Evaluating Formulas on Sparse Graphs

Part 4

Anuj Dawar

University of Cambridge Computer Laboratory

PhD Open, Warsaw, 6 November 2010

Results So Far

1. $\mathcal{T}_k$—the class of graphs of tree-width at most k.

   Courcelle (1990) shows that every MSO definable property is decidable in linear time on this class.

2. $\mathcal{D}_k$—the class of graphs of *degree* bounded by k.

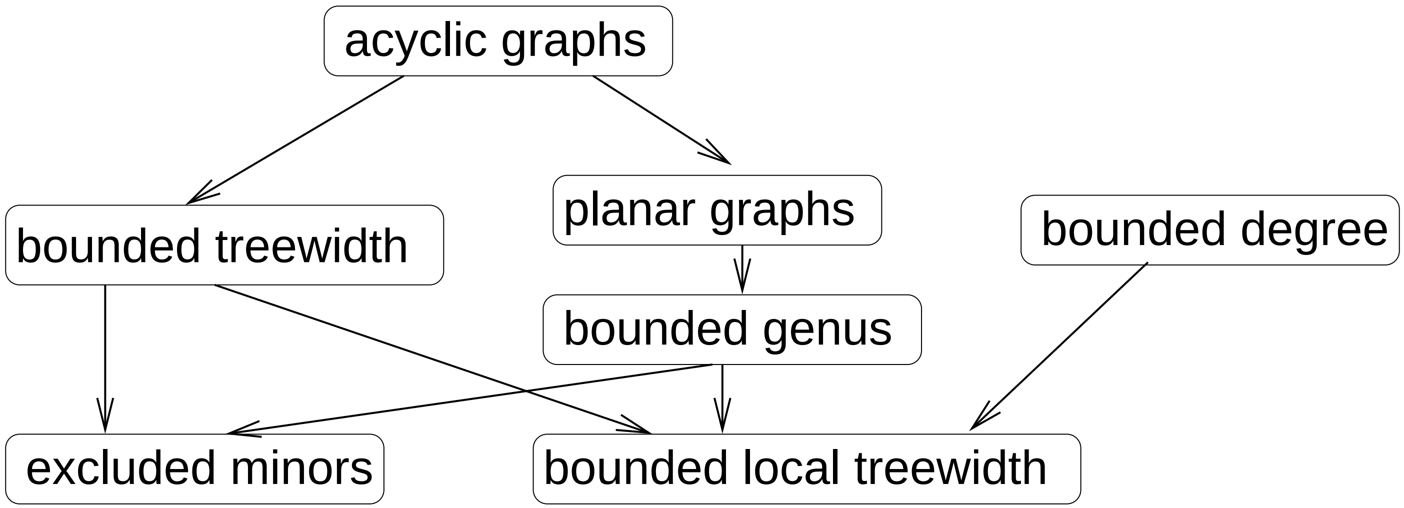
   Seese (1996) shows that every FO definable property is decidable in linear time.

3. LTW_t—the class of graphs of *local tree-width* bounded by a function t. **Frick and Grohe (2001)** show that every FO definable property is decidable in quadratic time.

4. $\mathcal{M}_k$—the class of graphs *excluding K_k as a minor*.

   Flum and Grohe (2001) show that every FO definable property is decidable in time $O(n^5)$.

Map of Classes



Incomparability

$\mathcal{D}_3$—the class of graphs of degree 3 does not exclude any minor.

The class of *apex graphs* does not have bounded local tree-width.

A graph G is an *apex* graph if there is a vertex v such that $G - v$ is *planar*.

Common Generalization

Let $t : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function.

LEM$_t$—the class of graphs $G = (V, E)$ such that for every $v \in V$:

$$K_{t(r)} \not\preceq N_r^G(v)$$

We say that $\mathcal{C}$ *locally excludes minors* if there is some function t such that $\mathcal{C} \subseteq$ LEM$_t$.

Theorem (D., Grohe, Kreutzer)

First-order logic is fixed-parameter tractable on every class $\mathcal{C}$ that locally excludes minors.

Application of Locality Method?

The result would be an easy application of the *locality method* if we had established:

> There is an algorithm deciding $\mathbb{A} \models \varphi$ in time $f(l, k)n^c$
> where k is the least value such that $K_k \not\preceq G$

While the **Flum-Grohe** theorem does give a $O(n^5)$ algorithm for the class of graphs that exclude K_k as a minor, *it is not clear if the constants are bounded by a computable function of* k.

Potential Sources of Uncomputability

1. The algorithm decomposing a graph in $\mathcal{M}_k$ over the class $\mathcal{L}_{\lambda,\mu}$ relies on a membership test in a *minor-closed superclass* of $\mathcal{M}_k$. It is not known whether the excluded minors for this class are given by a computable function of k.

2. The algorithm for reducing graphs in $\mathcal{L}_{\lambda,\mu}$ to $\mathcal{L}_{\lambda}$ relies on membership tests for $\mathcal{L}_{\lambda,\mu'}$ (for $\mu' \leq \mu$) and it is not known if the excluded minors for these classes are given by a computable function of k.

(D., Grohe, Kreutzer) gives constructive solutions to both these problems.

Constructive Decomposition

There is a *uniform in k* algorithm which computes a decomposition of a graph $G \in \mathcal{M}_k$ over $\mathcal{L}_{\lambda,\mu}$ in time $O(n^4)$.

Instead of *clique-sums*, the decomposition uses *neighbourhood-sums*.

$$(G_1, x) \odot_x (G_2, x)$$

is the graph obtained by taking the *disjoint sum* of G_1 and G_2 while identifying $N_1(x)$ and *deleting* x.

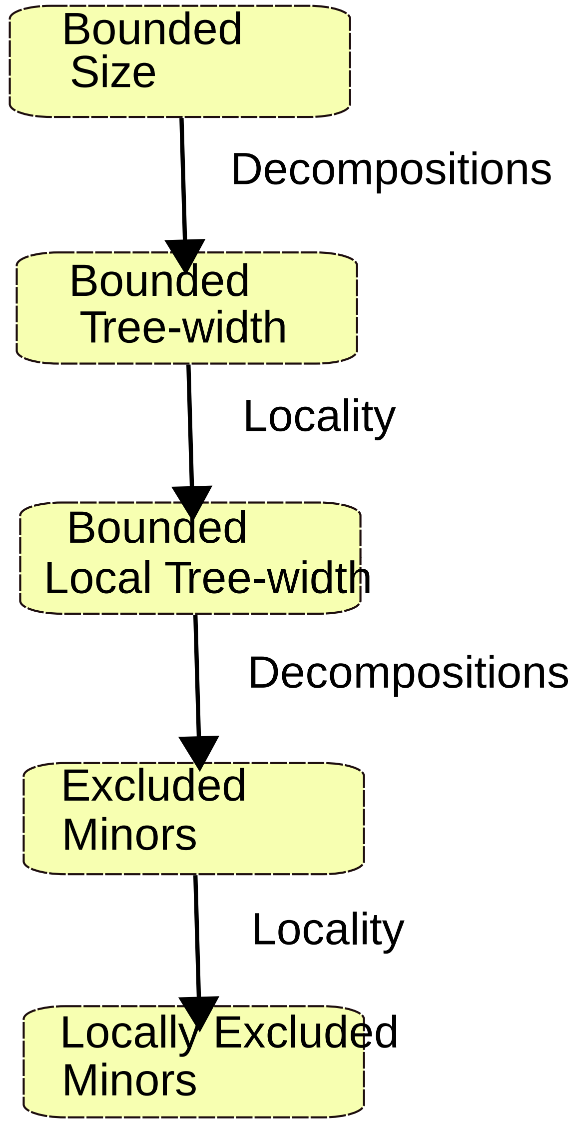
It is also shown that given $G \in \mathcal{L}_{\lambda,\mu}$, we can effectively find $v_1, \ldots, v_\mu$ such that $G \setminus \{v_1, \ldots, v_\mu\} \in \mathcal{L}_{\lambda'}$ for some λ' computable from λ.

Review

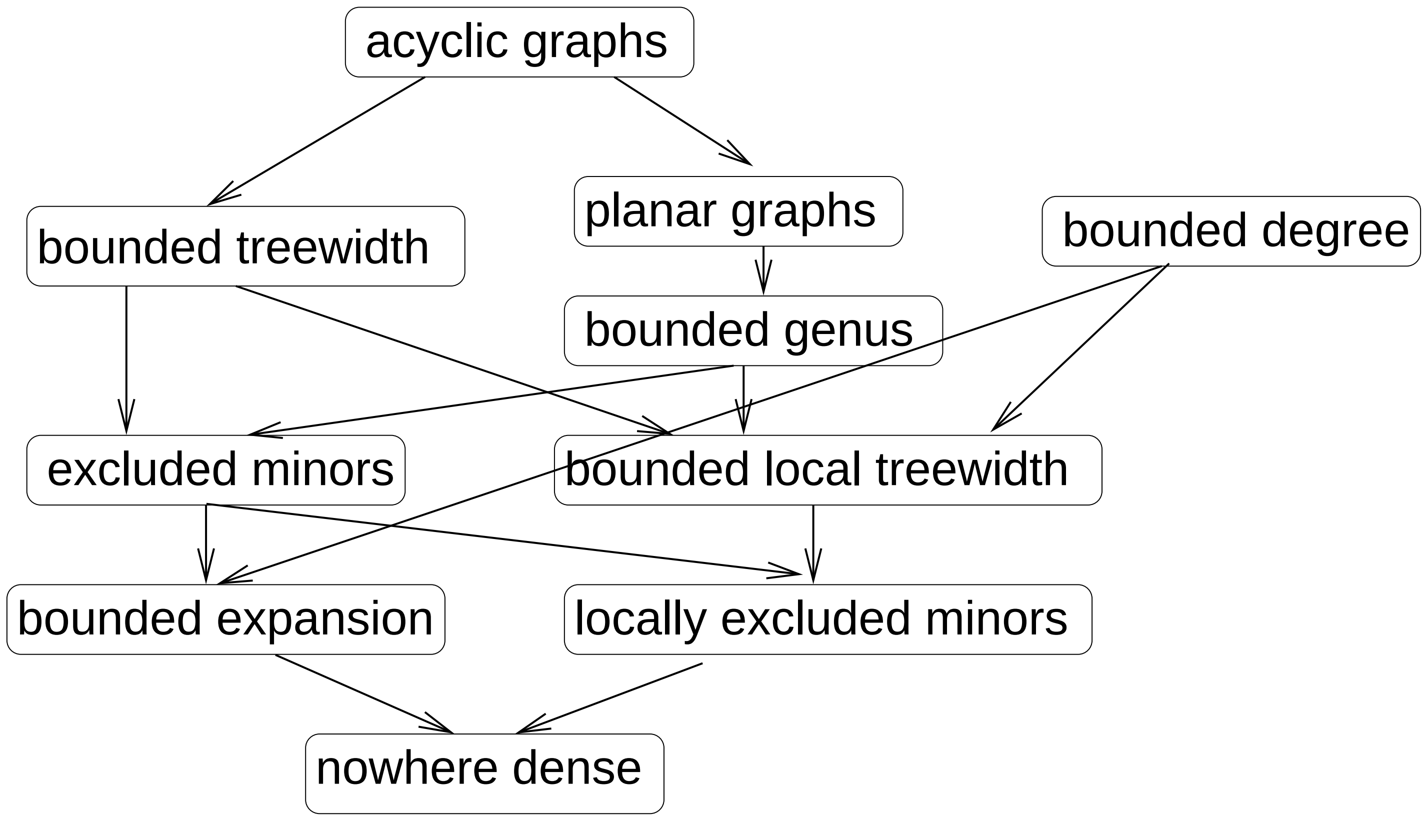
We have gone from graphs of *bounded size* to *locally excluded minors* in four steps, alternating decomposition steps with localisation steps.

This seems to be the limit of the methods. Are there interesting classes which admit decompositions over classes with locally excluded minors?

We next look at some other classes which allow for new methods.



Map of Classes



New Classes

Graph classes of *Bounded Expansion* were introduced in a series of papers by Nešetřil and Ossona de Mendez.

They are a common generalization of *bounded degree* and *excluded minor* classes.

They were used to study problems of *colourability* of *sparse* graphs.

What is a well-behaved notion of *sparseness*?

Sparse Classes

For a graph G, we write $|G|$ for the number of *vertices* in G and $||G||$ for the numer of *edges* in G.

If G has degree at most d, then $||G|| \leq \frac{d}{2}|G|$.

If G has tree-width at most k, then $||G|| \leq k|G|$.

Theorem (Kostochka)

There exists a constant c such that, for every r, every graph G of average degree $d \geq c\sqrt{r}\log r$ contains K_r as a minor.

Sparse Classes

Say a class $\mathcal{C}$ of graphs is *sparse* if there is a c so that for all $G \in \mathcal{C}$, $||G|| \leq c|G|$.

Equivalently, $\mathcal{C}$ has *bounded average degree*.

There are pathological sparse classes.

Take the class that contains, for every finite graph G the graph consisting of the union of G with $||G||$ isolated vertices.

Hereditarily Sparse Classes

Say a class $\mathcal{C}$ of graphs is *hereditarily sparse* if there is a c so that for all $G \in \mathcal{C}$, and every subgraph $H \subset G$ we have $||H|| \leq c|H|$.

With any graph $G = (V, E)$, we associate its *incidence graph* $I(G) = (V \cup E, F)$ where

$$F = \{(v, e) \mid v \in V, e \in E \text{ and } e \text{ is incident with } v\}.$$

The collection of all incidence graphs is hereditarily sparse but has all the complexity of the class of all graphs since the map $G \mapsto I(G)$ is an easy *first-order* reduction.

Shallow Minors

Recall that $H = (U, F)$ is a minor of $G = (V, E)$, if we can find a collection of *disjoint, connected* subgraphs of G: $(B_u \mid u \in U)$ such that whenever $(u_1, u_2) \in F$, there is an edge between some vertex in B_{u_1} and some vertex in B_{u_2}.

The graphs B_u are called *branch sets* witnessing that $H \preceq G$.

If the branch sets can be chosen so that for each u there is $b \in B_u$ and $B_u \subseteq N_r^G(b)$, we say that H is a minor *at depth* r of G and write $H \preceq_r G$

Bounded Expansion

A class $\mathcal{C}$ of graphs is said to have *bounded expansion* if, for each r, there is a ∇_r such that if $H \preceq_r G$ for some $G \in \mathcal{C}$ then $||H|| \leq \nabla_r|H|$.

In other words, $\mathcal{C}$ has bounded expansion if, for every r, the collection of depth-r minors of graphs in $\mathcal{C}$ is *sparse*.

$\mathcal{M}_k$ has bounded expansion by taking $\nabla_r = c\sqrt{k}\log k$ (where c is from Kostochka's theorem.

$\mathcal{D}_k$ has bounded expansion by taking $\nabla_r = k^r$.

FO on Bounded Expansion Classes

A very recent result (FOCS 2010) by **Dvořak, Král and Thomas** states:

> If $\mathcal{C}$ is a class of bounded expansion and φ is a first-order sentence, then $G \models \varphi$ can be decided, for $G \in \mathcal{C}$ in *linear time*.

Note: this improves the $O(n^5)$ bound for excluded minor classes.

The technique used is quite different to the locality and decomposition techniques we have seen. It relies on suitable *graph colourings*.

Tree Depth

The *tree-depth* of a graph G is defined to be the smallest k such that there is *directed forest* F of height k and

> G is a subgraph of the *undirected* graph underlying the *transitive closure* of F.

For any graph G, $\mathrm{tw}(G) \leq \mathrm{td}(G)$,

> where $\mathrm{tw}(G)$ is the tree-width of G and $\mathrm{td}(G)$ is the tree-depth of G.

Low Tree-Depth Colourings

Nešetřil and Ossona de Mendez prove a remarkable colouring property of classes of graphs of *bounded expansion*.

Theorem: (Nešetřil, Ossona de Mendez)
Let $\mathcal{C}$ be a class of graphs of bounded expansion. For any p there is an N such that any graph $G \in \mathcal{C}$ can be coloured using N colours in such a way that if $C_1, \ldots, C_p$ is any set of p colours then $G[C_1 \cup \cdots \cup C_p]$ has $tree - depth$ at most p.

Moreover, this colouring can be found efficiently—*in linear time*.

Evaluating Existential Formulas

Suppose φ is an *existential* first-order formula.

That is, it is of the form

$$\exists x_1 \cdots \exists x_q \theta$$

where θ is quantifier-free.

If $\mathcal{C}$ has bounded expansion, we can evaluate such formulas on graphs in $\mathcal{C}$ by the following process.

Find a colouring of G which guarantees that any q colours induce a graph of tree-depth at most q.

For each set of q colours, check whether φ can be evaluated in the subgraph induced by these colours.

This establishes that for *existential* formulas, satisfaction is FPT on $\mathcal{C}$.

Quantifier Alternation

To deal with formulas with quantifier alternation Dvořak et al. define a *quantifier elimination* procedure.

This requires us to replace, at each step, the graph G with an expansion including new *coloured* edges giving the low tree-depth decompositions.

Crucially, they prove that doing this on all graphs in $\mathcal{C}$ still gives a graph of bounded expansion, though with larger bounds.

Incomparibility

There are classes $\mathcal{C}$ with bounded expansion that do not have locally excluded minors.

> Let $\mathcal{C}$ be the collection of graphs G that contain a vertex v such that $G - v$ is in $\mathcal{D}_3$.

There are classes $\mathcal{C}$ that locally exclude minors but do not have bounded expansion.

Lemma (Erdos)

For any $d, k \in \mathbb{N}$ there is a graph G with *girth* $\geq k$ and *minimum degree* $\geq d$.

Nowhere Dense Graphs

We say that a class $\mathcal{C}$ of graphs is *nowhere-dense* if, for every r, the collection of graphs

$$\{H \mid H \preceq_r G \text{ for some } G \in \mathcal{C}\}$$

is *not* the class of all graphs.

In other words, for each r, there is a K_k that cannot be obtained as a depth-r minor of any graph in $\mathcal{C}$.

This clearly generalizes bounded expansion classes.

It also generalizes locally excluded minor classes because if $K_k \preceq_r G$ then there is a v in G such that $K_k \preceq N^G_{3r+1}(v)$.

Trichotomy Theorem

Associate with any infinite class $\mathcal{C}$ of graphs the following parameter:

$$d_{\mathcal{C}} = \lim_{r\to\infty} \limsup_{G \in \mathcal{C}_r} \frac{\log ||G||}{\log |G|},$$

where $\mathcal{C}_r$ is the collection of graphs obtained as minors of a graph in $\mathcal{C}$ by contracting neighbourhoods of radius at most r.

The *trichotomy theorem* of Nešetřil and Ossona de Mendez states that $d_{\mathcal{C}}$ can only take values 0, 1 and 2.

The nowhere-dense classes are exactly the ones where $d_{\mathcal{C}} \neq 2$.

This shows that these classes are a *natural limit* to one notion of sparseness.

FO on Nowhere Dense Classes

It is still an open question whether FO satisfaction is fixed-parameter tractable on nowhere-dense classes.

Some problems, defined by families of FO formulas, have been shown to be FPT on such classes.

- *Independent Set*;
- *Dominating Set*;
- *distance-d dominating set*

The proof for these is based on a technique distinct from those we have seen so far.

Wide Classes

A set of vertices A in a graph G is said to be r-scattered if for any $u, v \in A$, $N_r(u) \cap N_r(v) = \emptyset$.

Definition

A class of graphs $\mathcal{C}$ is said to be *wide* if for every r and m there is an N such that any graph in $\mathcal{C}$ with more than N vertices contains a r-scattered set of size m.

Example: Classes of graphs of bounded degree.

Non-Example: Trees

Almost Wide Classes

Definition

A class of graphs $\mathcal{C}$ is *almost wide* if there is an s such that for every r and m there is an N such that any graph in $\mathcal{C}$ with more than N vertices contains s elements whose removal leaves a r-scattered set of size m.

Example: Trees.

Examples: planar graphs?

Quasi-Wide Classes

Let $s : \mathbb{N} \to \mathbb{N}$ be a function. A class $\mathcal{C}$ of graphs is *quasi-wide with margin s* if for all $r \geq 0$ and $m \geq 0$ there exists an $N \geq 0$ such that if $G \in \mathcal{C}$ and $|G| > N$ then there is a set S of vertices with $|S| < s(r)$ such that $G - S$ contains an r-scattered set of size at least m.

We show that any class of nowhere-dense graphs is quasi-wide.

The proof also shows that any class that excludes K_k as a minor is *almost wide* with margin $k - 2$.

Finding Scattered Sets

Theorem

There is a computable N so that for any $k, r, m \geq 0$, if G is a graph with more than $N(k, r, m)$ vertices then

1. either $K_k \preceq_{r+1} G$; or
2. there is a set S of vertices with $|S| \leq k - 2$ such that $G - S$ contains an r-scattered set of size m.

Ramsey's Theorem

Write $[I]^k$ for the collection of k-element subsets of I.

Theorem (Ramsey)

For each k, l and m, there is an N such that, if $|I| > N$, and

$$I_1, \ldots, I_l$$

is a *partition* of $[I]^k$, there is a $J \subseteq I$ with $|J| > m$, such that $[J]^k \subseteq I_i$ for some i.

The special case when $k = l = 2$ is often stated in the following form

> For any m there is an N such that any graph with more than N vertices either contains an m-clique or an m-independent set.

Expanding Neighbourhoods

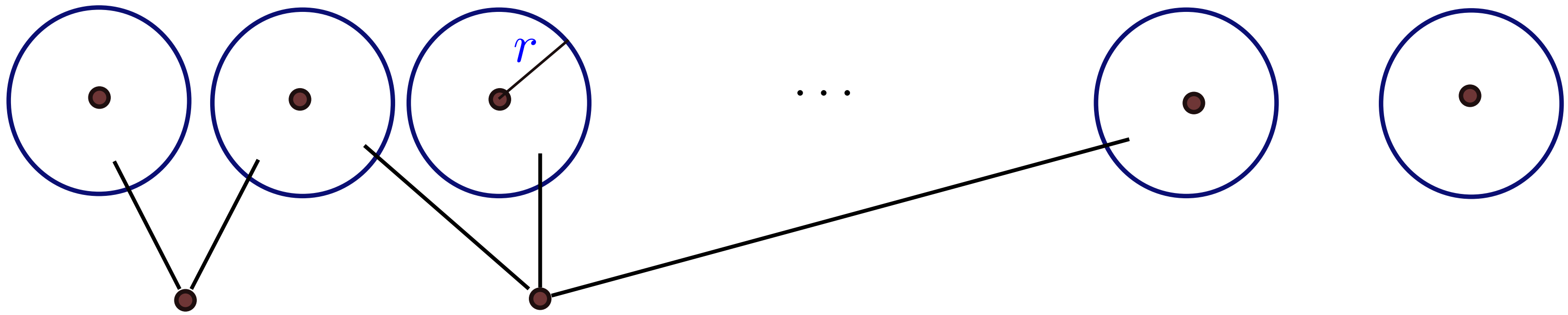
Start with the large set W, and take neighbourhoods of increasing radius around the vertices in W, while thinning out the set so that neighbourhoods are dijoint.

If there are more k neighbourhoods which pairwise have an edge between them, then $K_k \preceq_r G$.

So, there must be a large number of neighbourhoods that are independent (a simple application of Ramsey's theorem).

Expanding Neighbourhoods

r-neighbourhoods of vertices in W may also have common neighbours.



We can guarantee that we can find a *large* subset $W' \subseteq W$ which has a common neighbour.

The number of such common neighbours cannot be $k-1$.

Otherwise, we could find $K_{k-1,k-1}$ and therefore K_k as a minor (a more complicated application of Ramsey's theorem).

Equivalence

A class $\mathcal{C}$ of graphs is *uniformly quasi-wide* with margin s if for all $r \geq 0$ and all $m \geq 0$ there exists an $N \geq 0$ such that if $G = (V, E) \in \mathcal{C}$ and $W \subseteq V$ with $|W| > N$ then there is a set $S \subseteq V$ with $|S| < s(r)$ such that W contains an r-scattered set of size at least m in $G - S$.

Theorem: (Nešetřil, Ossona de Mendez)

The following are equivalent for any class $\mathcal{C}$ that is closed under taking induced subgraphs:

1. $\mathcal{C}$ is nowhere dense
2. $\mathcal{C}$ is quasi-wide
3. $\mathcal{C}$ is uniformly quasi-wide

Tractable Algorithm

The proof not only gives effective bounds on N, it provides an efficient (in the size of G) algorithm for finding the bottleneck and the scattered set.

Lemma: Let $\mathcal{C}$ be a nowhere-dense class of graphs and h be the function such that $K_{h(r)} \not\preceq_r G$ for all $G \in \mathcal{C}$. The following problem is solvable in time $\mathcal{O}(|G|^2)$.

Input:	$G = (V, E) \in \mathcal{C}$, $r, m \in \mathbb{N}$, $W \subseteq V$ with $\|W\| > N(h(r), r, m)$
Problem:	compute a set $S \subseteq V$, $\|S\| \leq h(r) - 2$ and a set $A \subseteq W$ with $\|A\| \geq m$, such that in $G - S$, A is r-scattered.

Network Centres Problem

The distance-d dominating set problem, also known as the *network centres* problem is the following:

> *Network Centres Problem:* Given a graph G and positive integers k, d, decide whether G contains a set X of k vertices such that every vertex of G is within distance d of a vertex in X.

We show that this problem is FPT (with parameter $k + d$) on any class $\mathcal{C}$ of graphs that is *effectively* nowhere-dense.

Kernelization

We prove that the following problem is fixed-parameter tractable for any *effectively nowhere-dense* class $\mathcal{C}$.

Input:	A graph $G = (V, E) \in \mathcal{C}$, $W \subseteq V$, $k, d \geq 0$
Parameter:	$k + d$
Problem:	Determine whether there is a set $X \subseteq V$ of k vertices which d-dominates W.

We replace W by a subset W' whose size depends only on the parameters, and such that any X that d-dominates W' d-dominates W

This suffices because the above problem is FPT (*on all graphs*) when parameterized by $k + d + |W|$.

Algorithm

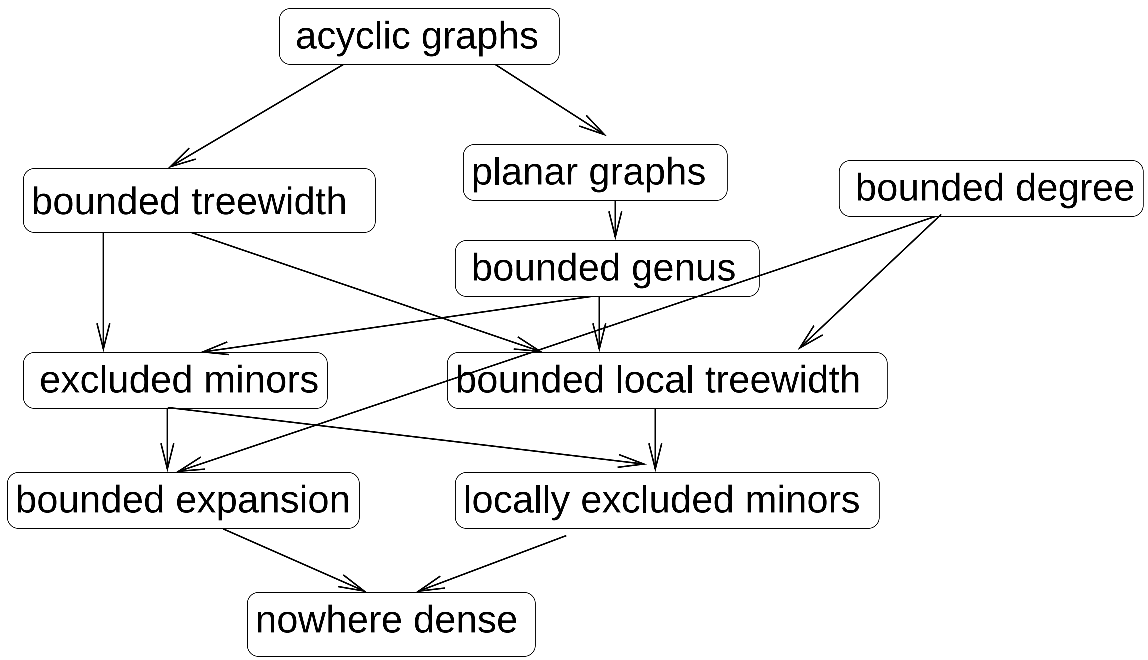
As long as W is big, we can find a bottleneck S and a large enough scattered set $A \subseteq W$ in $G - S$ so that there are $a_1, \ldots, a_{k+2}$ with the same *distance vector* to S.

> The *distance vector* of a is the s-tuple of elements from $\{0, \ldots, d, \infty\}$ indicating the distances of a from the elements of s.

If X d-dominates $W \setminus \{a_1\}$ then it d-dominates W.

Question: Can such a method be generalized to all of first-order logic?

Review



For all classes except the last one the picture, we have established that FO satisfaction is FPT.

For *nowhere dense classes* this remains an open question.

Techniques deployed use: *locality*, *decompositions*, *low tree-depth colourings* and *wideness*.