# Elementary Equivalence in Finite Structures

Anuj Dawar

University of Cambridge Computer Laboratory

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# When I First Met Yuri

When I was a graduate student, I sent Yuri a draft of the paper that would become:

> **Dawar, Lindell and Weinstein**.
> *Infinitary Logic and Inductive Definability over Finite Structures. Inf. Comput. (1995)*

and received generously extensive feedback.

# Abiteboul-Vianu Theorem

One of the main contributions of the paper was an alternative proof of the theorem of **Abiteboul-Vianu**:

### Theorem

FP = PFP *if, and only if,* $\mathrm{PTime} = \mathrm{PSpace}$.

Here:

- FP is *least fixed point logic*; and
- PFP is *partial fixed point logic*.

The proof was based on an *analysis* and *definability* of the equivalence relations $\equiv^k_L$.

# Finite Variable Equivalences

Write $L^k$ for the fragment of first-order logic using only variables $x_1, \ldots, x_k$.

For structures $\mathbb{A}$ and $\mathbb{B}$ write $\mathbb{A} \equiv_L^k \mathbb{B}$ to denote that they are not distinguished by *any* sentence of $L^k$.

By abuse of notation, for tuples $\mathbf{a}, \mathbf{a}' \in \mathbb{A}^k$ we write $\mathbf{a} \equiv_L^k \mathbf{a}'$ to denote that for every formula $\varphi$ of $L^k$,

$$\mathbb{A} \models \varphi[\mathbf{a}] \quad \text{if, and only if,} \quad \mathbb{A} \models \varphi[\mathbf{a}'].$$

# Fixed Point Logics

A class of structures $K$ is definable in FP *iff* there is some $k$ so that $K$ is closed under $\equiv^k$ *and*

> $\mathbb{A} \in K$ *is decided by an algorithm that runs in polynomial time on a quotient structure* $\mathbb{A}^k / \equiv_L^k$.

A class of structures $K$ is definable in PFP *iff* there is some $k$ so that $K$ is closed under $\equiv^k$ *and*

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# Oberwolfach 1994

In 1994, Yuri (together with **Heinz-Dieter Ebbinghaus** and **Jörg Flum**) was an organiser of a workshop on *Finite Model Theory* at *Oberwolfach*.

A *take-home message* from the workshop:

- *Classical model theory* is the study of the equivalence relation $\equiv$ of *elementary equivalence*.
  It tells us the limits of definability: *i.e.* properties that are not invariant are not definable.
- Can $\equiv_L^k$ play a similar role for finite structures?

Interesting work on $\equiv_L^k$ followed, but a more interesting notion of elementary equivalence emerged.

# Doing it with Counting

$C^k$ is the logic obtained from *first-order logic* by allowing:

- *counting quantifiers*: $\exists^i x\, \varphi$; and
- only the variables $x_1, \ldots . x_k$.

Every formula of $C^k$ is equivalent to a formula of first-order logic, albeit one with more variables.

We write $\mathbb{A} \equiv_C^k \mathbb{B}$ to denote that no sentence of $C^k$ distinguishes $\mathbb{A}$ from $\mathbb{B}$.

And similarly, for $\mathbf{a}, \mathbf{a}' \in \mathbb{A}^k$ we have $\mathbf{a} \equiv_C^k \mathbf{a}'$

This *family of equivalence relations* has many different natural formulations in *combinatorics, algebra,* and *logic*.

# Tractable Approximations of Isomorphism

If $\mathbb{A}, \mathbb{B}$ are $n$-element structures and $k < n$, we have:

$$\mathbb{A} \cong \mathbb{B} \quad \Leftrightarrow \quad \mathbb{A} \equiv_C^n \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_C^{k+1} \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_C^k \mathbb{B}.$$

$\mathbb{A} \equiv_C^k \mathbb{B}$ is decidable in time $n^{O(k)}$.

The equivalence relations $\equiv_C^k$ form a *family* of tractable approximations of isomorphism.

There is no fixed $k$ for which $\equiv_C^k$ coincides with isomorphism.
**(Cai, Fürer, Immerman 1992)**.

# Fixed-Point Logics with Counting

Analysis of $\equiv_C^k$ yields results analogous to the *Abiteboul-Vianu theorem*:

**Theorem**
FPC $=$ PFPC *if, and only if,* $\mathrm{PTime} = \mathrm{PSpace}$.

**Grädel-Otto**

**Grohe** has shown that FPC *captures* $\mathrm{PTime}$ on any *proper minor-closed* class of graphs.

In particular, for each such class $K$, there is a $k$ such that $\equiv_C^k$ is the same as *isomorphism* on $K$.

# Bijection Games

$\equiv^{C^k}$ is characterised by a $k$-pebble *bijection game*. **(Hella 96)**.

The game is played on structures $\mathbb{A}$ and $\mathbb{B}$ with pebbles $a_1, \ldots, a_k$ on $\mathbb{A}$ and $b_1, \ldots, b_k$ on $\mathbb{B}$.

- *Spoiler* chooses a pair of pebbles $a_i$ and $b_i$;
- *Duplicator* chooses a bijection $h : A \to B$ such that for pebbles $a_j$ and $b_j (j \neq i)$, $h(a_j) = b_j$;
- *Spoiler* chooses $a \in A$ and places $a_i$ on $a$ and $b_i$ on $h(a)$.

*Duplicator* loses if the partial map $a_i \mapsto b_i$ is not a partial isomorphism. *Duplicator* has a strategy to play forever if, and only if, $\mathbb{A} \equiv_C^k \mathbb{B}$.

# Weisfeiler-Lehman Test

The *$k$-dimensional Weisfeiler-Lehman* test for isomorphism (as described by **Babai**), gives a way of testing for $\equiv^k_C$.

We obtain, by successive refinements, an equivalence relation $\equiv^k$ on $k$-tuples of elements in a structure $\mathbb{A}$:

$$\equiv^k_0 \;\supseteq\; \equiv^k_1 \;\supseteq \cdots \supseteq\; \equiv^k_i \quad \cdots$$

$\mathbf{u} \equiv^k_0 \mathbf{v}$ if the two tuples induce isomorphic $k$-element structures.

The refinement is defined by an *easily checked* condition on tuples.
The refinement is guaranteed to terminate within $n^k$ iterations.

# Induced Partitions

Given an equivalence relation $\equiv_i^k$, each $k$-tuple $\mathbf{a}$ induces a *labelled partition* of the elements $A$, where each element $a$ is labelled by the $k$-tuple

$$\alpha_1, \ldots, \alpha_k$$

of $\equiv_i^k$-equivalence classes obtained by substituting $a$ in each of the $k$ positions in $\mathbf{a}$.

Define $\equiv_{i+1}^k$ to be the equivalence relation where $\mathbf{a} \equiv_{i+1}^k \mathbf{b}$ if, in the partitions they induce, the correponding labelled parts *have the same cardinality*.

# Graph Isomorphism Integer Program

Yet another way of approximating the *graph isomorphism relation* is obtained by considering it as a *0/1 linear program*.

If $A$ and $B$ are adjacency matrices of graphs $G$ and $H$, then $G \cong H$ if, and only if, there is a *permutation matrix* $P$ such that:

$$PAP^{-1} = B \quad \text{or, equivalently} \quad PA = BP$$

A *permutation matrix* is a $0$-$1$-matrix which has exactly one $1$ in each row and column.

# Integer Program

Introducing a variable $x_{ij}$ for each entry of $P$ , the equation $PA = BP$ becomes a system of *linear equations*

$$\sum_k x_{ik} a_{kj} = \sum_k b_{ik} x_{kj}$$

Adding the constraints:

$$\sum_i x_{ij} = 1 \quad \text{and} \quad \sum_j x_{ij} = 1$$

we get a system of equations that has a *0-1 solution* if, and only if, $G$ and $H$ are isomorphic.

# Sherali-Adams Hierarchy

If we have any *linear program* for which we seek a *0-1 solution*, we can relax the constraint and admit *fractional solutions*:

$$0 \leq x_{ij} \leq 1.$$

The resulting linear program can be solved in *polynomial time*, but admits solutions which are not solutions to the original problem.

**Sherali and Adams (1990)** define a way of *tightening* the linear program by adding a number of *lift and project* constraints.

Say that $G \cong^{f,k} H$ if the $k$th lift-and-project of the *isomorphism program* on $G$ and $H$ admits a solution.

# Sherali-Adams Isomorphism

For each $k$

$$G \equiv_C^{k+1} H \quad \Rightarrow \quad G \cong^{f,k} H \quad \Rightarrow \quad G \equiv_C^k H$$

**(Atserias, Maneva 2012)**

For $k > 2$, the reverse implications fail. **(Grohe, Otto 2012)**

# Coherent Algebras

**Weisfeiler and Lehman** presented their algorithm in terms of *cellular algebras*.

These are algebras of matrices on the *complex numbers* defined in terms of *Schur multiplication*:

$$(A \circ B)(i,j) = A(i,j)B(i,j)$$

They are also called *coherent configurations* in the work of **Higman**.

## Definition

A *coherent algebra* with index set $V$ is an algebra $\mathcal{A}$ of $V \times V$ matrices over $\mathbb{C}$ that is:

> *closed under Hermitian adjoints; closed under Schur multiplication; contains the identity $I$ and the all 1's matrix $J$.*

# Weisfeiler-Lehman method

Associate with any graph $G$, its *coherent invariant*, defined as the smallest coherent algebra $\mathcal{A}_G$ containing the adjacency matrix of $G$.

Say that two graphs $G_1$ and $G_2$ are *WL*-equivalent if there is an isomorphism between their *coherent invariants* $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$.

$G_1$ and $G_2$ are *WL*-equivalent if, and only if, $G_1 \equiv_C^3 G_2$.

**(D., Holm)** give a way of lifting this characterisation to any $k$.

Replacing the *complex field* $\mathbb{C}$ by *finite fields* gives a family of equivalences that can be used to analyse FPrk—*rank logic*.

# Homomorphisms

Recall a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a map $h : \mathbb{A} \to \mathbb{B}$ so that for any tuple $\mathbf{a}$ and any relation $R$,

$$R^{\mathbb{A}}(\mathbf{a}) \quad \Rightarrow \quad R^{\mathbb{B}}(h(\mathbf{a})).$$

$\mathbb{A} \cong \mathbb{B}$ *if, and only if,* there are homomorphisms $h : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$ such that

$$gh = \mathrm{id}_{\mathbb{A}} \quad \text{and} \quad hg = \mathrm{id}_{\mathbb{B}}.$$

# Local Consistency Maps

The problem of deciding if there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ is $\mathrm{NP}$-complete.

In practice, a commonly used test is the *local consistency test*.
There is one such for each $k$

Write $\mathbb{A} \Rrightarrow^k \mathbb{B}$ to denote that for any *existential, positive* sentence $\varphi$ of $L^k$

$$\text{if} \quad \mathbb{A} \models \varphi \quad \text{then} \quad \mathbb{B} \models \varphi.$$

# Existential Pebble Game

The relation $\mathbb{A} \Rrightarrow^k \mathbb{B}$ has a *pebble game* characterisation due to **Kolaitis-Vardi**:

The game is played on structures $\mathbb{A}$ and $\mathbb{B}$ with pebbles $a_1, \ldots, a_k$ on $\mathbb{A}$ and $b_1, \ldots, b_k$ on $\mathbb{B}$.

- *Spoiler* chooses a pair of pebbles $a_i$ and $b_i$;
- *Duplicator* chooses a *map* $h : A \to B$ such that for pebbles $a_j$ and $b_j (j \neq i)$, $h(a_j) = b_j$;
- *Spoiler* chooses $a \in A$ and places $a_i$ on $a$ and $b_i$ on $h(a)$.

*Duplicator* loses if the partial map $a_i \mapsto b_i$ is not a partial *homomorphism*. *Duplicator* has a strategy to play forever if, and only if, $\mathbb{A} \Rrightarrow^k \mathbb{B}$.

# Invertible Strategies

We can define *strategy composition* so that if $s : \mathbb{A} \Rrightarrow^k \mathbb{B}$ and $t : \mathbb{B} \Rrightarrow^k \mathbb{C}$ then

$$ts : \mathbb{A} \Rrightarrow \mathbb{C}$$

There is a pair of strategies $s : \mathbb{A} \Rrightarrow^k \mathbb{B}$ and $t : \mathbb{B} \Rrightarrow^k \mathbb{A}$ such that

$$ts = \mathrm{id}_{\mathbb{A}} \quad \text{and} \quad st = \mathrm{id}_{\mathbb{B}}$$

*if, and only if* $\mathbb{A} \equiv^k_C \mathbb{B}$.

# CSP Preservation

For a structure $\mathbb{B}$: $\mathrm{CSP}(\mathbb{B}) = \{\mathbb{A} \mid \mathbb{A} \to \mathbb{B}\}$

## Theorem

*If $\mathrm{CSP}(\mathbb{B})$ is closed under $\equiv^k_C$ for some $k$, then its complement is closed under $\Rrightarrow^{k'}$ for some $k'$.*

This follows from results of **(Atserias, Bulatov, D.)** and **(Barto, Kozik)**.

**Conjecture** (*Infinitary Homomorphism Preservation*)
If a class of structures $K$ is closed under homomorphisms and under $\equiv^k_C$ for some $k$, then it is closed under $\Rrightarrow^{k'}$ for some $k'$.

# Definability Dichotomy

A related result was presented at **(D., Wang, CSL 2015)** on *finite valued constraint satisfaction problems*.

These allow *"soft"* constraints that can be violated, but at a *cost*. The aim is to find a *minimum cost solution*.

Every finite valued CSP is **(Thapper-Živny) (D.-Wang)**

- *either*, in $\mathrm{PTime}$; closed under $\equiv^k_C$ for some $k$, and definable in FPC
- *or* $\mathrm{NP}$-complete; and not closed under $\equiv^k_C$ for any $k$.

# Summary

Notions of *elementary equivalence* are an essential tool for studying *definability* in finite structures.

The family of equivalence relations $\equiv_C^k$ arises naturally from many different sources; *and* turns out to to have many computational applications.