On Proofs of Equality as Paths

Andrew Pitts

joint work with Ian Orton

Logic & Semantics Seminar 2016-10-07
(Martin-Löf) Type Theory is formulated in terms of "judgements".

<table>
<thead>
<tr>
<th>a : A</th>
<th>a has type A</th>
</tr>
</thead>
<tbody>
<tr>
<td>a = b : A</td>
<td>a &amp; b are equal and of type A</td>
</tr>
</tbody>
</table>
**Type Theory**

is formulated in terms of hypothetical judgements

<table>
<thead>
<tr>
<th>(x : A, y : B(x) \vdash a(x, y) : C(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x : A, y : B(x) \vdash a(x, y) = b(x, y) : C(x, y))</td>
</tr>
</tbody>
</table>

*dependent types!*
Identity types

\[ x : A, y : A \vdash \text{Id}_A x y : U \]

type of proofs that \( x \) equals \( y \)
Identity types

\[ \text{x:A, y:A \rightarrow Id}_A x y : \mathcal{U} \]
Identity introduction

If $a : A$, then there is a proof

$$\text{refl} : \text{Id}_A a a$$

(reflexivity of equality)
Identity elimination

If \( a : A \) and \( x : A, p : \text{Id}_A a x \vdash B(x, p) : U \),
Identity elimination

If $a : A$ and $x : A$, $p : \text{Id}_A ax \vdash B(x, p) : U$, given any $x : A$ & $p : \text{Id}_A ax$, to construct an element of $B(x, p)$. 
Identity elimination

If \( a : A \) and \( x : A, p : \text{Id}_A ax \vdash B(x,p) : U \),
given any \( x : A \) \& \( p : \text{Id}_A ax \),
to construct an element of \( B(x,p) \),
it suffices to give some \( b : B(a, \text{refl}) \)

\[
J_{a,B} : B(a, \text{refl}) \to (x : A)(p : \text{Id}_A ax) \to B(x,p)
\]
Identity elimination & computation

If \( a : A \) and \( x : A, p : \text{Id}_Aax \vdash B(x,p) : U \), given any \( x : A \) & \( p : \text{Id}_Aax \), to construct an element of \( B(x,p) \), it suffices to give some \( b : B(a,\text{refl}) \).

\[
J_{a,B} : B(a,\text{refl}) \rightarrow (x : A)( \vdash \text{Id}_Aax ) \rightarrow B(x,p)
\]

\[
J_{a,B} b a \text{refl} = b : B(a,\text{refl})
\]
Higher identity proofs

\[ A \]
\[ \text{Id}_A \ a \ a' \]
\[ \text{Id} \ \text{Id}_A a a', p p' \]
\[ \text{Id} \ \text{Id} \ \text{Id}_A a a', p p', u u' \]
\[ \vdots \]
\[ \text{OMG!} \]
Homotopical view of Equality

\[ \text{Id}_A x y \quad [ \ x, y : A \ ] \]

type of proofs that \( x \) equals \( y \)

type of \( [\text{abstract}] \) paths from \( x \) to \( y \) in \( A \)

\[ A \]

\[ \bullet \quad p \]

\[ \bullet \quad y = p(1) \]

\[ \bullet \quad c = p(0) \]
Homotopical view of Equality

If “path” means “function \( I \to A \), what does an interval \( I \) (in a topos, say) have to satisfy to get a model of identity types?

Rest of the talk explores this question (cf. Michael Warren’s 2006 PhD thesis)
Homotopical view of Equality

If “path” means “function $I \to A$”, what does an interval $I$ (in a topos, say) have to satisfy to get a model of identity types?

Rest of the talk explores this question from a new angle...
Propositional identity types

Replace

judgemental computation rule

\[ J_{a, B} b a \text{refl} = b : B(a, \text{refl}) \]
Propositional identity types

Replace
judgemental computation rule

\[ J_{a, B} b \ a \ refl = b : B(a, \text{refl}) \]

by weaker propositional version

\[ H_{a, B} b : \text{Id} \ B(a, \text{refl}) (J_{a, B} b \ a \ refl) \ b \]
Propositional identity types

- for extensional TT, makes no difference
- Coquand-Danielsson: makes no difference in practice (?)
Propositional identity types

- for extensional TT, makes no difference
- Coquand-Danielsson: makes no difference in practice (?)
- Van Den Berg (Apr. 2016): category theoretic semantics
Propositional identity types

- for extensional TT, makes no difference

- Coquand-Danielsson: makes no difference in practice (?)

- VanDenBerg (Apr. 2016): category theoretic semantics

- Swan: prop. id.type $\rightarrow$ judg. id.type in a model of cubical type theory

What about in general?
Coquand’s axioms for propositional identity types

\[
\text{refl} : \ x \simeq x
\]

\[
\text{contr} : \ (x, \text{refl}) \simeq (y, p)
\]

\[
\cdot : \ x \simeq y \rightarrow Bx \rightarrow By
\]

\[
\text{refl} \cdot : \ \text{refl} \cdot b \simeq b
\]

From now on we write \(x \simeq y\) for \(\text{Id}_A x y\) (\(A\) implicit)
Coquand's axioms for propositional identity types

\[
\begin{align*}
\text{refl} & : x \sim x \\
\text{contr} & : (x, \text{refl}) \sim (y, p) \\
\_ \cdot \_ & : x \sim y \to Bx \to By \\
\text{refl} \cdot \_ & : \text{refl} \cdot b \sim b
\end{align*}
\]
Given

1 \xleftarrow{0} \xrightarrow{1} I

interval with end-points

in a topos \( \mathcal{E} \)

for each \( A \in \mathcal{E} \) we get

\[ x \sim y \overset{\text{def}}{=} \{ p : A^I \mid p_0 = x \land p_1 = y \} \]

\((x, y : A)\)

What's needed for this \( \sim \) to satisfy Coquand's axioms?
Coquand's axioms for propositional identity types

\[ \text{refl} : x \simeq x \]

\[ \text{contr} : (x, \text{refl}) \simeq (y, p) \]

\[ \vdots : x \simeq y \rightarrow B x \rightarrow By \]

\[ \text{refl} \cdot b \simeq b \]
Coquand’s axioms for propositional identity types

\[
\text{refl} : x \approx x
\]

\[
\text{refl} \overset{\text{def}}{=} \lambda i. x \quad \text{constant function}
\]
Coquand’s axioms for propositional identity types

✓ refl : \( x \simeq x \)

? contr : \((x, \text{refl}) \simeq (y, p)\)
Coquand's axioms for propositional identity types

$$\text{contr} : (x, \text{refl}) \simeq (y, p)$$

$$\text{contr} \overset{\text{def}}{=} \lambda i : I. (p_i, ?i)$$

$$\lambda i : x$$

$$p : x \simeq y$$

$$?: I \rightarrow (I \rightarrow A)$$

$$?o = \lambda j, x$$

$$?1 = p$$
Coquand's axioms for propositional identity types

\[ \text{contr} : (x, \text{refl}) \simeq (y, p) \]
\[ \text{contr} \overset{\text{def}}{=} \lambda i : I. (p_i, ?i) \]

\[ \land : I \to I \to I \]
\[ 0 \land i = 0 \]
\[ 1 \land i = i \]

\[ \text{take} \]
\[ A = I \]
\[ p = \text{id}_I \]
\[ \alpha = 0 \]
\[ y = 1 \]

\[ ? : I \to (I \to A) \]
\[ ?0 = \lambda j, x \]
\[ ?1 = p \]
Coquand's axioms for propositional identity types

\[ \text{contr} : (x, \text{refl}) \simeq (y, p) \]

\[ \text{contr} \overset{\text{def}}{=} \lambda i : I. (p_i, \lambda j. p(i \circ j)) \]

\[ \cap : I \rightarrow I \rightarrow I \]

0 \cap j = 0

1 \cap j = j

If we postulate that I has this "connection" structure, then we can satisfy contr like this.
Let's assume (for the moment) that I carries the following structure:

```
\[ \begin{array}{c}
0 \\
1 \\
\end{array} \quad I \quad \begin{array}{c}
\cap \\
\cup \\
\end{array} \quad I \times I
\]
```

```
0 \cap i = 0 = i \cap 0
1 \cap i = i = i \cap 1
0 \cup i = i = i \cup 0
1 \cup i = 1 = i \cup 1
```
Coquand’s axioms for propositional identity types

\[
\begin{align*}
\checkmark \quad \text{refl} & : \ x \simeq x \\
\checkmark \quad \text{contr} & : \ (x, \text{refl}) \simeq (y, p) \\
? \quad \text{elim} & : \ x \simeq y \to Bx \to By \\
? \quad \text{refl}_* & : \ \text{refl} \cdot b \simeq b
\end{align*}
\]
Path-subsitutive families

\[ \mathcal{B} = \sum_{\tau : A} \mathcal{B}_x \]

\[ \mathcal{B} \]

\[ A \]
Path-substitutive families
Path-substitutive families

\[ B \xrightarrow{b} B \]

\[ B \xrightarrow{\text{refl} \cdot b} B \]

\[ x \xrightarrow{\text{refl}} x \]
\[ \cdot \cdot : x \sim y \rightarrow Bx ightarrow By \]
\[ \text{refl} \cdot : \text{refl} \cdot b \sim b \]

Wanted: notion of fibration \( B \times A \)

Supporting \( \cdot \cdot \) & \( \text{refl} \cdot \), closed under \( \Sigma, \Pi, \sim, \ldots \)
\( \_ \cdot \_ : x \approx y \rightarrow B x \rightarrow B y \)

\( \text{id} \cdot \text{refl} \cdot b \approx b \)

Wanted: notion of fibration \( B \rightarrow A \)

Supporting \( \_ \cdot \_ \) & \( \text{refl} \cdot \), closed under \( \Sigma, \Pi, \approx, \ldots \)

Naïve approach: why can't we just take this as the definition of "fibration"?
\[ \_ \cdot \_ : x \equiv y \rightarrow Bx \rightarrow By \]

\[ \text{id}_\_ : \text{refl} \cdot \_ \equiv \_ \]

Wanted : notion of fibration \[ B \overset{\text{A}}{\rightarrow} \]

Supporting \[ \_ \cdot \_ \] & \[ \text{refl} \cdot \_ \], closed under \[ \Sigma, \Pi, \equiv, \ldots \]

Naïve approach : why can't we just take this as the definition of "fibration"?

[spoiler alert] Ans : we can!
To model propositional identity types, each
\[ \sum_{x, y} x \simeq y \]
has to be a path-substitutive family.

So we need:

\[ x \xrightarrow{\sim} x' \]
\[ y \xrightarrow{\sim} y' \]
To model propositional identity types, each
\[ \frac{x \simeq y}{\sum_{x,y} x \simeq y} \]
has to be a path-substitutive family.

So we need:

\[
\begin{align*}
x & \sim \to x' \\
y & \sim q \to y'
\end{align*}
\]

\[ (p, q) \cdot r \]
To model propositional identity types, each \( \sum_{x,y : A} x \simeq y \) has to be a path-substitutive family.

So we need:

\[
x \xrightarrow{p} x' \\
\downarrow{r} \\
y \xrightarrow{q} y'
\]

\( (p,q) \cdot r \)

plus

\( (\text{refl, refl}) \cdot r \xrightarrow{\text{refl}} r \)
Let's assume $I$ carries the following structure:

Connection + reversal

\[
\begin{align*}
0 & \xrightarrow{\text{rev}} \
1 & \xrightarrow{\text{rev}} \
1 & \xrightarrow{\text{rev}} I \
I & \xrightarrow{\text{rev}} I \times I \\
0 \cap i &= 0 = i \cap 0 \\
1 \cap i &= i = i \cap 1 \\
0 \cup i &= i = i \cup 0 \\
1 \cup i &= 1 = i \cup 1 \\
\text{rev } 0 &= 1 \\
\text{rev } 1 &= 0
\end{align*}
\]
\[ x \xrightarrow{p^{-1}} x' \]
\[ r \]
\[ y \xleftarrow{q} y' \]

\[ p' \overset{\text{def}}{=} p \circ \text{rev} \]
$x \xleftarrow{r} y \xrightarrow{q} y'$

$G$

$Idea: \quad (p,q) . r \overset{\text{def}}{=} q \circ (r \circ p^{-1})$

where $-\circ -$ is a weak form of path composition (weaker than in Warren's thesis)
Path composition

\[ p_0 \xrightarrow{p} p_1 = q_0 \xrightarrow{q} q_1 \]

\[ p_0 \xrightarrow{q \circ p} q_1 \]

If I were \([0,1]\), we could define

\[(q \circ p)(i) = \begin{cases} 
  p(2i) & \text{if } 0 \leq i \leq \frac{1}{2} \\
  q(2i-1) & \text{if } \frac{1}{2} \leq i \leq 1
\end{cases} \]
Path composition

\[ p_0 \xrightarrow{p} p_1 = q_0 \xrightarrow{q} q_1 \]

\[ p_0 \xrightarrow{q \circ p} q_1 \]

If I were \([0, 1]\), we could define

\[(q \circ p)(i) = \begin{cases} 
   p(\uparrow i) & \text{if } \downarrow i = 0 \\
   q(\downarrow i ) & \text{if } \uparrow i = 1
\end{cases} \]

where \( \uparrow i \) is defined as if \( i \leq \frac{1}{2} \) then \( 2i \) else 1

\( \downarrow i \) is defined as if \( i \leq \frac{1}{2} \) then 0 else \( 2i - 1 \)
Axioms for $\uparrow, \downarrow : I \to I$

<table>
<thead>
<tr>
<th>$\uparrow 0 = 0$</th>
<th>$\downarrow 0 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\uparrow 1 = 1$</td>
<td>$\downarrow 1 = 1$</td>
</tr>
</tbody>
</table>

$\forall i : I. \downarrow i = 0 \lor \uparrow i = 1$
Axioms for $\uparrow, \downarrow : I \rightarrow I$

<table>
<thead>
<tr>
<th>$\uparrow 0$</th>
<th>$\downarrow 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 0$</td>
<td>$= 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\uparrow 1$</th>
<th>$\downarrow 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 1$</td>
<td>$= 1$</td>
</tr>
</tbody>
</table>

$\forall i : I. \downarrow i = 0 \lor \uparrow i = 1$

Then for any $p, q : I \rightarrow A$ with $p 1 = q 0$ we get $q 0 p : I \rightarrow A$ satisfying

$\forall i : I. \downarrow i = 0 \Rightarrow (q \circ p) i = p (\uparrow i)$

$\forall i : I. \uparrow i = 1 \Rightarrow (q \circ p) i = q (\downarrow i)$
Need

$x \overset{p}{\rightarrow} x'$

$r \overset{(p, q) \cdot r}{\rightarrow} (refl, refl) \cdot r \overset{refl}{\rightarrow} r$

plus
Have 

\[
\begin{align*}
x \xRightarrow{p} & \quad \quad x' \\
\downarrow \quad r \quad \downarrow & \quad \quad q \quad \downarrow \\
y \xRightarrow{q} & \quad \quad y'
\end{align*}
\]

\[q \circ (r \circ p^{-1})\]

but what about 

\[\text{refl} \circ (r \circ \text{refl}^{-1}) \xRightarrow{\text{refl}} r\]

?
Is there a path \( \text{refl} \circ (r \circ \text{refl}^{-1}) \sim r \) ?
Is there a path $\text{refl} \circ (r \circ \text{refl}^{-1}) \approx r$?

$\text{refl}^{-1} = (\lambda i. x) \cdot \text{rev} = \lambda i. x = \text{refl}$

$r \circ \text{refl} = r \circ \downarrow$

$\text{refl} \circ r = r \circ \uparrow$
Is there a path \( \text{refl} \circ (r \circ \text{refl}^{-1}) \sim r \)?

\[ \text{refl}^{-1} = (\lambda i.x) \circ \text{rev} = \lambda i.x = \text{refl} \]

\[ r \circ \text{refl} = r \circ \downarrow \]

\[ \text{refl} \circ r = r \circ \uparrow \]

So we just need a path \( \downarrow \circ \uparrow \simeq \text{id}_I \), or more generally, for any \( p, q : 0 \simeq 1 \) a path \( p \simeq q \).
When $I$ is $[0,1]$, for $c_i$ we can use a convex combination

$$(pi)(1-k) + (gi)k$$

as $k$ ranges over $[0,1]$.  

A path $p \approx q$, for any $p, q : 0 \leq 1$
Interval Axioms

\[ 0, 1 : I \rightarrow I \rightarrow I \rightarrow I \rightarrow I \uparrow, \downarrow : I \rightarrow I \]

\[ i \rightarrow 0 \rightarrow k = i \quad i \rightarrow 1 \rightarrow k = k \]

\[ i \rightarrow j \rightarrow i = i \quad 0 \rightarrow j \rightarrow 1 = j \]

(simple properties of \( i, j, k \rightarrow (1-j)i + jk \) when \( I \) is the unit interval \([0,1]\))
Interval Axioms

\[ 0, 1 : I \xrightarrow{\cdot} I \xrightarrow{\cdot} I \xrightarrow{\cdot} I \xrightarrow{\cdot} I \]

\[ \uparrow, \downarrow : I \rightarrow I \]

\[ i \cdot 0 \cdot k = i \quad i \cdot 1 \cdot k = k \]

\[ i \cdot j \cdot i = i \quad 0 \cdot j \cdot 1 = j \]

Subsumes

\[
\begin{align*}
\text{inv} i &= 1 \cdot i \cdot 0 \\
\end{align*}
\]
**Interval Axioms (IA)**

- $0, 1 : I \rightarrow I \rightarrow I \rightarrow I$
- $\uparrow, \downarrow : I \rightarrow I$

<table>
<thead>
<tr>
<th>$i + 0 + k$</th>
<th>$i + 1 + k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$k$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i + j + i$</th>
<th>$0 + j + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$j$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\uparrow 0$</th>
<th>$\downarrow 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\uparrow 1$</th>
<th>$\downarrow 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

| $\forall i : I$, $\downarrow i = 0 \lor \uparrow i = 1$ |
Theorem. In any topos with a model of IA, the path-substitutive families give a model of intensional ML type theory with

- \( \Sigma \)-types
- \( \Pi \)-types
- Propositional identity types
- Coproducts
- \( W \)-types
- \( \emptyset, 1 \)
Theorem: In any topos with a model of IA, the path-substitutive families give a model of intensional M-L type theory.

Proof:
- was developed using Agda
- does not use the impredicative aspect of topos logic/type theory.
Theorem: In any topos with a model of \textit{IA}, the path-substitutive families give a model of intensional M-L type theory with ...

Logical consistency: Giraud's gros topos contains a model of \textit{IA} for which \texttt{true} $\neq$ \texttt{false} (i.e. path \texttt{p : I \rightarrow B} with \texttt{p o = true \land p 1 = false})
Theorem. In any topos with a model of IA, the path-substitutive families give a model of intensional M-L type theory with ...

Don't yet know whether we can get an instance of Voevodsky's univalent universe this way (obvious place to look is the classifying topos of IA)
Theorem. In any topos with a model of \( \text{IA} \), the path-substitutive families give a model of intensional M-L type theory with ... 

Advantage over cubical sets of Coquand et al.: no Kan filling conditions and (hence) the interval is a first-class object of the type theory (i.e. \( I \) is fibrant).
Theorem. In any topos with a model of \( \text{IA} \), the path-substitutive families give a model of intensional M-L type theory.

Advantage over cubical sets of Coquand et al.: the interval is a first-class object of the type theory (i.e. \( I \) is fibrant)

Is there a useful "interval type theory" analogous to cubical type theory?
IA "coherent" theory of the interval
Summary

IA coherent theory of the interval

Sh(IA) classifying topos of IA

Ul

model of intensional M-L type theory

with propositional identity types that are based on equality-as-path and without any "Kan-filling" conditions
Summary

IA coherent theory of the interval

$\text{Sh(IA)}$ classifying topos of IA

$\text{Ul}$ model of intensional M-L type theory with propositional identity types that are based on equality-as-path

- first-class interval type
- function extensionality automatic
- universe extensionality ... in progress