Fixed Point Logics on Finite and Infinite Structures

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Fixed Point Logics

Fixed Point Logics

extensions of first order logic with fixed point operators.

- Extending study of inductive definability.
- Extensively studied in finite model theory.
- Related to questions of computational complexity.

In this talk

- Review results on fixed point logics on finite structures.
- Discuss expressive power of these logics on infinite structures.

LFP

Least fixed point logic

Close first order logic simultaneously under first order operations and

$$\mathbf{lfp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

where,

- R is a k-ary relation variable.
- $\mathbf{x}$ is a k-tuple of variables.
- $\mathbf{t}$ is a k-tuple of terms.
- R occurs only positively in φ.

The operator **lfp** binds all occurrences of R and variables in $\mathbf{x}$ that occur in φ, but not those in $\mathbf{t}$

Semantics

The semantics of the **lfp** operator are given by:

$$\mathcal{A} \models \textbf{lfp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

if

> $\mathbf{t}^{\mathcal{A}}$ is in the least fixed point of the monotone operator defined by $\varphi(R, \mathbf{x})$ on $|\mathcal{A}|^k$:

$$\Phi(R^{\mathcal{A}}) = \{\mathbf{a} \mid \mathcal{A} \models \varphi[R^{\mathcal{A}}, \mathbf{a}]\}$$

The least fixed point can be reached by iterating the operator:

$$\begin{aligned} \varphi_0 &= \emptyset \\ \varphi_{\alpha+1} &= \varphi(\varphi_\alpha/R) \\ \varphi_\lambda &= \bigcup_{\beta<\lambda} \varphi_\beta \qquad \text{for limit } \lambda \end{aligned}$$

The closure ordinal of φ is the least ordinal α such that

$$\varphi_\alpha = \varphi_{\alpha+1}.$$

Example 1

Example:

Let $\varphi(R, x, y)$ be $x = y \vee \exists z(E(x, z) \wedge R(z, y))$

Then, $\mathbf{lfp}_{R,xy}\varphi(u, v)$ is a formula in two free variables that defines the transitive closure of E.

φ_n defines the set of pairs with paths of length at most n.

On finite graphs, the closure ordinal is the length of the longest path.

The closure ordinal is $\leq \omega$ on all structures.

Example 2

Let $\varphi(P, x)$ be

$$\forall y(y < x \to P(y)).$$

$\mathbf{lfp}_{P,x}\varphi$ defines the well-founded part of the binary relation $<$.

If $<$ is a linear order, this is the longest well-ordered initial segment.

The closure ordinal is the length of this well-order.

For any formula $\varphi(\mathbf{x})$ *of* LFP*,*

the set

$$\{(\mathcal{A}, \mathbf{c}) \mid \mathcal{A} \models \varphi[\mathbf{c}]\}$$

is decidable in polynomial time.

Conversely, in the presence of a distinguished binary relation $<$ *interpreted as a linear order on all structures:*

Every polynomial time query is definable in LFP.

(Immerman; Vardi 1982)

Normal Form

For every formula of LFP, there is an equivalent (*over finite structures*) one of the form:

$$\exists x \, \mathbf{lfp}_{R,\mathbf{y}} \varphi(x, \ldots, x). \tag{1}$$

where φ is first order.

In contrast, on

$$(\omega, <)$$

the interleaving of **lfp** and $\neg$ yields a strict hierarchy of expressive power.

This is shown by a diagonalisation, using truth predicates.

A normal form like (1) holds whenever fixed points are not reached at limit ordinals.

Inflationary Fixed Points

IFP is similar to LFP except it has an operator

$$\textbf{ifp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

to denote that $\mathbf{t}$ is in the union of the sequence of relations:

$$\varphi_\alpha = \bigcup_{\beta<\alpha} \varphi(\varphi_\beta/R).$$

where,

$$\varphi(\varphi_\beta/R)$$

is the relation defined by φ when φ_β is fixed as the interpretation of R.

Note: R need not be positive in φ.

If φ defines a monotone operator, then

$$\mathbf{ifp}_{R,\mathbf{x}}\varphi$$

defines the same relation as

$$\mathbf{lfp}_{R,\mathbf{x}}\varphi$$

So, in general:

$$\mathrm{LFP} \leq \mathrm{IFP}$$

On finite structures:

$$\mathrm{LFP} = \mathrm{IFP}$$

(Gurevich–Shelah 1986).

On $(\omega, <)$ there are first order $\varphi(R)$ such that

$$\textbf{ifp}_{R,\mathbf{x}}\varphi$$

is not obtained as the least fixed point of any *monotone* first order formula.

If a well-founded relation of ordinal height α is definable in IFP, then every LFP formula with all closure ordinals $< \alpha$ is definable.

Yet,

$$\text{LFP} = \text{IFP}$$

on $(\omega, <)$.

On what structures is IFP more powerful than LFP?

IFP and LFP

If $\mathcal{F}$ is a collection of operators on ω, then the class Γ of relations definable as

$$\textbf{ifp}_R f(R) \quad (f \in \mathcal{F})$$

is the smallest *Spector* class containing $\mathcal{F}$.

(Moschovakis)

Taking $\mathcal{F}$ to be the LFP definable operators, verify that the collection of relations

$$\textbf{lfp}_R f(R) \quad (f \in \mathcal{F})$$

is a Spector class.

Moreover, by adapting an argument of Gurevich and Shelah, we also have that any $\textbf{ifp}_{R,\mathbf{x}}\varphi$ is equivalent to a formula $\textbf{ifp}_{R,\mathbf{x}}\psi$, where ψ is monotone.
This equivalence holds over all structures.

Second Order Logic

A collection of finite structures is definable in Σ^1_1

if, and only if,

it is decidable in NP.

(Fagin 1974).

On finite structures:

$$\mathsf{LFP} = \mathsf{IFP} \leq \Delta^1_1.$$

either by Fagin's theorem or by a direct translation.

Translation

Translating $\textbf{ifp}_{R,\mathbf{x}}\varphi(\mathbf{t})$:

$$\begin{array}{ll}
\forall(\exists) & \text{a linear order on } k\text{-tuples} \\
\forall(\exists) & \text{a sequence of relations } R^i \text{ indexed by } k\text{-tuples} \\
& [R^0 = \emptyset \wedge \;\; R^{i+1} = R^i \cup \varphi(R^i)] \\
& \qquad\qquad \rightarrow (\wedge)\ \exists i\, \mathbf{t} \in R^i.
\end{array}$$

Without the restriction to finite structures:

$$\mathsf{LFP} \leq \mathsf{IFP} \leq \Delta^1_2$$

$\forall(\exists)$ a sequence of relations indexed by k-tuples
$\forall(\exists)$ a well ordering of k-tuples
$\quad\ldots$

This cannot be improved as

$$\Pi^1_1 \cup \Sigma^1_1 \leq \mathsf{LFP}$$

on $(\omega, <)$.

Partial Fixed Points

For a formula $\varphi(R)$,

(not necessarily monotone in R)

define the sequence of relations:

$$\begin{aligned} \varphi_0 &= \emptyset \\ \varphi_{i+1} &= \varphi(\varphi_i/R) \end{aligned}$$

Define:

$$\mathbf{pfp}_{R,\mathbf{x}}\varphi$$

to be the relation:

$$\begin{array}{ll} \varphi_m & \text{if } \varphi_m = \varphi_{m+1} \\ \emptyset & \text{if there is no such } m. \end{array}$$

In the presence of linear order, the PFP definable properties of finite structures are exactly the PSPACE computable ones.

Thus, on finite, ordered structures:

$$\mathsf{SO} \leq \mathsf{PFP}$$

Without the requirement of order, the two logics are incomparable (?).

IFP = PFP if, and only if, P = PSPACE

(Abiteboul–Vianu, 1992).

PFP inductions may require an exponential number of steps:

$$[R(x) \wedge \exists y(y < x \wedge \neg R(y))] \vee$$

$$[\neg R(x) \wedge \forall y(y < x \rightarrow R(y))] \vee$$

$$\forall y\, R(y).$$

The semantics of PFP *does not easily extend to infinite structures.*

- What happens at limit ordinals?
- Can an induction on a well ordering cycle through all subsets?

Depends on well-ordering the power-set?

Non-Deterministic Fixed Points

Consider two first order formulas:

$$\varphi_0(R, \mathbf{x}) \text{ and } \varphi_1(R, \mathbf{x})$$

We define a sequence of relations φ^b on a *finite* structure, indexed by *binary strings* b.

$$\begin{aligned} \varphi^{\epsilon} &= \emptyset \\ \varphi^{b\cdot 0} &= \varphi_0(\varphi^b/R) \cup \varphi^b \\ \varphi^{b\cdot 1} &= \varphi_1(\varphi^b/R) \cup \varphi^b \end{aligned}$$

$\mathbf{nfp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)$ is then the relation

$$\bigcup_b \varphi^b$$

On ordered structures:

A property is definable as:

$$\exists x\, \mathbf{nfp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)(x, \ldots, x)$$

where φ_0 and φ_1 are first order

if, and only if, it is NP.

The above is a normal form for alternation-free NFP.

A property is definable in NFP (*closing under negations and* **nfp**) if, and only if, it is definable in second order logic.

Without linear order,

$$\mathsf{NFP} < \mathsf{SO}$$

still,

$$\mathsf{IFP} = \mathsf{NFP} \text{ if, and only if, } \mathsf{P} = \mathsf{NP}.$$

(Abiteboul, Vardi and Vianu)

Infinite Structures

For any ordinal α, a binary string b of length α is a map

$$b : \alpha \to \{0, 1\}.$$

For a pair of formulas $\varphi_0(R, \mathbf{x})$ and $\varphi_1(R, \mathbf{x})$, define:

$$\begin{aligned}
\varphi^{\epsilon} &= \emptyset \\
\varphi^{b\cdot 0} &= \varphi_0(\varphi^b/R) \cup \varphi^b \\
\varphi^{b\cdot 1} &= \varphi_1(\varphi^b/R) \cup \varphi^b \\
\varphi^{b} &= \bigcup_{c<b} \varphi^c \qquad \text{when } b \text{ is of limit ordinal length}
\end{aligned}$$

here, $c < b$ means that c is a proper initial segment of b.

Well Ordered Structures

On well-ordered structures:

$$\mathsf{NFP} = \mathsf{SO}$$

NFP $\leq$ SO holds always

by a direct translation (*viz.* IFP).

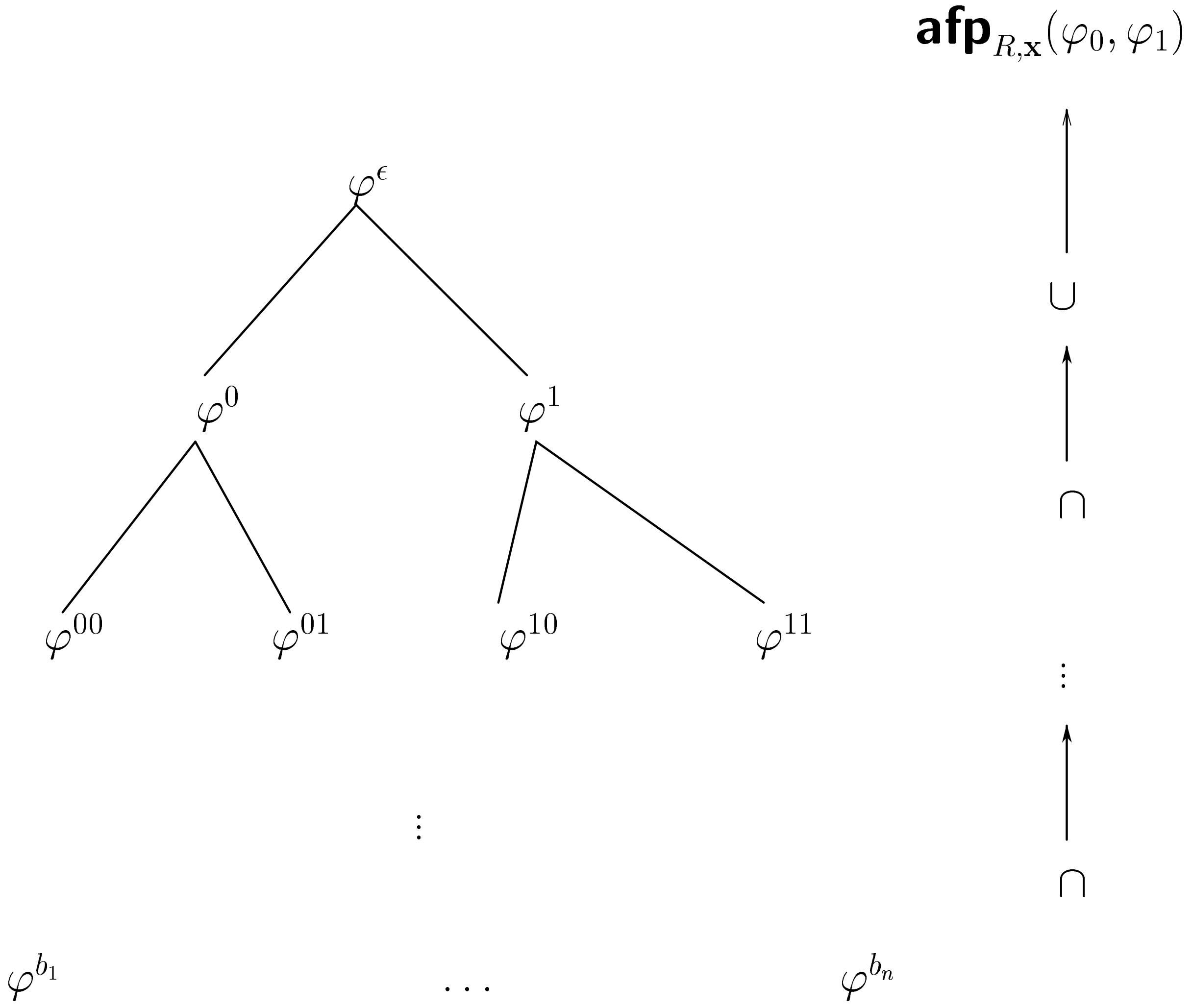
In the other direction, to translate $\exists R\,\theta$,

Intuitively, we can write φ_0 and φ_1 so that at each stage,

φ_0 adds the next element in the order to R.
φ_1 does not.

Alternations of **nfp** and $\neg$ correspond exactly to *quantifier alternations*.

Alternating Fixed Point Logic



$$\mathrm{AFP} = \mathrm{PFP}$$

Games

Player 1 and Player 2 take turns in deciding which of the two formulas

$$\varphi_0 \text{ or } \varphi_1$$

to apply.

Player 1 plays at limit ordinals.

A tuple $\mathbf{t}$ is in

$$\textbf{afp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)$$

if Player 1 has a winning strategy for placing it in the constructed relation.

In what situations is this game determined?

Determinacy

There are (uncountable) structures for which there are formulas with undetermined games.

(adapting Gale-Stewart. Requires AC)

If a fixed point is always reached at ω, then the game is determined.

The set of ω-length binary strings b such that $\mathbf{x} \in \varphi^b$ is necessarily an open set

(again Gale-Stewart.)

What happens on countable structures where the game does not close at ω?

There is a countable structure, and a pair of first order formulas such that the determinacy of the game is independent of ZFC.

There is a countable structure and a pair of first-order formulas where the closure ordinal is not bounded by any countable ordinal