Mathematical Tripos Part III
Finite Model Theory

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Exercise Sheet 2

1. A second-order Horn sentence ($\mathsf{SO\text{-}Horn}$ sentence, for short) is one of the form
$$Q_1R_1 \dots Q_pR_p(\forall \mathbf{x} \bigwedge_i C_i)$$
where, each Q_i is either $\exists$ or $\forall$, each R_i is a relational variable and each C_i is a *Horn* clause, which is defined for our purposes as a disjunction of atomic and negated atomic formulae such that it contains at most one positive occurrence of a relational variable. A sentence is said to be $\mathsf{ESO\text{-}Horn}$ if it is as above, and all Q_i are $\exists$.

   (a) Show that any $\mathsf{ESO\text{-}Horn}$ sentence in a relational signature defines a class of structures decidable in polynomial time.

   (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and
$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$
is decidable in polynomial time, then there is a $\mathsf{ESO\text{-}Horn}$ sentence that defines K.

   (c) Show that any $\mathsf{SO\text{-}Horn}$ sentence is equivalent to a $\mathsf{ESO\text{-}Horn}$ sentence.

2. The *directed graph reachability problem* is the problem of deciding, given a structure (V, E, s, t) where E is an arbitrary binary relation on V, and $s, t \in V$, whether (s, t) is in the reflexive-transitive closure of E. This problem is known to be decidable in NL.

   Transitive closure logic is the extension of first-order logic with an operator $\mathbf{tc}$ which allows us to form formulae
$$\phi \equiv [\mathbf{tc}_{\mathbf{x},\mathbf{y}}\psi](\mathbf{t}_1, \mathbf{t}_2)$$
where $\mathbf{x}$ and $\mathbf{y}$ are k-tuples of variables and $\mathbf{t}_1$ and $\mathbf{t}_2$ are k-tuples of terms, for some k; and all occurrences of variables $\mathbf{x}$ and $\mathbf{y}$ in ψ are bound in ϕ. The semantics is given by saying, if $\mathbf{a}$ is an interpretation for the free variables of ϕ, then $\mathcal{A} \models \phi[\mathbf{a}]$ just in case $(\mathbf{t}_1^{\mathbf{a}}, \mathbf{t}_2^{\mathbf{a}})$ is in the reflexive-transitive closure of the binary relation defined by $\psi(\mathbf{x}, \mathbf{y})$ on A^k.

   (a) Show that any class of structures definable by a sentence ϕ, as above, where ψ is first-order, is decidable in NL.

(b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

is decidable in NL, then there is a sentence of transitive-closure logic that defines K.

3. For a binary relation E on a set A, define its *deterministic transitive closure* to be the set of pairs (a, b) for which there are $c_1, \ldots, c_n \in A$ such that $a = c_1$, $b = c_n$ and for each $i < n$, c_{i+1} is the *unique* element of A with $(c_i, c_{i+1}) \in E$.

   Let DTC denote the logic formed by extending first-order logic with an operator **dtc** with syntax analogous to **tc** above, where $[\mathbf{dtc}_{\mathbf{x},\mathbf{y}}\psi]$ defines the deterministic transitive closure of $\psi(\mathbf{x}, \mathbf{y})$.

   (a) Show that every sentence of DTC defines a class of structures decidable in L.

   (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

   is decidable in L, then there is a sentence of DTC that defines K.

4. Show that every sentence of PFP defines a class of structures decidable in PSPACE, and that PFP captures PSPACE on *ordered* structures, in the same sense as above.

5. Suppose ϕ is formula of PFP, R is a relational variable, and $\mathcal{O}$ is the class of structures that interpret the symbol $<$ as a linear order. Show there is a formula of PFP that is equivalent to $\exists R\phi$ on all structures in $\mathcal{O}$. Use this fact to conclude that a class K of structures is definable by a sentence of the form $\exists R\phi$ (where ϕ is in PFP) if, and only if, $\{[\mathcal{A}]_< \mid \mathcal{A} \in K \text{ and } < \text{ is any order on } A\}$ is in PSPACE.

6. For a signature σ, a canonical labelling function for σ-structures is a function l on strings such that, if $\mathcal{A}$ is a finite σ-structure and $<$ an order on its universe, then $l([\mathcal{A}]_<) = [\mathcal{A}]_{<'}$, for some order $<'$; and if $<_1$ and $<_2$ are any orders on the universe of $\mathcal{A}$, $l([\mathcal{A}]_{<_1}) = l([\mathcal{A}]_{<_2})$.

   Show that, if there is a polynomial-time computable canonical labelling function for σ-structures, then the polynomial-time properties of σ-structures are recursively enumerable.