquand's axioms for propositional identity types

$$\mathrm{contr} : (x, \mathrm{refl}) \simeq (y, p)$$

$$\mathrm{contr} \overset{\text{def}}{=} \lambda i : I.\, (p\, i,\; ?\, i)$$

$$\sqcap : I \to I \to I$$
$$0 \sqcap i = 0 = i \sqcap 0$$
$$1 \sqcap i = i = i \sqcap 1$$

take
$A = I$
$p = \mathrm{id}_I$
$x = 0$
$y = 1$

$$? : I \to (I \to A)$$
$$?\, 0 = \lambda j.\, x$$
$$?\, 1 = p$$

Coquand's axioms for propositional identity types

$\mathrm{contr} : (x, \mathrm{refl}) \simeq (y, p)$

$\mathrm{contr} \overset{\mathrm{def}}{=} \lambda i : \mathbb{I}.\,(p\,i,\ \lambda j.\, p(i \sqcap j))$

$$\boxed{\begin{array}{l} \sqcap : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \\ 0 \sqcap i = 0 = i \sqcap 0 \\ 1 \sqcap i = i = i \sqcap 1 \end{array}}$$

If we postulate that $\mathbb{I}$ has this "connection" structure, then we can satisfy contr like this

Let's assume (for the moment) that I carries the following structure

"Connection algebra"

$$\begin{tikzcd} 1 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & I & I \times I \arrow[l, shift right, "\sqcap"'] \arrow[l, shift left, "\sqcup"] \end{tikzcd}$$

$$\begin{aligned}
0 \sqcap i &= 0 = i \sqcap 0 \\
1 \sqcap i &= i = i \sqcap 1 \\
0 \sqcup i &= i = i \sqcup 0 \\
1 \sqcup i &= 1 = i \sqcup 1
\end{aligned}$$

Coquand's axioms for propositional identity types

✓ refl : $x \simeq x$

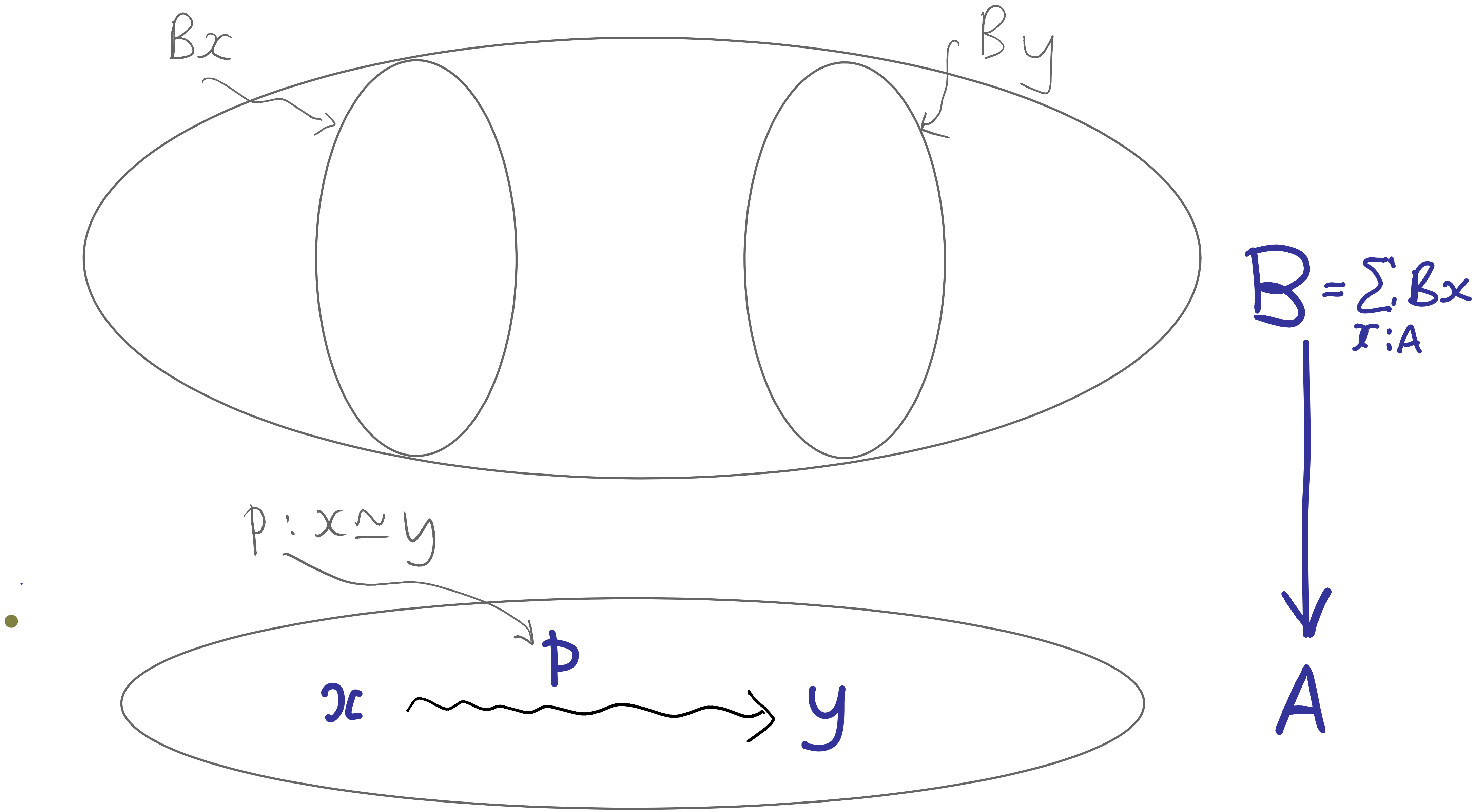
✓ contr : $(x, \mathrm{refl}) \simeq (y, p)$

? $\_\bullet\_$: $x \simeq y \to Bx \to By$

? refl• : $\mathrm{refl} \bullet b \simeq b$

fibre of $\begin{array}{c} B \\ \downarrow \\ A \end{array}$ over $y : A$

Transport along paths



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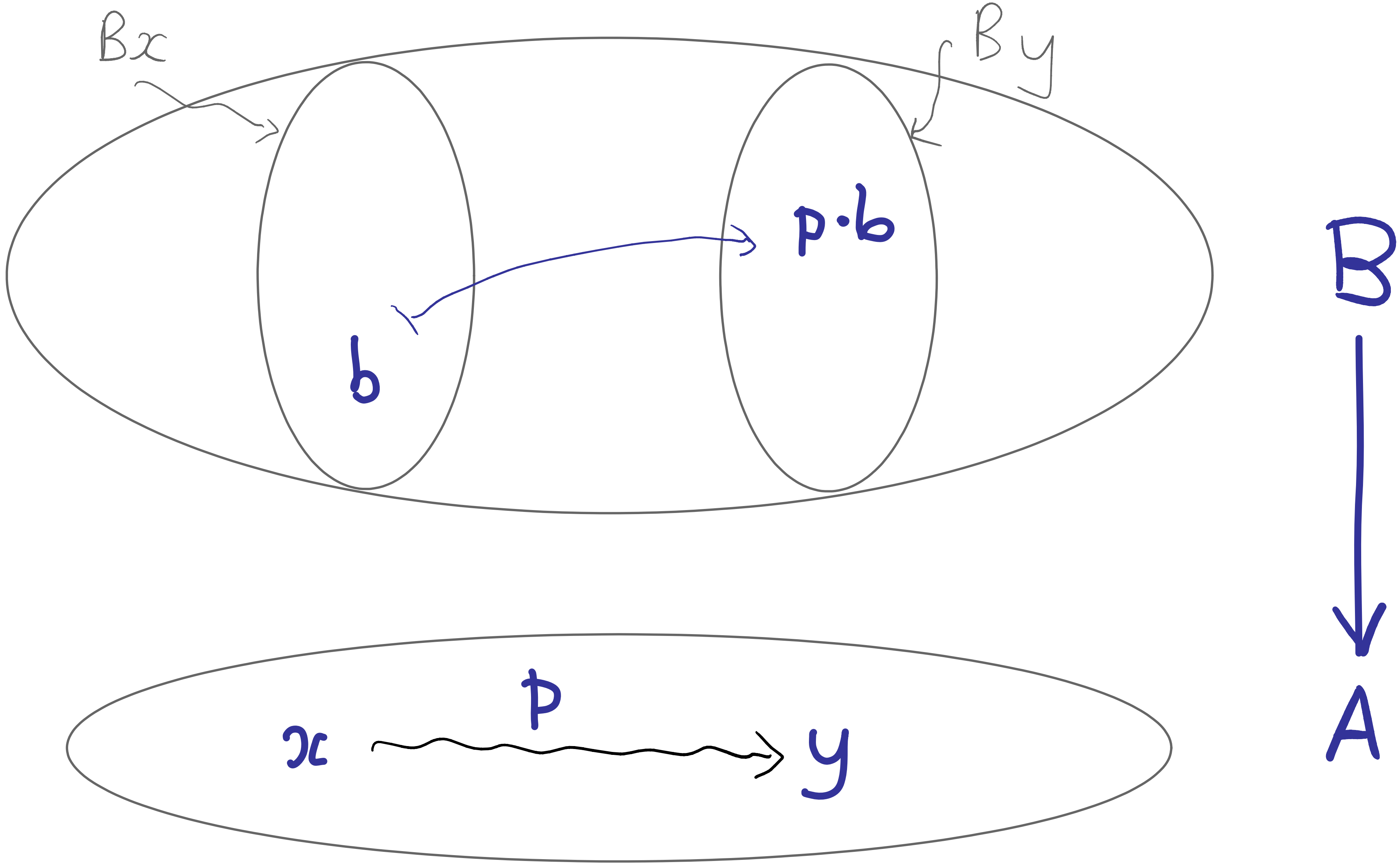

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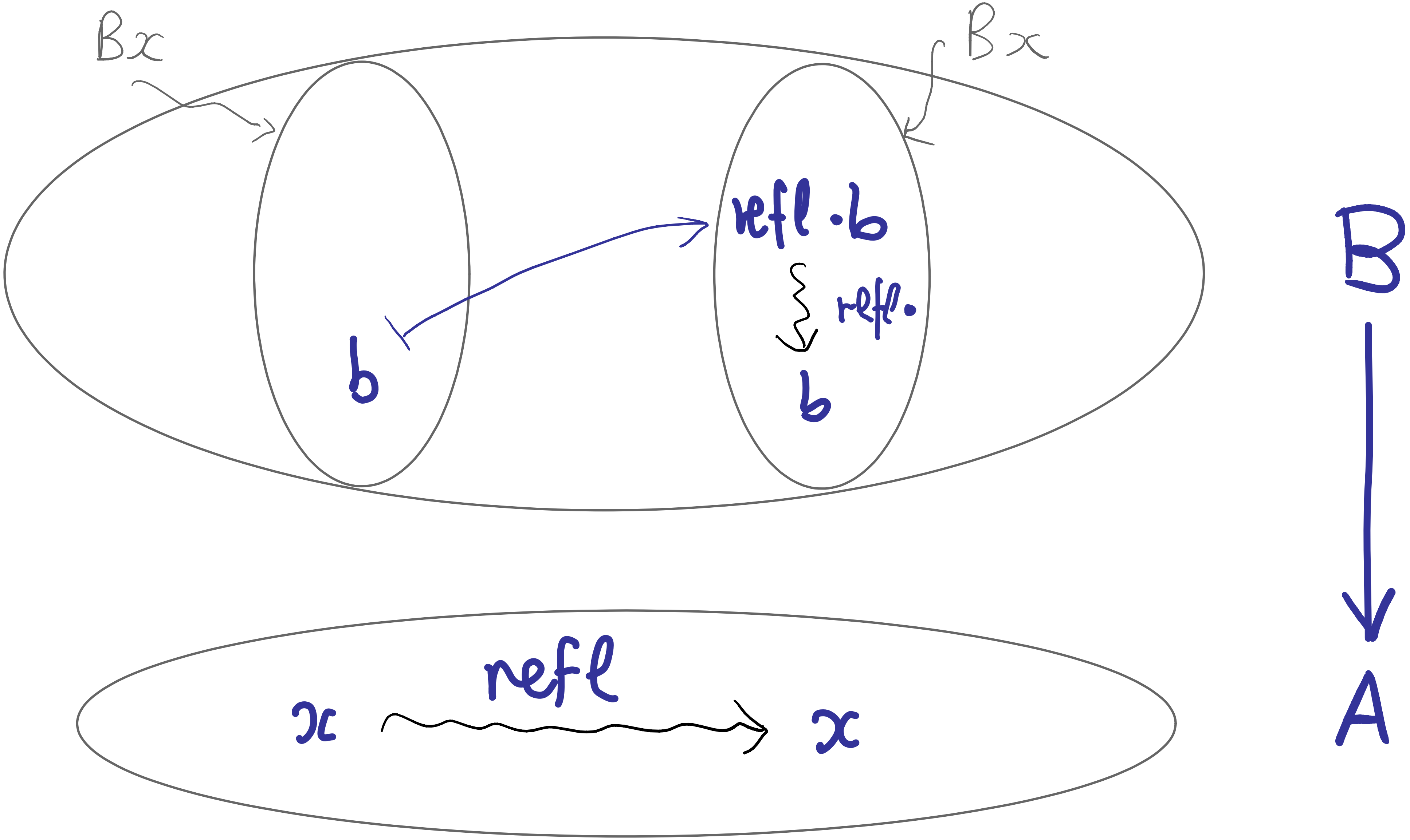

$$\_\bullet\_ \quad : \quad x \simeq y \to Bx \to By$$

$$\mathrm{refl}\bullet \quad : \quad \mathrm{refl} \bullet b \simeq b$$

Wanted: notion of fibration $\begin{array}{c} B \\ \downarrow \\ A \end{array}$

supporting $\_\bullet\_$ & $\mathrm{refl}\bullet$, closed under $\Sigma, \Pi, \simeq, \ldots$

TAP: $\boxed{\begin{array}{lll} \_\bullet\_ & : & x \simeq y \to Bx \to By \\ \mathrm{id}_\bullet & : & \mathrm{refl} \bullet b \simeq b \end{array}}$

Wanted: notion of fibration $\begin{array}{c} B \\ \downarrow \\ A \end{array}$

supporting $\_\bullet\_$ & $\mathrm{refl}_\bullet$, closed under $\Sigma, \Pi, \simeq, \ldots$

Naïve approach: why can't we just take TAP as the definition of "fibration"?

TAP: $\boxed{\begin{array}{lll} \_\bullet\_ & : & x \simeq y \to Bx \to By \\ \mathrm{id}_\bullet & : & \mathrm{refl} \bullet b \simeq b \end{array}}$

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Naïve approach: why can't we just take TAP as the definition of "fibration"?

$\boxed{\text{[spoiler alert] Ans: we can!}}$

To model propositional identity types, each

$$\begin{tikzcd} \sum_{x,y} x \simeq y \arrow[d] \\ A \times A \end{tikzcd}$$

has to be a family with TAP

So we need :

$$\begin{tikzcd} x \arrow[r, squiggly, "p"] \arrow[d, squiggly, "r"'] & x' \\ y \arrow[r, squiggly, "q"'] & y' \end{tikzcd}$$

To model propositional identity types, each

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So we need :

$$\begin{tikzcd} x \arrow[r, squiggly, "p"] \arrow[d, squiggly, "r"'] & x' \arrow[d, squiggly, "(p,q)\cdot r"] \\ y \arrow[r, squiggly, "q"'] & y' \end{tikzcd}$$

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$$\sum_{x,y} x \simeq y \downarrow A \times A$$

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plus $(\mathrm{refl},\mathrm{refl})\cdot r \overset{\mathrm{refl}\cdot}{\rightsquigarrow} r$

Let's assume for the moment I carries the following structure

Connection + reversal

$$
\begin{tikzcd}
1 \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] & I \arrow[loop above, "\mathrm{rev}"] & I \times I \arrow[l, bend right, "\sqcap"'] \arrow[l, bend left, "\sqcup"]
\end{tikzcd}
$$

$$0 \sqcap i = 0 = i \sqcap 0$$

$$1 \sqcap i = i = i \sqcap 1$$

$$0 \sqcup i = i = i \sqcup 0$$

$$1 \sqcup i = 1 = i \sqcup 1$$

$$\mathrm{rev}\ 0 = 1 \qquad \mathrm{rev}\ 1 = 0$$

$$
\begin{tikzcd}
x \arrow[r, squiggly, "p"] \arrow[d, squiggly, "r"'] & x' \\
y \arrow[r, squiggly, "q"'] & y'
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \arrow[d, squiggly, "r"'] & & x' \arrow[ll, squiggly, "p^{-1}"'] \\
y \arrow[rr, squiggly, "q"'] & & y'
\end{tikzcd}
$$

$$p^{-1} \overset{\text{def}}{=} p \circ \mathrm{rev}$$

$$\begin{tikzcd}
x & x' \arrow[l, squiggly, "p^{-1}"'] \arrow[d, squiggly] \\
y \arrow[u, phantom] \arrow[r, squiggly, "q"'] \arrow[ur, phantom, "\circlearrowright"] & y'
\end{tikzcd}$$

with $x \overset{r}{\rightsquigarrow} y$

Idea:

$$(p,q) \cdot r \overset{\text{def}}{=} q \odot (r \odot p^{-1})$$

where $\_\odot\_$ is a weak form of path composition (weaker than in Warren's thesis)

Path composition

$$\frac{p0 \xrightarrow{p} p1 = q0 \xrightarrow{q} q1}{p0 \xrightarrow{q \odot p} q1}$$

If I were $[0,1]$, we could define

$$(q \odot p)(i) = \begin{cases} p(2i) & \text{if } 0 \leq i \leq \frac{1}{2} \\ q(2i-1) & \text{if } \frac{1}{2} \leq i \leq 1 \end{cases}$$

Path composition

$$\frac{p0 \xrightarrow{p} p1 = q0 \xrightarrow{q} q1}{p0 \xrightarrow{q \odot p} q1}$$

If I were $[0,1]$, we could define

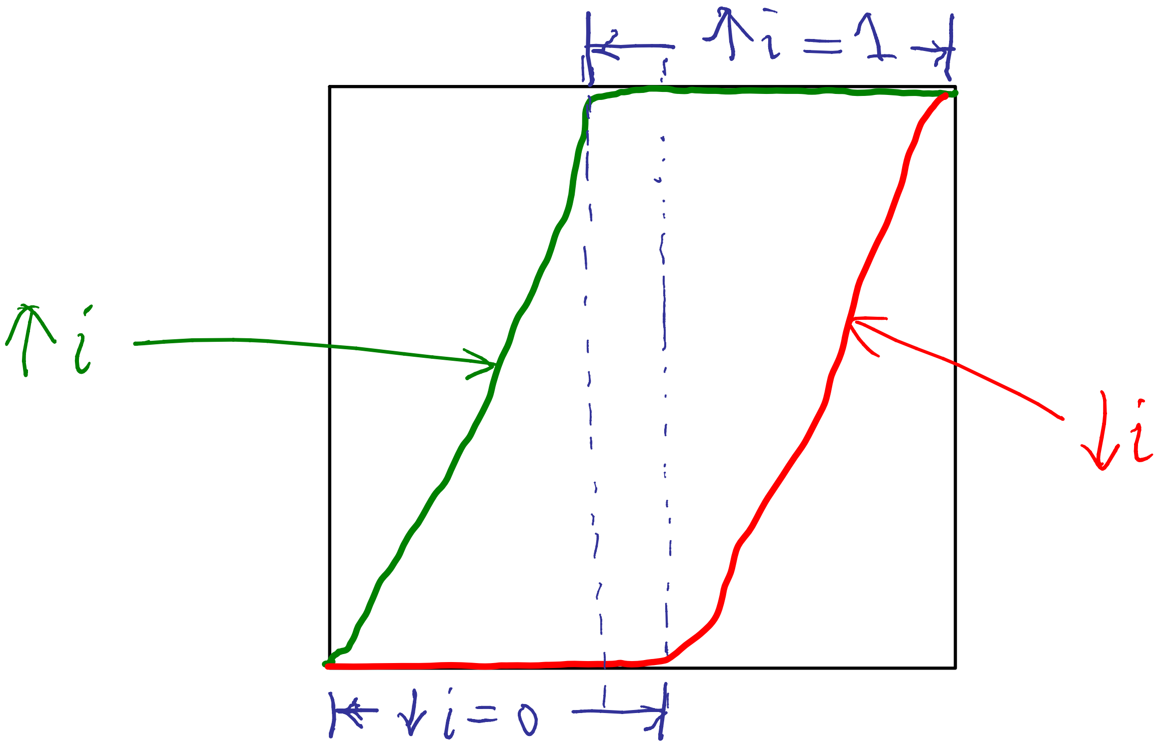
$$(q \odot p)(i) = \begin{cases} p(\uparrow i) & \text{if } \downarrow i = 0 \\ q(\downarrow i) & \text{if } \uparrow i = 1 \end{cases}$$

where $\begin{cases} \uparrow i \stackrel{\text{def}}{=} \text{if } i \leq \frac{1}{2} \text{ then } 2i \text{ else } 1 \\ \downarrow i \stackrel{\text{def}}{=} \text{if } i \leq \frac{1}{2} \text{ then } 0 \text{ else } 2i-1 \end{cases}$

Axioms for $\uparrow, \downarrow : \mathbb{I} \to \mathbb{I}$

$\uparrow 0 = 0$	$\downarrow 0 = 0$
$\uparrow 1 = 1$	$\downarrow 1 = 1$

$\forall i : \mathbb{I}.\ \downarrow i = 0 \lor \uparrow i = 1$



Axioms for $\uparrow, \downarrow : \mathbb{I} \to \mathbb{I}$

$\uparrow 0 = 0$	$\downarrow 0 = 0$
$\uparrow 1 = 1$	$\downarrow 1 = 1$
$\forall i:\mathbb{I}.\ \downarrow i = 0 \vee \uparrow i = 1$	

Then for any $p, q : \mathbb{I} \to A$ with $p\,1 = q\,0$ we get
$q \odot p : \mathbb{I} \to A$ satisfying

$$\forall i:\mathbb{I}.\ \downarrow i = 0 \Rightarrow (q \odot p)\, i = p(\uparrow i)$$
$$\forall i:\mathbb{I}.\ \uparrow i = 1 \Rightarrow (q \odot p)\, i = q(\downarrow i)$$

Need

$$\begin{tikzcd} x \arrow[r, rightsquigarrow, "p"] \arrow[d, rightsquigarrow, "r"'] & x' \arrow[d, rightsquigarrow, "(p,q)\bullet r"] \\ y \arrow[r, rightsquigarrow, "q"'] & y' \end{tikzcd}$$

plus

$$(\mathrm{refl},\mathrm{refl})\bullet r \overset{\mathrm{refl}\bullet}{\rightsquigarrow} r$$

Have

$$\begin{tikzcd}
x \arrow[r, squiggly, "p"] \arrow[d, squiggly, "r"'] & x' \arrow[d, squiggly, "q\odot(r\odot p^{-1})"] \\
y \arrow[r, squiggly, "q"'] & y'
\end{tikzcd}$$

but what about

$$\mathrm{refl}\odot(r\odot \mathrm{refl}^{-1}) \xrightarrow{\mathrm{refl}\bullet} r$$

?

Is there a path $\mathrm{refl} \odot (r \odot \mathrm{refl}^{-1}) \simeq r$?

Is there a path $\mathrm{refl} \odot (r \odot \mathrm{refl}^{-1}) \simeq r$?

$$\mathrm{refl}^{-1} = (\lambda i.\, x) \circ \mathrm{rev} = \lambda i.\, x = \mathrm{refl}$$

$$r \odot \mathrm{refl} = r \circ \downarrow$$

$$\mathrm{refl} \odot r = r \circ \uparrow$$

Is there a path $\mathrm{refl} \odot (r \odot \mathrm{refl}^{-1}) \simeq r$?

$$\mathrm{refl}^{-1} = (\lambda i.\, x) \circ \mathrm{rev} = \lambda i.\, x = \mathrm{refl}$$

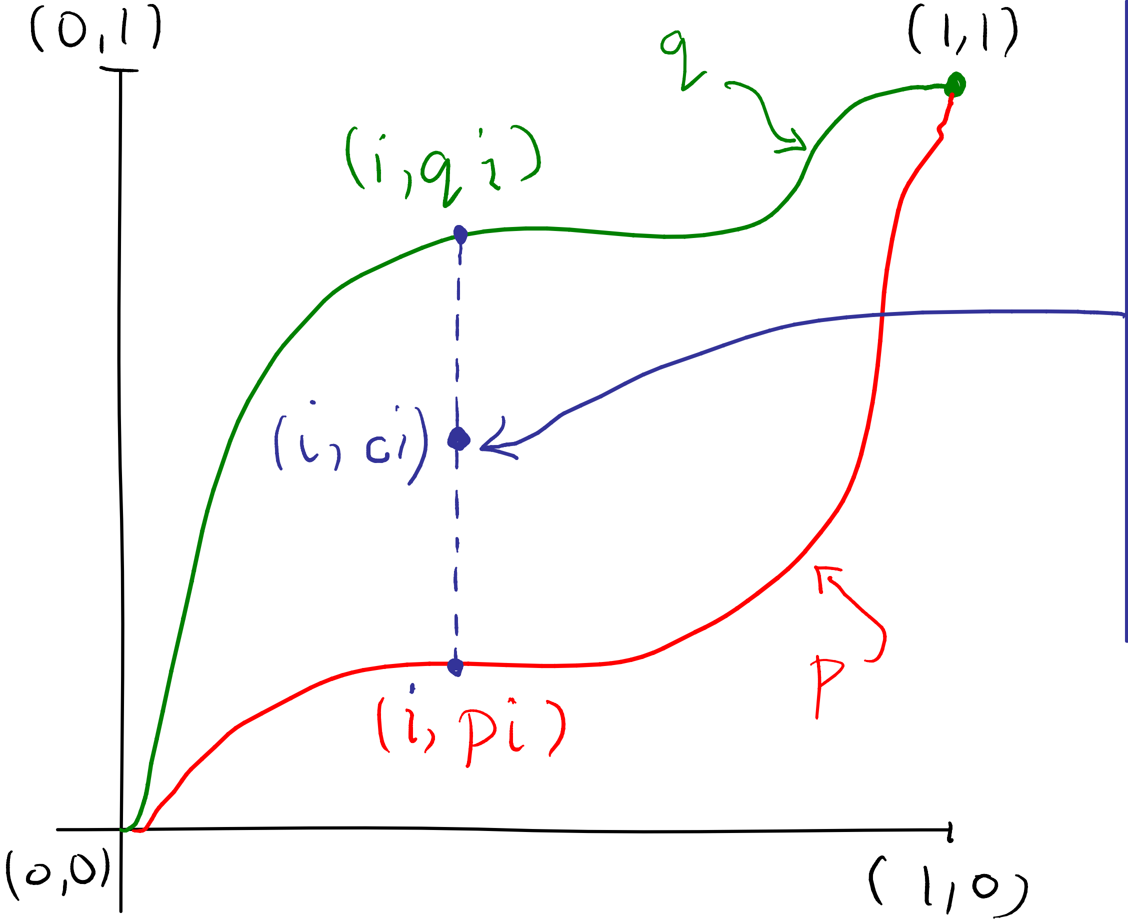
$$r \odot \mathrm{refl} = r \circ \downarrow$$

$$\mathrm{refl} \odot r = r \circ \uparrow$$

So we just need a path $\downarrow \circ \uparrow \simeq \mathrm{id}_I$,

or more generally, for any $p, q : 0 \simeq 1$

a path $p \simeq q$



When I is $[0,1]$, for c_i we can use a **convex combination**

$$(p_i)(1-k)+(q_i)k$$

as k ranges over $[0,1]$

a path $p \simeq q$, for any $p, q : 0 \simeq 1$

Interval Axioms

$0, 1 : I \qquad \_\dashv\_\vdash\_ : I \to I \to I \to I \qquad \uparrow, \downarrow : I \to I$

$$i \dashv 0 \vdash k = i \qquad i \dashv 1 \vdash k = k$$

$$i \dashv j \vdash i = i \qquad 0 \dashv j \vdash 1 = j$$

(simple properties of $i, j, k \mapsto (1-j)\,i + jk$
when I is the unit interval $[0,1]$)

Interval Axioms

$$0, 1 : I \qquad \_\dashv\_\vdash\_ : I \to I \to I \to I \qquad \uparrow, \downarrow : I \to I$$

$$i \dashv 0 \vdash k = i \qquad i \dashv 1 \vdash k = k$$

$$i \dashv j \vdash i = i \qquad 0 \dashv j \vdash 1 = j$$

$$\text{Subsumes} \begin{cases} i \sqcap j = 0 \dashv i \vdash j \\ i \sqcup j = j \dashv i \vdash 1 \\ \mathrm{rev}\ i = 1 \dashv i \vdash 0 \end{cases}$$

Interval Axioms IA

$0, 1 : I \quad \_ \dashv \_ \vdash \_ : I \to I \to I \to I \quad \uparrow, \downarrow : I \to I$

$i \dashv 0 \vdash k = i$	$i \dashv 1 \vdash k = k$
$i \dashv j \vdash i = i$	$0 \dashv j \vdash 1 = j$
$\uparrow 0 = 0$	$\downarrow 0 = 0$
$\uparrow 1 = 1$	$\downarrow 1 = 1$

$\forall i : I,\ \downarrow i = 0 \vee \uparrow i = 1$

<u>Theorem</u> In any topos with a model of IA, the families with TAP give a model of intensional M-L type theory with

- Σ-types
- Π-types
- propositional identity types
- coproducts
- W-types
- $\emptyset$, 1

Theorem In any topos with a model of IA, the families with TAP give a model of intensional M-L type theory with ...

Proof

- was developed using Agda
- does not use the impredicative aspect of topos logic/type theory

Theorem In any topos with a model of IA, the families with TAP give a model of intensional M-L type theory with ...

Logical consistency: Giraud's gros topos contains a model of IA for which true $\not\simeq$ false (i.e. $\nexists$ path $p : I \to \mathbb{B}$ with $p\,0 = \mathrm{true} \wedge p\,1 = \mathrm{false}$)

Theorem In any topos with a model of IA, the families with TAP give a model of intensional M-L type theory with ...

Dont yet know whether we can get an instance of Voevodsky's univalent universe this way.

Theorem In any topos with a model of IA, the families with TAP give a model of intensional M-L type theory with ...

Advantage over cubical sets of Coquand et al: no Kan filling conditions and (hence) the interval is a first-class object of the type theory (i.e. $\mathbb{I}$ is fibrant).

Theorem In any topos with a model of IA, the families with TAP give a model of intensional M-L type theory with ...

Advantage over cubical sets of Coquand et al: the interval is a first-class object of the type theory (i.e. I is fibrant)

Is there a useful "interval type theory" analogous to cubical type theory?

Summary

IA "coherent" theory of the interval

Summary

IA coherent theory of the interval

Models of IA in toposes give models of intensional M-L type theory with propositional identity types that are based on equality-as-path and without any "Kan-filling" conditions

Summary

IA coherent theory of the interval

Models of IA in toposes give models of intensional M-L type theory with propositional identity types that are based on equality-as-path

- first-class interval type
- function extensionality automatic
- universe extensionality ...in progress