Mathematical Tripos Part III
Finite Model Theory

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Exercise Sheet 3

1. The 0-1 law for first-order logic was obtained for purely relational vocabularies. It does not hold in the presence of constant or function symbols: consider the sentence $P(c)$ in a language with constant c and a unary predicate P. However, if there are no function symbols of positive arity (i.e. constant symbols are permitted), one can establish the following, which specialises to the 0-1 law in the case of relational vocabularies:

   (a) *Almost everywhere quantifier elimination:* For every sentence ϕ, there is a quantifier-free sentence θ such that $\mu(\phi \leftrightarrow \theta) = 1$.

   (b) Is this also true of $L^{\omega}_{\infty\omega}$?

   This, too, fails if we have a function symbol with arity at least 1.

   (c) Show that there is no quantifier-free sentence that is "almost everywhere equivalent" (in the sense of part (a)) to the sentence $\forall x(f(x) \neq x)$.

2. For any constant $0 < p < 1$, we use $\mathcal{G}_{n,p}$ to denote the probability distribution on directed graphs on the set of vertices $\{1, \ldots, n\}$ obtained by assigning, independently, to each pair (i, j) an edge with probability p. For any sentence ϕ in the language of graphs, we write $\mu_{n,p}(\phi)$ for the probability that ϕ is true in $\mathcal{G}_{n,p}$.

   Show that, for all ϕ, $\lim_{n\to\infty} \mu_{n,p}(\phi)$ is defined and is either 0 or 1.

3. Recall that a graph is 2-colourable if, and only if, it contains no cycle of odd length. Show (using Hanf's locality condition or otherwise) that 2-colourability is not definable in first-order logic. Do the same for 3-colourability.

4. If σ is a relational signature, and $\mathcal{A}$ and $\mathcal{B}$ are σ-structures, write $\mathcal{A}+\mathcal{B}$ for the structure whose universe is the disjoint union of the universes of $\mathcal{A}$ and $\mathcal{B}$ and where each relation symbol R is interpreted by the corresponding union of its interpretations in $\mathcal{A}$ and $\mathcal{B}$. Similarly, write $n\mathcal{A}$ for the disjoint union of n copies of $\mathcal{A}$.

   (a) Show that, if $\mathcal{A} \equiv^k \mathcal{A}'$ and $\mathcal{B} \equiv^k \mathcal{B}'$, then $\mathcal{A} + \mathcal{B} \equiv^k \mathcal{A}' + \mathcal{B}'$.

   (b) Show that, for $n, m \geq k$, $n\mathcal{A} \equiv^k m\mathcal{A}$.

   With $\mathcal{A}$ and $\mathcal{B}$ as above, define $\mathcal{A} \times \mathcal{B}$ to be the structure whose universe is the Cartesian product of A and B, and where an m-ary relation symbol R is interpreted by the set of tuples $((a_1, b_1), \ldots, (a_m, b_m))$ such that $(a_1, \ldots, a_m) \in R^{\mathcal{A}}$, and $(b_1, \ldots, b_m) \in R^{\mathcal{B}}$. Similarly, write $\mathcal{A}^k$ for the structure that is the product of k disjoint copies of $\mathcal{A}$.

(c) Show that, if $\mathcal{A} \equiv^k \mathcal{A}'$ and $\mathcal{B} \equiv^k \mathcal{B}'$, then $\mathcal{A} \times \mathcal{B} \equiv^k \mathcal{A} \times \mathcal{B}'$.

(d) Show that, for each p, there is an n_p such that if $n, m \geq n_p$, then $\mathcal{A}^n \equiv_p^k \mathcal{A}^m$.

5. Consider a structure $\mathcal{E} = (A, E)$ where E is an equivalence relation on the set A, and let e_i denote the number of equivalence classes of E with exactly i elements. Define the k-index of E to be the k-tuple $(n_1, \ldots, n_k)$ where, for $i < k$, $n_i = \min(k, e_i)$ and $n_k = \min(k, \sum_{i \geq k} e_i,)$.

   (a) Show that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two such structures with the same k-index, then $\mathcal{E}_1 \equiv^k \mathcal{E}_2$.

   For an isomorphism-closed class of finite structures $\mathcal{C}$, we say that $\mathcal{C}$ is k-compact if there are finitely many $\equiv^k$-equivalence classes of structures in $\mathcal{C}$.

   (b) Show that, on any k-compact class $\mathcal{C}$, $L^k_{\infty\omega}$ is no more expressive than L^k, i.e. for each sentence ϕ of the former, there is a sentence ψ of the latter such that $\phi \leftrightarrow \psi$ holds on all structures in $\mathcal{C}$.

   (c) Deduce from (a) and (b) that LFP is no more expressive than first-order logic on the class of finite equivalence relations.

6. Show that $(\mathbb{Z}, <) \equiv^2_{\infty\omega} (\mathbb{Q}, <)$ but $(\mathbb{Z}, <) \not\equiv^3 (\mathbb{Q}, <)$.

7. Let $\mathcal{A} = (A, <)$ and $\mathcal{B} = (B, <)$ be two linear orders (not necessarily finite), and $\mathbf{a}$ and $\mathbf{b}$ be two n-tuples of elements from A and B respectively, in increasing order. Suppose that for each $i < n$, $(\mathcal{A}, a_i, a_{i+1}) \equiv^3_p (\mathcal{B}, b_i, b_{i+1})$. Show that $(\mathcal{A}, \mathbf{a}) \equiv_p (\mathcal{B}, \mathbf{b})$ (i.e. without restriction on the number of variables).

   Conclude that, on linear orders, every first-order sentence is equivalent to a sentence of L^3.

   Generalise the above to linear orders with additional unary relations.

8. Let γ be the signature of two binary relations: $<$ and E.

   (a) Show that any isomorphism-closed class of finite γ-structures in which $<$ is a linear order is definable in $L^3_{\infty\omega}$.

   Let χ_n denote the first-order sentence that asserts that $<$ is a linear order, and E is an equivalence relation with at least n distinct equivalence classes.

   (b) Show that, if infinite structures are admitted, χ_{k+1} is not equivalent to any sentence of $L^k_{\infty\omega}$. (Hint: consider structures where $<$ is dense, and the equivalence classes of E are all dense in each other.)

   (c) Show that, on finite structures, χ_n is equivalent to a sentence of L^3, for any n.