Modal Logic and Inflationary Fixed Points

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Modal Logic

Hennessy-Milner logic:

Fixing a set P of atomic propositions, and a set A of actions

The formulas of the logic are given by:

- T and F
- $p \qquad\qquad (p \in P)$
- $\varphi \wedge \psi$; $\varphi \vee \psi$; $\neg\varphi$
- $[a]\varphi$; $\langle a \rangle \varphi \qquad (a \in A)$.

Transition Systems

Interpret formulas in a *Transition system* (or *Kripke structure* or *labeled directed graph*):

$$M = (S, A^M, \mathcal{P})$$

- S is a set of *states*;
- For each $a \in A$, $a^M \subseteq S \times S$ is a *transition relation* in A^M;
- $\mathcal{P} : S \to \mathsf{Pow}(P)$;

For any $s \in S$

- $(M, s) \models p$ if $p \in \mathcal{P}(s)$.
- $(M, s) \models \langle a \rangle \varphi$ if for some t with $(s, t) \in a^M$, $(M, t) \models \varphi$.
- $(M, s) \models [a]\varphi$ if for every t with $(s, t) \in a^M$, $(M, t) \models \varphi$.
- Boolean connectives as usual.

μ-calculus

An additional collection $X_1, X_2, \ldots$ of variables.

Formulas of μ-calculus:

- all the rules of modal logic
- X
- $\mu_X\varphi$; $\nu_X\varphi$ — when X occurs only positively in φ.

$(M, \bar{X}^M, s) \models \mu_X\varphi$ if
s is in the least set X such that $X \leftrightarrow \varphi$ in $M, \bar{X}$

Dually, $(M, \bar{X}^M, s) \models \nu_X\varphi$ if
s is in the greatest set X such that $X \leftrightarrow \varphi$ in $M, \bar{X}$

Iterative Evaluation

By the restriction to positive formulas, the map taking the set of states in which X holds to the set where $\varphi(X)$ holds is monotone, so the least and greatest fixed points are well-defined.

The least fixed point can be found bottom-up:

$$\begin{aligned} X^0 &= \emptyset \\ X^{\alpha+1} &= \varphi(X^\alpha) \\ X^\lambda &= \bigcup_{\alpha<\lambda} X^\alpha \text{ for limit } \lambda \end{aligned}$$

$\nu_X\varphi$ is equivalent to $\neg\mu_X\neg\varphi(\neg X)$.

Examples

- $\mu X.(p \vee \langle a \rangle X)$ "there is an a-path to a point where p holds"
- $\mu X.\Box X$ "no infinite paths" (well-foundedness)

Inflationary Fixed Points

An alternative way of introducing fixed point constructions is to define

$$\textbf{ifp}X.\varphi$$

to be the limit of the sequence

$$\begin{aligned} X^0 &= \emptyset \\ X^{\alpha+1} &= X^\alpha \cup \varphi(X^\alpha) \\ X^\lambda &= \bigcup_{\alpha<\lambda} X^\alpha \qquad \text{for limit } \lambda \end{aligned}$$

This is the same as $\mu_X\varphi$ when φ is positive in X.

LFP and IFP

In finite model theory,

LFP is the extension of first order predicate logic with a least fixed point operator

IFP is the extension of first order predicate logic with an inflationary fixed point operator

IFP and LFP are equivalent in expressive power.

(Gurevich-Shelah)

IFP is generally more robust and often more intuitive.

Expressive Power

There are modal **ifp** formulas that are not equivalent to any formula of the μ-calculus.

Consider

$$\textbf{ifp} X.((a \wedge \pi) \vee (b \wedge \Box X))$$

where π is the formula

$$\textbf{ifp} Y.(\Diamond(b \wedge \neg X) \vee \Diamond(a \wedge X \wedge Y))$$

This, interpreted on strings in $a^\star b^\star$ defines the language

$$\{a^n b^m \mid n \leq m\}.$$

Modal Iteration Calculus

More generally, we consider simultaneous inductive definitions.

We define the modal iteration calculus (MIC), by extending modal logic with the following formula formation rule.

For a system S of rules,

$$S = \begin{cases} X_1 \leftarrow \varphi_1 \\ \quad\vdots \\ X_k \leftarrow \varphi_k \end{cases}$$

where X_i are variables and φ_i are formulas,

$(\textbf{ifp} X_i : S)$ is a formula.

$\textbf{ifp} X.\varphi$ is shorthand for $(\textbf{ifp} X : X \leftarrow \varphi)$.

Questions

Comparing with the μ-calculus

- **Does MIC have the finite model property?**
  L_μ has the finite model property.
- **Is Satisfiability decidable?**
  Sat(L_μ) $\textsc{Exptime}$-complete (Emerson, Jutla)
- **What is the complexity of model-checking?**
  Model-Checking for L_μ in NP $\cap$ co-NP (Open Problem: in P?)
- **Automata evaluation strategies for MIC?**
  Automata based algorithms important for L_μ
- **What kind of languages are definable in MIC?**
  L_μ defines exactly the regular languages.
- **Can simultaneous inductions be eliminated?**
  They can in L_μ

Finite Model Property

Every formula of the μ-calculus that is satisfiable has a finite model.

This fails for the modal iteration logic.
Let S be

$$
\begin{aligned}
X &\leftarrow \Box F \vee (\Box X \wedge \Diamond \neg Y) \\
Y &\leftarrow X
\end{aligned}
$$

then,

$$\theta \equiv (\textbf{ifp} X : S)$$

is true in a well-founded tree if, and only if, it has finite height.

$\neg\theta \wedge wf$ is satisfiable, but has no finite models.

Satisfiability

The satisfiability problem for MIC is undecidable. In fact, it is not even arithmetical.

There is a reduction of first order arithmetic to the satisfiability of MIC.

In a well-founded tree of height ω (which we can define),

represent $n \in \omega$ by $N = \{v \in \mathcal{T} \mid \mathrm{height}(v) < n\}$.

We can then define formulas:

$\mathrm{number}(X)$—X represents a number,

$\mathrm{plus}(X, Y)$—representing $x + y$,

$\mathrm{times}(X, Y)$—representing $x \cdot y$

$\mathrm{equal}(X, Y)$—$x = y$,

$\exists X \varphi$.

Arithmetic

$$\begin{aligned}\mathrm{plus}(S,T) := \textbf{ifp} Y :\ & X \leftarrow \Box X \\ & Y \leftarrow S \vee (\ \Box Y \wedge \mathrm{nonempty}(X) \\ & \qquad\qquad\qquad \wedge \mathrm{all}(X \rightarrow T)).\end{aligned}$$

where,

$$\mathrm{nonempty}(\varphi) := (\textbf{ifp} X \leftarrow \varphi \vee \Diamond X)$$

and,

$$\mathrm{all}(\varphi) := (\textbf{ifp} X \leftarrow \varphi \wedge \Box X)$$

Simultaneous vs. Simple Inductions

The above coding of arithmetic *requires* simultaneous inductions.

The formula $\omega-\mathrm{height}$ is not definable with simple inductions alone.

Consider the canonical tree T_ω of height ω.

Or, $\sim$-equivalently, the structure $\mathcal{O} = (\omega + 1, >)$.

For any formula φ of 1MIC, if $\mathcal{O}, \omega \models \varphi$, there is a finite N, such that for all $n > N$,

$$\mathcal{O}, n \models \varphi.$$

On the other hand, we can still reduce first order arithmetic to the satisfiability of 1MIC.

Model Checking

The complexity of checking $\mathcal{M} \models \varphi$, for

a system M with n states, and

a formula φ with a depth d of nesting of **ifp** operators, each applied to a system of at most k simultaneous rules, is:

$$O((kn)^d|\varphi| \cdot n) \quad \text{time}$$

and

$$O(n|\varphi|) \quad \text{space}$$

In short, the model checking problem is in PSPACE, and for a fixed sentence φ, the collection of its models is in polynomial time and linear space.

Model Checking

The model checking problem for MIC (or even 1MIC) is PSPACE-complete.

Even on a fixed two element structure

for 1MIC

Or a fixed one element structure if we allow simultaneous inductions.

The proof is by reduction from QBF.

Languages

There are languages that are not context-free that are definable in 1MIC.

$$L = \{cwdw \mid w \in \{a,b\}^*\}.$$

is defined by the conjunction of:

$$c \wedge \Box\mathrm{empty}(c) \wedge \mathrm{nonempty}(d)$$
$$\wedge\mathrm{all}(d \to \Box\mathrm{empty}(d))$$

and the negation of

$$\mathbf{ifp} X \leftarrow ([\neg c \wedge (\Box X \vee \Box d)] \vee [c \wedge \mathrm{nonempty}(\psi)])$$

where ψ is

$$\neg X \wedge \Box X \wedge [(b \wedge \mathrm{nonempty}(a \wedge \Box X \wedge \neg X)) \vee$$
$$(a \wedge \mathrm{nonempty}(b \wedge \Box X \wedge \neg X))]$$

Every language in DTIME$(O(n))$ is MIC definable.

Automaticity

Automaticity is a measure of complexity on languages.

A language $L \subseteq \Sigma^*$ has automaticity $O(f)$, if there is a constant c, such that:

> for each n, there is a finite automaton with no more than $c \cdot f(n)$ states which accepts a string of length n or less if, and only if, it is in L.

A language is regular if, and only if, it has constant automaticity.

The example non-CFL definable in MIC has exponential automaticity.

Automaticity can be generalised to finite trees, where n is now the height of the tree.

> There are bisimulation-invariant classes of trees, decidable in polynomial time, of non-elementary automaticity.
>
> Every MIC definable class of finite trees has exponential autmaticity.

Conclusion

Inflationary fixed points are more powerful than least fixed points in the context of modal logic.

Simultaneous inductions are even more powerful than simple ones.

This additional expressive power is accompanied with algorithmic intractability.

Useful?