An Equivalent Presentation of the Bezem-Coquand-Huber Category of Cubical Sets

Andrew Pitts
University of Cambridge Computer Laboratory

17 September 2013 [updated 21 December 2013]

Abstract

Staton has shown that there is an equivalence between the category of presheaves on (the opposite of) finite sets and partial bijections and the category of nominal restriction sets: see [2, Exercise 9.7]. The aim here is to see that this extends to an equivalence between the category of cubical sets introduced in [1] and a category of nominal sets equipped with a '01-substitution' operation. It seems to me that presenting the topos in question equivalently as 01-substitution sets rather than cubical sets will make it easier (and more elegant) to carry out the constructions and calculations needed to build the intended univalent model of intentional constructive type theory.

1 Nominal sets

I will use notation for nominal sets as in [2]. In particular:

- **Nom** is the category of nominal sets and equivariant functions over a countably infinite set of names $\mathbb{A}$.
- $\mathrm{Perm}\,\mathbb{A}$ is the group of finite permutations of the countably infinite set $\mathbb{A}$.
- $\mathbb{A}$ also denotes the nominal set of names (permutation action: $\pi \cdot a = \pi\, a$).
- $2 \triangleq \{0, 1\}$ is the discrete nominal set with two elements (trivial permutation action: $\pi \cdot i = i$).
- $\mathrm{supp}\, x$ denotes the smallest finite subset of $\mathbb{A}$ that supports an element x of a nominal set.
- $a \# x$ $[a \in \mathbb{A}, x \in X]$ is the freshness relation associated with $X \in \mathbf{Nom}$ (which holds by definition iff $a \notin \mathrm{supp}\, x$).
- $[\mathbb{A}]X$ is the nominal set of name abstractions $\langle a \rangle x$ of elements $x \in X$ of a nominal set X: see [2, chapter 4].
- $\mathrm{P}_{\mathrm{fs}} X$ is the nominal set of finitely supported subsets of a nominal set X; see [2, Defintion 2.26].

2 01-Substitution operations

Let X be a nominal set. A 01-*substitution operation* on X is a morphism

$$s \in \mathbf{Nom}(X \times \mathbb{A} \times 2, X)$$

satisfying the following properties, where we write $x(a := i)$ for $s(x, a, i)$:

$$a \mathbin{\#} x(a := i) \tag{1}$$

$$a \mathbin{\#} x \;\Rightarrow\; x(a := i) = x \tag{2}$$

$$a \mathbin{\#} a' \;\Rightarrow\; x(a := i)(a' := i') = x(a' := i')(a := i) \tag{3}$$

Note that since s is a morphism is **Nom**, we also have

$$\pi \cdot (x(a := i)) = (\pi \cdot x)(\pi\, a := i) \tag{4}$$

for all $\pi \in \mathrm{Perm}\,\mathbb{A}$.

Remark 2.1. Property (1) tells us that s corresponds to a pair of morphisms in $\mathbf{Nom}([\mathbb{A}]X, X)$

$$\langle a \rangle x \mapsto x(a := 0) \quad \text{and} \quad \langle a \rangle x \mapsto x(a := 1)$$

and the other two properties imply that these are in fact name restriction operations in the sense of [2, section 9.1].

Definition 2.2 (the category of 01-substitution sets). The category **01Sub** has objects that are nominal sets equipped with a 01-substitution operation and morphisms $f \in \mathbf{01Sub}(X, Y)$ that are equivariant functions $f \in \mathbf{Nom}(X, Y)$ preserving the 01-substitution operation:

$$f(x(a := i)) = (f\,x)(a := i)\,. \tag{5}$$

Composition and identities are as for ordinary functions. (Note that (5) makes sense from the point of view of a property of substitution, only because f is equivariant, which is to say that it has empty support as a member of the exponential object $X \to_{\mathrm{fs}} Y$ in **Nom**, that is, $(\forall a \in \mathbb{A})\ a \mathbin{\#} f \in X \to_{\mathrm{fs}} Y$.)

3 Cubical sets

Let $\mathbf{C}$ be the small category whose objects A are finite subsets of $\mathbb{A}$ and whose morphisms $f \in \mathbf{C}(A, B)$ are functions $f \in \mathbf{Set}(A, B + 2)$ satisfying

$$(\forall a, a' \in f^{-1}B)\ f\,a = f\,a' \;\Rightarrow a = a'\,. \tag{6}$$

The identity morphism $\mathrm{id}_A \in \mathbf{C}(A, A)$ is the inclusion function $A \hookrightarrow A + 2$:

$$(\forall a \in A)\ \mathrm{id}_A\, a = a \tag{7}$$

and the composition of $f \in \mathbf{C}(A, B)$ with $g \in \mathbf{C}(B, C)$ is $g \circ f \in \mathbf{Set}(A, C + 2)$ given by:

$$(\forall a \in A)\ (g \circ f)\, a = \begin{cases} g(f\,a) & \text{if } a \in f^{-1}B \\ f\,a & \text{if } a \in A - f^{-1}B. \end{cases} \tag{8}$$

The *category of cubical sets* is the category $[\mathbf{C}, \mathbf{Set}]$ of presheaves on $\mathbf{C}^{\mathrm{op}}$.

4 01Sub and $[\mathbf{C}, \mathbf{Set}]$ are equivalent categories

Let $\mathbf{I}$ be the subcategory of $\mathbf{C}$ with the same objects, but whose morphisms are those $f \in \mathbf{C}(A, B)$ satisfying $f^{-1}B = A$; in other words, $\mathbf{I}(A, B)$ consists of all injective functions from A to B. The category $\mathbf{I}$ has all pullbacks, created by the inclusion of $\mathbf{I}$ into $\mathbf{Set}$. The full subcategory of $[\mathbf{I}, \mathit{Set}]$ consisting of pullback-preserving functors is one presentation of the *Schanuel topos* and is in particular equivalent to **Nom**. Section 6.3 of [2] contains a detailed account of this equivalence, which I will make use of here.

When we restrict a functor $F \in [\mathbf{C}, \mathbf{Set}]$ along the inclusion $i : \mathbf{I} \to \mathbf{C}$ we get a pullback-preserving functor $i^*F = F \circ i : \mathbf{I} \to \mathbf{Set}$ because of the following elementary piece of category theory (this was observed by Staton and Levy for the category of finite sets and partial bijections, but works just the same for $\mathbf{C}$):

Lemma 4.1. *In any category, suppose*

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd} \tag{9}$$

is a commuting square of monomorphisms for which p and g have left inverses p' and g' (so that $p' \circ p = \mathrm{id}_D$ and $g' \circ g = \mathrm{id}_B$) making

$$\begin{tikzcd} D \arrow[d, "q"'] & A \arrow[l, "p'"'] \arrow[d, "f"] \\ B & C \arrow[l, "g'"] \end{tikzcd} \tag{10}$$

commute. Then (9) *is a pullback square.*

Proof. Exercise! (Hint: use the fact that f is a monomorphism.) $\square$

To apply this lemma to $\mathbf{C}$, note that if the morphisms in (9) are all in $\mathbf{I}$, then they have left inverses in $\mathbf{C}$: given $f \in \mathbf{I}(A, C)$, we can take $f' \in \mathbf{C}(C, A) = \mathbf{Set}(C, A + 2)$ to be

$$f'\,c \triangleq \begin{cases} a & \text{if } f\,a = c \text{ for some (unique) } a \in A \\ 0 & \text{if } a \in C - fA. \end{cases}$$

Corollary 4.2 (Staton, Levy)**.** *Composing any functor $\mathbf{C} \to \mathbf{Set}$ with the inclusion $i : \mathbf{I} \to \mathbf{C}$ yields a pullback-preserving functor.*

Proof. It is not hard to see that if (9) is a pullback square in $\mathbf{I}$, then (10) commutes in $\mathbf{C}$. So applying any functor $\mathbf{C} \to \mathbf{Set}$ to (9) preserves the monomorphisms (because they all have left inverses) and gives a square in $\mathbf{Set}$ satisfying the hypotheses of the lemma – hence which is a pullback. $\square$

So we have the following picture:

$$\begin{tikzcd} I^* : [\mathbf{C}, \mathbf{Set}] \arrow[r, dashed] \arrow[dr, "i^*"'] & \mathbf{Sch} \arrow[d, hook] & \simeq & \mathbf{Nom} \\ & {[\mathbf{I}, \mathbf{Set}]} \end{tikzcd} \tag{11}$$

where **Sch** is the full subcategory of pullback-preserving functors and the equivalence $\mathbf{Sch} \simeq \mathbf{Nom}$ is described in [2, section 6.3]. From that description of the equivalence we get the following explicit construction for the functor $I^* : [\mathbf{C}, \mathbf{Set}] \to \mathbf{Nom}$:

Definition 4.3 (the functor $I^* : [\mathbf{C}, \mathbf{Set}] \to \mathbf{Nom}$). Given $F \in [\mathbf{C}, \mathbf{Set}]$, the nominal set I^*F consists of equivalence classes $[A, x]$ of pairs $(A \in \mathbf{C}, x \in F\,A)$ for the equivalence relation relating (A, x) and (A', x') when there is some $B \supseteq A \cup A'$ with $F(A \hookrightarrow B)\,x = F(A' \hookrightarrow B)\,x'$. The permutation action on equivalence classes is given by $\pi \cdot [A, x] = [\pi A, F(\pi|_A)\,x]$, where $\pi|_A \in \mathbf{I}(A, \pi A) \subseteq \mathbf{C}(A, \pi A)$ is the injective function that π gives from the set A to the set $\pi A = \{\pi\, a \mid a \in A\}$. It is not hard to see that A supports $[A, x]$ with respect to this action, so that I^*F is a nominal set.

Given $\varphi : F \to F'$ in $[\mathbf{C}, \mathbf{Set}]$, $I^*\varphi \in \mathbf{Nom}(I^*F, I^*F')$ is the function $I^*\varphi : [A, x] \mapsto [A, \varphi_A\, x]$, which is (well-defined and) equivariant because φ_A is natural in A.

Remark 4.4. Since from Corollary 4.2 we know that each $F \in [\mathbf{C}, \mathbf{Set}]$ preserves the pullback

$$\begin{tikzcd} A \cap A' \arrow[r, hook] \arrow[d, hook] & A' \arrow[d, hook] \\ A \arrow[r, hook] & A \cup A' \end{tikzcd}$$

the equivalence relation defining I^*F relates (A, x) and (A', x) iff there is some $y \in F(A \cap A')$ with $F(A \cap A' \hookrightarrow A)\, y = x$ and $F(A \cap A' \hookrightarrow A')\, y = x'$.

We will show that $I^* : [\mathbf{C}, \mathbf{Set}] \to \mathbf{Nom}$ factors through the forgetful functor $\mathbf{01Sub} \to \mathbf{Nom}$ to give an equivalence of categories.

Definition 4.5 (the 01-substitution operation on I^*F). Given $F \in [\mathbf{C}, \mathbf{Set}]$ and $[A, x] \in I^*F$, for each $a \in \mathbb{A}$ and $i \in 2$ we define

$$[A, x](a := i) \triangleq [A - \{a\}, F(f_{A,a,i})\, x] \tag{12}$$

where $f_{A,a,i} \in \mathbf{C}(A, A - \{a\})$ is the morphism mapping a to i if $a \in A$ and otherwise acting like the identity. It is easy to see that this definition is independent of the choice of representative (A, x). It is equivariant (4) and satisfies property (3) because the diagrams

$$\begin{tikzcd} A \arrow[r, "f_{A,a,i}"] \arrow[d, "\pi|_A"'] & A - \{a\} \arrow[d, "\pi|_{A-\{a\}}"] \\ \pi A \arrow[r, "f_{\pi A, \pi a, i}"'] & \pi A - \{\pi\, a\} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "f_{A,a,i}"] \arrow[d, "f_{A,a',i'}"'] & A - \{a\} \arrow[d, "f_{A-\{a\},a',i'}"] \\ A - \{a'\} \arrow[r, "f_{A-\{a'\},a,i}"'] & A - \{a, a'\} \end{tikzcd} \quad (a \mathrel{\#} a')$$

commute in $\mathbf{C}$. Since $\operatorname{supp}[A,x] \subseteq A$, definition (12) also satisfies property (1). Finally, it remains to see that it also satisfies property (2). Note that

$$a \notin A \;\Rightarrow\; [A,x](a := i) = [A,x] \tag{13}$$

because when $a \notin A$, then $A - \{a\} = A$ and $f_{A,a,i} = \mathrm{id}_A$. So if $a \mathrel{\#} [A,x]$, then picking any $a' \notin A \cup \{a\}$, we have $a' \mathrel{\#} [A,x]$ and hence $[A,x] = (a\ a') \cdot [A,x] = [(a\ a')A, F((a\ a')|_A)\,x]$. Now $a \notin (a\ a')A$, so by (13) $[(a\ a')A, F((a\ a')|_A)\,x](a := i) = [(a\ a')A, F((a\ a')|_A)\,x]$; hence $[A,x](a := i) = [A,x]$, as required for (2).

Given $\varphi : F \to F'$ in $[\mathbf{C},\mathbf{Set}]$ and $[A,x] \in I^*F$, using naturality of φ and Definition 4.3 we can calculate that

$$\begin{aligned} I^*\varphi([A,x](a := i)) &= [A - \{a\}, \varphi_{A-\{a\}}(F(f_{A,a,i})\,x)] \\ &= [A - \{a\}, F(f_{A,a,i})(\varphi_A\,x)] \\ &= (I^*\varphi[A,x])(a := i)\,. \end{aligned}$$

So each $I^*\varphi$ is a morphism in **01Sub** and I^* lifts to give a functor $I^* : [\mathbf{C},\mathbf{Set}] \to \mathbf{01Sub}$.

Lemma 4.6. $I^* : [\mathbf{C},\mathbf{Set}] \to \mathbf{01Sub}$ *is a faithful functor.*

Proof. Since $i : \mathbf{I} \to \mathbf{C}$ is the identity on objects, $i^* : [\mathbf{C},\mathbf{Set}] \to [\mathbf{I},\mathbf{Set}]$ is a faithful functor and hence so is $I^* : [\mathbf{C},\mathbf{Set}] \to \mathbf{Nom}$ (*cf.* diagram (11)). Therefore I^* is faithful as a functor from $[\mathbf{C},\mathbf{Set}]$ t **01Sub**. □

Lemma 4.7. $I^* : [\mathbf{C},\mathbf{Set}] \to \mathbf{01Sub}$ *is a full functor.*

Proof. First note that in view of Remark 4.4 we have for any $F \in [\mathbf{C},\mathbf{Set}]$ that

$$[A,x] = [A,x'] \in I^*F \;\Rightarrow\; x = x' \in F\,A\,. \tag{14}$$

Furthermore

$$(\forall d \in I^*F)\ \operatorname{supp} d \subseteq A \;\Rightarrow\; (\exists x \in F\,A)\ d = [A,x]\,. \tag{15}$$

For if $\operatorname{supp}[B,y] \subseteq A$, letting $b_1,\ldots,b_n$ be the distinct elements of $B - A$, then $b_i \mathrel{\#} [B,y]$, so by property (2) for the 01-substitution set I^*F and (12) we have

$$[B,y] = [B,y](b_1 := 0)\cdots(b_n := 0) = [B - \{b_1,\ldots,b_n\}, y'] = [A, F(A \cap B \hookrightarrow A)\,y']$$

for some $y' \in F(B - \{b_1,\ldots,b_n\}) = F(A \cap B)$.

So now suppose $F, F' \in [\mathbf{C},\mathbf{Set}]$ and $g \in \mathbf{01Sub}(I^*F, I^*F')$. For each $A \in \mathbf{C}$ and $x \in F\,A$, since g is equivariant we have $\operatorname{supp}(g[A,x]) \subseteq \operatorname{supp}[A,x] \subseteq A$. So by (15), there is some $\varphi_A\,x \in F'A$ with $g[A,x] = [A, \varphi_A\,x]$; and by (14), $\varphi_A\,x$ is uniquely determined from A and x by this property. So we get functions $\varphi_A : F\,A \to F'A$ for each $A \in \mathbf{C}$. If we can prove they are natural in A, then $\varphi \in [\mathbf{C},\mathbf{Set}](F,F')$; and $I^*\varphi = g$ by construction, as required for fullness.

To prove naturality we have to express the F and F' action of an arbitrary morphism $f \in \mathbf{C}(A,B)$ in terms of permutation action and 01-substitution. Note that because of (6), f restricts to a bijection between $f^{-1}B$ and $f(f^{-1}B)$. Pick a finite permutation $\pi \in \operatorname{Perm}\mathbb{A}$ that agrees with f on $f^{-1}B$ and which is the identity outside the finite set $f^{-1}B \cup f(f^{-1}B)$.

(We can always find such a π – see the Homogeneity Lemma 1.14 in [2].) Let $a_1, \ldots, a_n$ list the distinct elements of $A - f^{-1}B$. Then for any $x \in F\,A$

$$[B, (F\,f)\,x] = \pi \cdot ([A, x](a_1 := f\,a_0) \cdots (a_n := f\,a_n))$$

and similarly for $x' \in F'A$. Therefore since g is a morphism of 01-substitution sets we get:

$$\begin{aligned}
[B, \varphi_B((F\,f)\,x)] &= g[B, (F\,f)\,x] \\
&= g(\pi \cdot ([A, x](a_1 := f\,a_1) \cdots (a_n := f\,a_n))) \\
&= (\pi \cdot (g[A, x]))(a_1 := f\,a_1) \cdots (a_n := f\,a_n))) \\
&= (\pi \cdot [A, \varphi_A\,x])(a_1 := f\,a_1) \cdots (a_n := f\,a_n))) \\
&= [B, (F'f)(\varphi_A\,x)]
\end{aligned}$$

and hence by (14) we do indeed have $\varphi_B \circ (F\,f) = (F'f) \circ \varphi_A$. $\square$

Theorem 4.8. $I^* : [\mathbf{C}, \mathbf{Set}] \to \mathbf{01Sub}$ *is an equivalence of categories.*

Proof. In view of Lemmas 4.6 and 4.7, it suffices to check that I^* is essentially surjective, that is, for each $X \in \mathbf{01Sub}$ there is some $I_*X \in [\mathbf{C}, \mathbf{Set}]$ and an isomorphism $\varepsilon_X : I^*(I_*X) \cong X$ in $\mathbf{01Sub}$.

Given $X \in \mathbf{01Sub}$, for each $A \in \mathbf{C}$ define

$$I_*\,X\,A \triangleq \{x \in X \mid \operatorname{supp} x \subseteq A\} \in \mathbf{Set}\,.$$

Then for each $f \in \mathbf{C}(A, B)$, we wish to construct a function $I_*\,X\,f \in \mathbf{Set}(I_*\,X\,A, I_*\,X\,B)$. Given f, picking π and $a_1, \ldots, a_n$ as in the proof of Lemma 4.7, for each $x \in I_*\,X\,A$ we define

$$I_*\,X\,f\,x \triangleq \pi \cdot (x(a_1 := f\,a_1) \cdots (a_n := f\,a_n))\,. \tag{16}$$

(In the case $n = 0$, we take $x(a_1 := f\,a_0) \cdots (a_n := f\,a_n)$ to just mean x.) Note that since $\operatorname{supp} x \subseteq A$ and using (1), we have

$$\operatorname{supp}(x(a_1 := f\,a_1) \cdots (a_n := f\,a_n)) \subseteq A - \{a_1, \ldots, a_n\} = f^{-1}B \tag{17}$$

and that by choice of π, $\pi(f^{-1}B) = f(f^{-1}B) \subseteq B$. So the support of the element on the right-hand side of (16) is contained in B and hence it is an element of $I_*\,X\,B$. In view of (17), the right-hand side of (16) is independent of the choice of π; and by (3) it independent of the order in which the elements of $A - f^{-1}B$ are listed. So (16) gives a well-defined function $I_*\,X\,f \in \mathbf{Set}(I_*\,X\,A, I_*\,X\,B)$. One can check that $f \mapsto I_*\,X\,f$ preserves identities and composition `[the proof for composition seems very tedious - I have not checked it properly]` and so we get $I_*\,X \in [\mathbf{C}, \mathbf{Set}]$.

Note that when f is an inclusion $A \hookrightarrow B$, then in (16) we can take $\pi = \mathrm{id}$ and $n = 0$, so that

$$\operatorname{supp} x \subseteq A \;\Rightarrow\; I_*\,X(A \hookrightarrow B)\,x = x\,. \tag{18}$$

If (A, x) and (A', x') both represent the same element of $I^*(I_*X)$, then for some $x'' \in I_*\,X\,(A \cap A')$ we have

$$x = I_*\,X\,(A \cap A' \hookrightarrow A)\,x'' \quad \text{and} \quad x' = I_*\,X\,(A \cap A' \hookrightarrow A')\,x''$$

so that by (18), $x = x'' = x'$. Therefore we get a well-defined function $\varepsilon_X : I^*(I_* X) \to X$ satisfying

$$(\forall A \in \mathbf{C}, x \in I_* X A)\ \varepsilon_X[A, x] = x\,. \tag{19}$$

It follows immediately that ε_X is a bijection. So it just remains to check that is it also a morphism in **01Sub**.

To see that it is equivariant, note that in (16) when $f = \pi|_A$ we have $n = 0$ and

$$I_* X\,(\pi|_A)\, x = \pi \cdot x \tag{20}$$

so that

$$\begin{aligned}
\pi \cdot (\varepsilon_X[A, x]) &= \pi \cdot x \\
&= I_* X\,(\pi|_A)\, x \\
&= \varepsilon_X[\pi A, I_* X\,(\pi|_A)\, x] \\
&= \varepsilon_X(\pi \cdot [A, x])\,.
\end{aligned}$$

Finally, to see that ε_X also preserves the 01-substitution operation, note that in (16) when $f = f_{A,a,i}$ (Definition 4.5), then we can take $\pi = \mathrm{id}$, $n = 1$ and $a_1 = a$ and get

$$I_* X\,(f_{A,a,i})\, x = x(a := i) \tag{21}$$

and hence

$$\begin{aligned}
\varepsilon_X([A, x](a := i)) &= \varepsilon_X[A - \{a\}, I_* X\,(f_{A,a,i})\, x] \\
&= \varepsilon_X[A - \{a\}, x(a := i)] \\
&= x(a := i) \\
&= (\varepsilon_X[A, x])(a := i)\,.
\end{aligned}$$

□

Remark 4.9. An immediate corollary of the theorem is that **01Sub** is a Grothendieck topos. In fact Staton [3, section 6.4] has shown that for a quite general notion of 'substitution action', categories of nominal sets equipped with such actions are all Grothendieck toposes.

5 The uniform-Kan condition

Definition 5.1 (open boxes). Given a non-empty finite subset $A \subseteq_{\mathrm{fin}} \mathbb{A}$ with a distinguished element $a \in A$, an 1-*open* (A, a)-*box* in a 01-substitution set $X \in \mathbf{01Sub}$ is a function

$$u : (A \times 2) - \{(a, 1)\} \to X$$

satisfying for all $(b, i), (b', i') \in (A \times 2) - \{(a, 1)\}$

$$b \mathrel{\#} u(b, i) \tag{22}$$

$$u(b, i)(b' := i') = u(b', i')(b := i)\,. \tag{23}$$

Note that any $x \in X$ gives rise to a 1-open (A, a)-box u_x with $u_x(b, i) = x(b := i)$ for all $(b, i) \in (A \times 2) - \{(a, 1)\}$. We call x a *filling* for the 1-open (A, a)-box u if $u = u_x$. Reversing the role of 0 and 1 in these definitions, we get the notion of 0-*open* (A, a)-*boxes* and their fillings.

If $c \# A$ and $j \in 2$, then we get another 1-open (A, a)-box $u(c := j)$ mapping each $(b, i) \in (A \times 2) - \{(a, 1)\}$ to

$$(u(c := j))\,(b, i) = u(b, i)(c := j)\,. \tag{24}$$

Note also that, using the usual permutation action on functions

$$\pi \cdot u = \lambda x \to \pi \cdot (u\,(\pi^{-1} \cdot x)) \tag{25}$$

if u is a 1-open (A, a)-box, then $\pi \cdot u$ is a 1-open $(\pi\, A, \pi\, a)$-box.

Definition 5.2 (**uniform-Kan objects in 01Sub**)**.** A 01-substitution set $X \in$ **01Sub** is *uniform-Kan* if it comes equipped with operations mapping 1-open (respectively 0-open) (A, a)-boxes u in X for any (A, a), to fillings $\uparrow u$ (respectively $\downarrow u$) in X. These operations are required to be equivariant

$$\begin{aligned} \pi \cdot \uparrow u &= \uparrow(\pi \cdot u) \quad \text{if } u \text{ is 1-open} \\ \pi \cdot \downarrow u &= \downarrow(\pi \cdot u) \quad \text{if } u \text{ is 0-open} \end{aligned} \tag{26}$$

and to commute with substitution in the sense that if u is an open (A, a)-box, $c \# A$ and $j \in 2$, then

$$\begin{aligned} (\uparrow u)(c := j) &= \uparrow(u(c := j)) \quad \text{if } u \text{ is 1-open} \\ (\downarrow u)(c := j) &= \downarrow(u(c := j)) \quad \text{if } u \text{ is 0-open.} \end{aligned} \tag{27}$$

Note 5.3. If X is uniform-Kan, the filling operation $\uparrow$ gives rise to an operation $u \mapsto {}^{+}u$ sending a 1-open (A, a)-box u to the 1-face of its filling $\uparrow u$ that is orthogonal to the distinguished dimension a:

$${}^{+}u = (\uparrow u)(a := 1)\,. \tag{28}$$

Similarly we get an operation $u \mapsto {}^{-}u$ sending a 0-open (A, a)-box u to the 0-face of its filling:

$${}^{-}u = (\downarrow u)(a := 0)\,. \tag{29}$$

Note 5.4. The above definition of uniform-Kan can be reformulated in a less 'nominal' fashion by making use of name abstraction [2, chapter 4], as follows (but it is not clear that is any more useful when formulated that way):

Given $X \in$ **01Sub**, the 01-substitution operation on X lifts to the nominal set $[\mathbb{A}]X$ of name abstractions, where it satisfies

$$a \# a' \;\Rightarrow\; (\langle a \rangle x)(a' := i) = \langle a \rangle (x(a' := i)) \tag{30}$$

and hence gives a 01-substitution operation for $[\mathbb{A}]X$. Denote the resulting object of **01Sub** by $\square X$. Iterating this construction, define

$$\begin{cases} \square_0 X &= X \\ \square_{n+1} X &= \square(\square_n X)\,. \end{cases} \tag{31}$$

Thus $\square_n X$ is the nominal set of n-ary name abstractions $\langle a_1, \ldots, a_n \rangle x$ (with $a_1, \ldots, a_n$ mutually distinct), with 01-substitution operation satisfying the evident generalization of (30) to n-ary name abstractions.

We can think of the elements of $\square X$ as *intervals* in X: given $\langle a\rangle x \in \square X$ its endpoints in X are $\delta^1_0\langle a\rangle x = x(a := 0)$ and $\delta^1_1\langle a\rangle x = x(a := 0)$. Note that the functions δ^1_0 and δ^1_1 are morphisms in **01Sub** from $\square X$ to X. In general at higher dimensions, think of the elements of $\square_n X$ as *n-cubes* in X. There are face morphisms

$$\begin{aligned} \delta^m_i &\in \mathbf{01Sub}(\square_n X, \square_{n-1} X) \qquad (1 \le m \le n, i = 0, 1) \\ \delta^m_i \langle a_1, \dots, a_n\rangle x &= \langle a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_n\rangle x(a_m := i) \end{aligned} \tag{32}$$

and degeneracy morphisms

$$\begin{aligned} \iota^m &\in \mathbf{01Sub}(\square_n X, \square_{n+1} X) \qquad (0 \le m \le n) \\ \iota^m \langle a_1, \dots, a_n\rangle x &= \langle a_1, \dots, a_{m-1}, a, a_{m+1}, \dots, a_n\rangle x \\ &\qquad \text{for some/any } a \,\#\, (a_1, \dots, a_n, x). \end{aligned} \tag{33}$$

We get a 01-substitution set $\sqcup_n X$ of 1-open boxes in X of dimension $n+1$: it elements are $(n+1)$-ary name abstractions $\langle a\vec{a}\rangle u$ where $a\vec{a}$ are $n+1$ distinct names and u is a 1-open $(\{a\vec{a}\}, a)$-box. Recalling from Definition 5.1 that each $x \in X$ gives a 1-open $(\{a\vec{a}\}, a)$-box u_x, we get a morphism in **01Sub**:

$$\begin{aligned} p_n &\in \mathbf{01Sub}(\square_{n+1} X, \sqcup_n X) \\ p_n(\langle a\vec{a}\rangle x) &= \langle a\vec{a}\rangle u_x \end{aligned} \tag{34}$$

Symmetrically, there is a 01-substitution set $\sqcap_n X$ of open 0-boxes in X of dimension $n+1$, together with a morphism $q_n \in \mathbf{01Sub}(\square_{n+1} X, \sqcap_n X)$.

Then X is uniform-Kan iff each p_n and each q_n is split, that is, there are morphisms

$$i_n \in \mathbf{01Sub}(\sqcup_n X, \square_{n+1} X) \quad \text{and} \quad j_n \in \mathbf{01Sub}(\sqcap_n X, \square_{n+1} X)$$

with $p \circ i = \mathrm{id}_{\sqcup_n X}$ and $q \circ j = \mathrm{id}_{\sqcap_n X}$.

Definitions 5.1 and 5.2 generalize to indexed families in **01Sub** as follows:

Definition 5.5 (uniform-Kan fibrations). Given $p : X \to Y$ in **01Sub**, a 1-open (A, a)-box in X *lies over* $y \in Y$ if $p(u(b, i)) = y(b := i)$ for all $(b, i) \in (A \times 2) - \{(a, 1)\}$. Such a u has a *filling over* y if it has a filling $x \in X$ with $p\, x = y$. Note that if u is a 1-open (A, a)-box over y, then $\pi \cdot u$ is a 1-open $(\pi\, A, \pi\, a)$-box over $\pi \cdot y$; and if $c \,\#\, (A, y)$ and $j \in 2$, then $u(c := j)$, defined as in (24), is also 1-open (A, a)-box over y (since $y(c := j) = y$).

Then $p : X \to Y$ is a *uniform-Kan fibration* if for each $y \in Y$ there are operations mapping any 1-open (respectively, 0-open) box u over y to an element $\uparrow u$ (respectively $\downarrow u$) in X that is filling over y for u; furthermore, the operations $\uparrow$ and $\downarrow$ are required to be equivariant (26) and commute with substitutions (27) for names c satisfying not only $c \,\#\, A$, but also $c \,\#\, y$.

References

[1] M. Bezem, T. Coquand, and S. Huber. A model of type theory in cubical sets. Preprint., September 2013.

[2] A. M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*, volume 57 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2013.

[3] S. Staton. *Name-Passing Process Calculi: Operational Models and Structural Operational Semantics*. PhD thesis, University of Cambridge, 2007. Available as University of Cambridge Computer Laboratory Technical Report Number UCAM-CL-TR-688.