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SHORT COMMUNICATION

FUZZY SETS DO NOT FORM A TOPOS

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The relationship between fuzzy sets and H-valued sets is briefly examined and a claim of Eytan's that the former comprise the objects of a topos is corrected.

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0. Introduction

In a paper recently published in this journal [1], M. Eytan makes the claim that a certain category Fuz(H), whose objects are fuzzy sets with values in a complete Heyting algebra H (in the sense of Goguen [4]) is a topos, and goes on to compare it with the topos H-Set of H-valued sets (equivalent to the category of sheaves on H; c.f. [3, 5]). Unfortunately this claim is false: Fuz(H) is a topos only in the special (and to fuzzy set theorists, rather degenerate) case that H is Boolean and is then equivalent to H-Set.

Since Eytan's assertion seems to be gaining some currency among fuzzy set theorists (e.g. it is repeated in the first version of [6]) it seems worthwhile to present this note which is devoted to setting the record straight.

1. Preliminaries

Let us briefly recall the definitions of H-valued sets and functions. (For more details, [3] is a good reference.) Suppose then that H is a complete Heyting algebra. An *H-valued set* $(X, =)$ is given by a set X together with a map $X \times X \to H$, written $x, x' \mapsto [\![x = x']\!]$ and satisfying

$$[\![x = x']\!] \leq [\![x' = x]\!] \qquad \text{(symmetric)}, \tag{1}$$

$$[\![x = x']\!] \wedge [\![x' = x'']\!] \leq [\![x = x'']\!] \qquad \text{(transitive)}, \tag{2}$$

for all x, x', x'' in X. The value $[\![x = x]\!]$ is usually written $[\![x \in X]\!]$ (or Ex). An *H-valued map* between such H-valued sets, $f: (X, =) \to (Y, =)$, is a map

$f: X \times Y \to H$ satisfying

$$f(x, y) \leq [\![x \in X]\!] \wedge [\![y \in Y]\!] \quad \text{(strict)}, \tag{3}$$

$$[\![x = x']\!] \wedge f(x, y) \wedge [\![y = y']\!] \leq f(x', y') \quad \text{(extensional)}, \tag{4}$$

$$f(x, y) \wedge f(x, y') \leq [\![y = y']\!] \quad \text{(single-valued)}, \tag{5}$$

$$[\![x \in X]\!] \leq \bigvee_y f(x, y) \quad \text{(total)}, \tag{6}$$

for all x, x' in X and y, y' in Y.

These H-valued sets and maps comprise a category, denoted H-Set. The identity for $(X, =)$ is $x, x' \mapsto [\![x = x']\!]$, whilst the composite of $f: (X, =) \to (Y, =)$ and $g: (Y, =) \to (Z, =)$ is given by $x, z \mapsto \bigvee_y f(x, y) \wedge g(y, z)$.

It is well known that H-Set is a topos and is equivalent to Sh(H), the topos of sheaves over H (see [3, Theorem 5.9]). Note that a map $f: (X, =) \to (Y, =)$ is a monomorphism in H-Set iff for all x, x' in X and y in Y we have

$$f(x, y) \wedge f(x', y) \leq [\![x = x']\!]. \tag{7}$$

Similarly, f is an epimorphism in H-Set iff

$$[\![y \in Y]\!] \leq \bigvee_x f(x, y) \tag{8}$$

holds for all y in Y. Since H-Set is a topos, it is *balanced*, i.e. f is an isomorphism iff it is both a monomorphism and an epimorphism, i.e. iff it satisfies all of (3) to (8).

2. Fuz(H) and H-Set

We now wish to identify the category Fuz(H) defined in [1] with a full subcategory of H-Set. Given a set I, let $\delta: I \times I \to H$ *be the function*

$$\delta(i, i') = \bigvee \{\top \mid i = i'\} = \begin{cases} \top & \text{if } i = i', \\ \bot & \text{otherwise} \end{cases}$$

(where $\top$, $\bot$ are the top and bottom elements of H respectively). Since δ trivially satisfies (1) and (2), we obtain an H-valued set (I, δ) which is usually written ΔI and called the *constant H-valued set on I*. We shall say that an H-valued set $(X, =)$ is *subconstant* iff there is a monomorphism from it into some ΔI in H-Set.

If now $\varphi: I \to H$ is a fuzzy set, we obtain an H-valued set $\|\varphi\|$, namely $(I, =_\varphi)$ where

$$[\![i =_\varphi i']\!] = \delta(i, i') \wedge \varphi(i)$$

for all i, i' in I. Furthermore $[\![\circ =_\varphi \circ]\!]: I \times I \to H$ satisfies the requirements (3)–(7) of a monomorphism $\|\varphi\| \rightarrowtail \Delta I$ in H-Set, so that $\|\varphi\|$ is subconstant. We note the following:

(i) An arrow $f: \varphi \to \psi$ between fuzzy sets $\varphi: I \to H$ and $\psi: J \to H$ as defined in

3.1 of [1] is exactly the same thing as an H-valued map $f:\|\varphi\| \to \|\psi\|$. Composition and identities agree under this identification.

(ii) If $(X, =)$ is subconstant it is isomorphic in H-Set to $\|\varphi\|$ for some fuzzy set $\varphi: I \to H$. (For if $m:(X, =) \rightarrowtail \Delta I$ is a mono, we may take φ to be $i \mapsto \bigvee_x m(x, i)$ and then m is an isomorphism from $(X, =)$ to $\|\varphi\|$ in H-Set.)

We deduce from (i) and (ii) that Fuz(H) is equivalent to the full subcategory of H-Set whose objects are the subconstant H-valued sets.

3. Main proposition

If $(X, =)$ is an H-valued set, letting $\varphi: X \to H$ be

$$\varphi(x) = [\![x \in X]\!],$$

we find that $[\![\circ = \circ]\!]: X \to H$ gives an epimorphism $\|\varphi\| \twoheadrightarrow (X, =)$ in H-Set. Thus a typical object $(X, =)$ of H-Set may be presented as a quotient of a subconstant object:

$$\begin{array}{l} \|\varphi\| \twoheadrightarrow (X, =) \\ \downarrow \\ \Delta X \end{array} \tag{9}$$

Accordingly we might expect Fuz(H) to lack sufficient quotients (and hence fail to be a topos). In general this is the case; and if we adjoin the missing quotients, up to equivalence we obtain H-Set. This process does not generate any new objects only in the case that H is Boolean:

Proposition. Fuz(H) *is a topos iff H is Boolean and in that case it is equivalent to* H-Set.

Proof[1]. First suppose that H is Boolean. Then H-Set (equivalently Sh(H)) satisfies the (*external*) *Axiom of Choice*, i.e. every epimorphism splits (see [5, Theorem 5.39]). Splitting the epimorphism in (9) we see that every $(X, =)$ is subconstant, so that from Section 2, Fuz(H) is equivalent to H-Set.

Now suppose that Fuz(H) is a topos. Given $d \in H$, regarding it as a fuzzy set $d: 1 \to H$ (where $1 = \{0\}$, a one-point set), we obtain $\|d\|$ in H-Set and a map $m_d: \|d\| \to \Delta 1$ which is monomorphic in H-Set and hence also in Fuz(H) (which we identify with the full subcategory of subconstant H-valued sets). But if d is $\neg\neg$-dense (i.e. $\neg\neg d = \top$) then m_d is also an epimorphism in Fuz(H). To see this, it is sufficient to show that for any set I and any pair of H-valued maps

$$\Delta 1 \underset{g}{\overset{f}{\rightrightarrows}} \Delta I,$$

if $f \circ m_d = g \circ m_d$ then $f = g$. Now $f: 1 \times I \to H$ is an H-valued map $\Delta 1 \to \Delta I$ iff

[1] The second half of the proof was suggested by P.T. Johnstone.

$(f(0,i)\mid i\in I)$ is a partition of H, i.e.

$$f(0,i)\wedge f(0,i')=\bot \quad \text{if } i\neq i'$$

and

$$\bigvee_i f(0,i)=\top.$$

In particular each $f(0,i)$ has a complement (namely $\bigvee_{i'\neq i} f(0,i')$), so that $\neg\neg f(0,i)=f(0,i)$. But if $f\circ m_d=g\circ m_d$, then by definition of composition

$$d\wedge f(0,i)=d\wedge g(0,i)$$

for each i in I. Taking $\neg\neg$ of both sides thus yields $f(0,i)=g(0,i)$ all i, and hence $f=g$.

Therefore if $d\in H$ is $\neg\neg$-dense, m_d is both a monomorphism and an epimorphism in Fuz(H) and hence an isomorphism there (since we are supposing that Fuz(H) is a topos); but then it is an isomorphism in H-Set and so by (8) $d=\top$. Applying this to $d=h\vee\neg h$, for any $h\in H$, we find that H is Boolean, as required. □

4. Conclusions

It is known (c.f. [2]) that the notion of topos is equivalent to a certain kind of (typed, intuitionistic) set theory. The failure in general of Fuz(H) to be a topos should perhaps lead us to question whether fuzzy sets are a sound generalisation of the concepts of set theory. The philosophy behind the notion of fuzzy set is that we cannot always expect to say with precision whether an element belongs to a set. But since classical set theory is founded on two primitive predicates—equality and membership—we cannot expect to get a mathematically or philosopically satisfying theory if we 'fuzzify' one of these and not the other. The notion of H-valued set, in which both predicates are fuzzified, does provide a satisfactory set theory—in as much as H-Set is a topos. Accordingly we suggest (as does Eytan in [1]) that this notion, rather than that of an object of Fuz(H), is the one that should lie at the heart of fuzzy mathematics.

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