Grid approximation of a finite set of points

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We consider the problem of approximating a finite set of points \( X \subset \mathbb{R}^n \) by a rectangular grid \( G(\alpha, E) = \{ \sum_{i=1}^{n} \alpha z_i e_i | z \in \mathbb{Z} \} \), where \( E = \{ e_i \}_{i=1}^{n} \) is an orthonormal basis of \( \mathbb{R}^n \) and \( \alpha \in \mathbb{R} \). Each point of \( X \) is approximated by the nearest node of \( G(\alpha, E) \). Let the distance between \( x_k \in X \) and its nearest node be \( d_k(G) \), \( k = 1, \ldots, |X| \), and define the “distance” from \( X \) to \( G \) to be \( d(X, G) = \max_{i=1,\ldots,m} d_i(G) \). For any given \( \epsilon > 0 \) and basis \( E \) let \( A_\epsilon(E) \) be the set of all \( \alpha \) such that \( d(X, G(\alpha, E)) < \epsilon \). The optimal \( \epsilon \)-grid \( G_\epsilon(\alpha^*, E^*) \) is such a grid that \( \alpha^* = \max_{E} \max_{\alpha} A_\epsilon(E) \) and \( E^* = \arg \max_{E} \max_{\alpha} A_\epsilon(E) \). In this problem \( E \) is the orientation of the grid and \( \alpha \) is its spacing. The motivation behind the considered problem is in finding grids which are not too dense.

We use the optimisation results of Brucker and Meyer [1] to provide a numerical recipe of finding a good candidate for the optimal \( \epsilon \)-grid \( G_\epsilon(\alpha^*, E^*) \). Furthermore, we discuss the problem of approximation of a finite set of points in \( \mathbb{R}^n \) by truly rectangular grid \( G(\alpha, E) = \{ \sum_{i=1}^{n} \alpha_i z_i e_i | z \in \mathbb{Z} \} \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \). The complexity of the suggested procedures is also discussed.

Besides natural applications in interpolation and solution methods for partial differential equations, when calculations can be done on a grid and the initial values have to be approximated by points of a grid, the discussed algorithms can be used in the analysis of spatial patterns and in optimal design problems for spatial interaction models.

References: