TO APPEAR IN A SPECIAL ISSUE OF *Annals of Pure and Applied Logic* OF PAPERS FROM LICS '94.

Nonaxiomatisability of equivalences over finite state processes

Peter Sewell
Computer Laboratory, University of Cambridge,
New Museums Site, Pembroke Street, Cambridge, CB2 3QG.
Peter.Sewell@cl.cam.ac.uk

May 12, 1997

Abstract

This paper considers the existence of finite equational axiomatisations of behavioural equivalences over a calculus of finite state processes. To express even simple properties such as $\mu xE = \mu xE[E/x]$ some notation for substitutions is required. Accordingly the calculus is embedded in a simply typed lambda calculus, allowing such schemas to be expressed as equations between terms containing first order variables. A notion of first order trace congruence over such terms is introduced and used to show that no finite set of such equations is sound and complete for any reasonable equivalence finer than trace equivalence. The intermediate results are then applied to give two nonaxiomatisability results over calculi of regular expressions.

Keywords: Nonaxiomatisability, Equational Logic, Process Algebra, Regular Expressions, Behavioural Equivalences

1 Introduction

Nondeterministic finite state machines are, in their various formalisations, the basis for models or specifications of many computational phenomena. A common formalisation is the labelled transition system, consisting of a (finite) set equipped with an indexed family of binary relations over it. Typically the set is thought of as the possible states that a modelled system may be in, with the relations as the allowable changes of state. In applications it is often desirable to identify labelled transition systems that are in some sense behaviourally equivalent. Among the notions of behavioural equivalence that have been proposed are the trace equivalence of Hoare

[Hoa85] and the bisimulation equivalence of Park [Par81]. A survey of these and other notions, differing in their treatment of nondeterministic choice and termination, has been given by van Glabbeek [Gla90]. Given the additional structure of a termination predicate on states one can also define the language equivalence of Kleene [Kle56].

Direct presentations of labelled transition systems as sets and relations are awkward to work with. Accordingly, syntactic forms have been introduced to represent them, including a variety of process calculi and regular expressions. We will largely be concerned with a simple syntax, the μ-expressions of [Mil84], with zero, prefix, summation and a binding operator for recursion.

Definition The *μ-expressions* are those of the grammar

$$E ::= 0 \;\Big|\; x \;\Big|\; aE \;\Big|\; E + E \;\Big|\; \mu x E$$

where x and a are drawn from countably infinite sets V and Act of variables and actions and μ is a binding operator. We adopt standard notions of free and bound variables and substitution and work up to alpha conversion. The scope of a binder is generally as far to the right as possible. Sum is taken to have lower precedence than prefix so $aE + F$ is $(aE) + (F)$. For $n \geq 1$ we define $a^{n+1}E = aa^nE$ and $a^1E = aE$.

There is an extensive literature concerned with the axiomatisation of behavioural equivalences over the μ-expressions (and other simple process calculi), with several motivations. The most obvious is that any sound system may be useful for human or machine manipulation of terms, particularly but not necessarily if it is complete. This must be qualified by the existence of efficient decision procedures over finite labelled transition systems. Completeness results also permit a comparison of different equivalences and with the alternative view that takes a set of axioms as primary. For this paper a more important motivation is that axiomatisability results (and especially the proofs of completeness or nonexistence) shed light on the nature of the equivalences involved and on the expressiveness of the calculus as compared with the expressiveness of the metalanguage of axioms. It is obviously desirable to have completeness results using as weak (and nonexistence results using as strong) a metalanguage as possible.

A number of complete systems have been given that contain an impure Horn clause expressing the fact that certain equations have unique solutions (together with a finite set of equational axioms). The first seems to be that for language equivalence of $*$-expressions by Salomaa [Sal66]. For μ-expressions there are complete systems for bisimulation [Mil84], weak bisimulation congruence [Mil89], branching bisimulation congruence [Gla93a], divergence bisimulation [Gla93b] and trace congruence [Rab93]. The system of Milner for bisimulation [Mil84] is typical, using the implication

$$E = F[E/x] \wedge x \text{ guarded in } F \quad \rightarrow \quad E = \mu x F$$

where x is guarded in F if every free occurrence of x in F is contained in a subexpression aG. The use of this auxiliary predicate was shown to be unnecessary by Bloom and Ésik, who give in [BÉ94] a finite pure Horn clause system for bisimulation using the 'GA implication':

$$\mu z E[zz/xy] = \mu z F[zz/xy] \quad \rightarrow \quad \mu z E[zz/xy] = \mu x F[\mu y E\,/y]$$

in which it is assumed that z is not free in E or F.

In this paper we confirm the intuition that the use of an implication is essential, showing that there is no finite equational axiomatisation for any of a wide class of equivalences over μ-expressions.

To state the result a precise definition of the equational axiomatisations under consideration is required, preferably as permissive as possible. For a syntax with variable binding, such as the μ-expressions, there does not seem to be a canonical definition. To equationally express anything of interest about fixed points, such as the simple properties below, some notation for substitution is required.

$$\begin{aligned}
\mu x E &= \mu x E[E/x] \\
\mu x E &= E[\mu x E\,/x] \\
\mu x E[F/x] &= E[\mu x F[E/x]\,/x] \\
\mu x\ E[x,x,x,x] &= \mu x\ E[x,x,\mu y\ E[x,y,x,y],\mu y\ E[x,y,x,y]]
\end{aligned}$$

Instead of considering axioms containing substitutions explicitly we will embed the μ-expressions in a simply typed lambda calculus and work up to $\beta\eta$ equality. Axioms such as the above can be written as equations containing variables of higher type rather than as equation schemes, with substitution appearing only in the rules defining $\beta\eta$ equality. This simplifies the technical development and also gives added significance to some of the intermediate results as the terms of higher type can be viewed as a fragment of a higher order process calculus (such as the higher order π calculus of Sangiorgi [San93]).

The main theorem, stated in §2 and proved in §3,4, asserts the nonexistence of finite axiomatisations containing at most first order variables. These axiomatisations may contain (the embeddings of) equation schemes such as those above.

The results of §3,4 can be applied to give a range of non-finite-axiomatisability results over finite state processes expressed with iteration instead of explicit recursion, as regular expressions of various kinds. This is done in §5.

An overview of some of the literature and a discussion of possible generalisations are contained in §6.

This work is a development of that presented in [Sew94, Sew95]. It differs primarily in the main result has been generalized to all reasonable equivalences finer than trace equivalence, rather than only those finer than bisimulation.

2 Basic definitions

This section contains the basic definitions required for a statement of the main non-axiomatisability theorem. We first define trace equivalence and bisimulation over the closed μ-expressions, via a definition of the labelled transition system denoted by a μ-expression.

Definition The relations $\xrightarrow{a} \mid a \in Act$ are the least over the μ-expressions such that

$$\frac{}{aE \xrightarrow{a} E} \qquad \frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} \qquad \frac{F \xrightarrow{a} F'}{E + F \xrightarrow{a} F'} \qquad \frac{E \xrightarrow{a} E'}{\mu y E \xrightarrow{a} E'[\mu y E\,/y]}$$

The rule for μ differs from the more usual

$$\frac{E[\mu y E\,/y] \xrightarrow{a} E'}{\mu y E \xrightarrow{a} E'}$$

but is slightly more convenient. It is straightforward to check that (in the absence of parallel composition) it is equipotent.

Among the finest of behavioural equivalences is bisimulation, at the top of the linear-branching time hierarchy of van Glabbeek [Gla90]. It takes full account of the nondeterministic branching structure of the transition relations.

Definition Bisimulation, written $\sim$, is the largest relation over the closed μ-expressions such that if $E \sim F$ then for all $a \in Act$

- If $E \xrightarrow{a} E'$ then $\exists F' \,.\, F \xrightarrow{a} F' \wedge E' \sim F'$.
- If $F \xrightarrow{a} F'$ then $\exists E' \,.\, E \xrightarrow{a} E' \wedge E' \sim F'$.

At the bottom of the linear-branching time hierarchy are various forms of trace or language equivalence.

Definition The *trace set* of a closed μ-expression E is the subset of Act^* containing $a_1 \dots a_m$ if there exist $E_1, \dots, E_m$ such that $E \xrightarrow{a_1} E_1 \cdots \xrightarrow{a_m} E_m$. Two expressions are *trace equivalent* , written $E =_{\mathrm{tr}} E'$, if they have the same trace sets.

Our interest in the μ-expressions, as opposed to the regular expressions, is partly due to the expressiveness results of Milner [Mil84]. It is shown there that the μ-expressions suffice to express all finite labelled transition systems up to bisimulation (and hence up to all coarser equivalences) but that the regular expressions do not.

2.1 Lambda calculus

We now embed the μ-expressions into a simply typed lambda calculus, in which interesting equations can be expressed. We take a single base type P and a set Con of constants, ranged over by c, as follows.

$$\begin{array}{l} 0 : P \\ a : P \to P \quad \text{for each } a \in Act \\ + : P \to P \to P \\ \mathit{fix} : (P \to P) \to P \end{array}$$

We take a set Var of variables equipped with an assignment of types, with a countable infinity of variables mapped onto each type and $\{ x \mid x : P \in Var \} = V$. The typed terms are given by the rules in Figure 1. The free variables $\mathrm{fv}(M)$ of a typed term M are as usual. A typed equation $M = N : \sigma$ consists of a type and a pair of terms such that $M : \sigma$ and $N : \sigma$. Typed $\beta\eta$ equality is given by the rules in Figure 2. If $\mathcal{E}$ is a set of typed equations we write $\mathcal{E} \vdash M = N : \sigma$ if $M = N : \sigma$ is derivable in the system for $\beta\eta$ equality augmented with the rule

$$\frac{(M = N : \sigma) \in \mathcal{E}}{M = N : \sigma}\ ax\ .$$

We will work up to $\beta\eta$ equality, using abstraction to allow parameterised equations. This is in contrast to taking β-reduction to be of comparable computational interest

$$\frac{x : \sigma \in Var}{x : \sigma}\ var \qquad \frac{c : \sigma \in Con}{c : \sigma}\ cst$$

$$\frac{x : \sigma \quad M : \tau}{(\lambda x{:}\sigma.\, M) : \sigma \to \tau} \to \mathit{Intro} \qquad \frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \to \mathit{Elim}$$

Figure 1: Lambda terms

$$\frac{M:\sigma}{M=M:\sigma}\ \mathit{ref} \qquad \frac{M=N:\sigma}{N=M:\sigma}\ \mathit{sym}$$

$$\frac{L=M:\sigma \quad M=N:\sigma}{L=N:\sigma}\ \mathit{tran} \qquad \frac{M:\tau \quad x:\sigma \quad y:\sigma \quad y\notin \mathrm{fv}(M)}{\lambda x{:}\sigma.\,M=\lambda y{:}\sigma.\,M[y/x]:\sigma\to\tau}\ \alpha$$

$$\frac{M=N:\tau \quad x:\sigma}{(\lambda x{:}\sigma.\,M)=(\lambda x{:}\sigma.\,N):\sigma\to\tau}\ \xi \qquad \frac{M=M':\sigma\to\tau \quad N=N':\sigma}{MN=M'N':\tau}\ \mu$$

$$\frac{M:\tau \quad N:\sigma \quad x:\sigma}{(\lambda x{:}\sigma.\,M)N=M[N/x]:\tau}\ \beta \qquad \frac{M:\sigma\to\tau \quad x:\sigma \quad x\notin \mathrm{fv}(M)}{\lambda x{:}\sigma.\,(Mx)=M:\sigma\to\tau \quad x:\sigma}\ \eta$$

Figure 2: $\beta\eta$ equality

to the labelled transitions, e.g. in the work of Nielson [Nie89]. Some candidate axioms (corresponding to the axiom schemes given earlier) are below, taking variables $x:P$, $y:P$, $e:P\to P$, $f:P\to P$ and $z:P\to P\to P\to P\to P$.

$$\mathit{fix}\ e=\mathit{fix}\ \lambda x{:}P.\,e(ex):P$$

$$\mathit{fix}\ e=e(\mathit{fix}\ e):P$$

$$\mathit{fix}\ \lambda x{:}P.\,e(fx)=e(\mathit{fix}\ \lambda x{:}P.\,f(e(x))):P$$

$$\mathit{fix}\ \lambda x{:}P.\,zxxxx=\mathit{fix}\ \lambda x{:}P.\,zxx(\mathit{fix}\ \lambda y{:}P.\,zxyxy)(\mathit{fix}\ \lambda y{:}P.\,zxyxy):P$$

These equations only contain variables of base or first order types. The proof of the main theorem will depend strongly upon a restriction to such equations. To state this restriction precisely we define the order of a type as usual:

$$\begin{aligned}\mathrm{order}(P) &= 0\\ \mathrm{order}(\sigma\to\tau) &= \max\{1+\mathrm{order}(\sigma),\mathrm{order}(\tau)\}\end{aligned}$$

and take the order of a set $\mathcal{E}$ of typed equations to be the least upper bound (in the integers extended with limit points $-\infty,+\infty$) of the orders of the types of variables occurring (free or bound) in $\mathcal{E}$. If $m\geq 0$ we write T^m for the set of alpha-equivalence classes of terms E in long $\beta\eta$ normal form such that $E:P$ and $\mathrm{order}(\mathrm{fv}(E))<m$. There is an evident bijection between the closed μ-expressions and the terms in T^1, with for example

$$\mu x\ a0+x \leftrightarrow \mathit{fix}\ \lambda x{:}P.\,+\ (a0)\ (x).$$

Any equivalence $\simeq$ over closed μ-expressions thus induces an equivalence over T^1. If $\mathcal{E}$ is a set of typed equations then $\mathcal{E}$ is sound for $\simeq$ if

$$\forall E,F\in T^1\ .\ \mathcal{E}\vdash E=F:P \Rightarrow E\simeq F$$

and complete for $\simeq$ if

$$\forall E, F \in T^1 \,.\, \mathcal{E} \vdash E = F : P \Leftarrow E \simeq F.$$

The main theorem can now be stated.

Theorem 1 *If $\simeq$ is an equivalence over the closed μ-expressions that is finer than (or identical to) trace equivalence and for some $a \in Act$ and all $n \geq 1$ satisfies*

$$\mu x\ ax \quad \simeq \quad \mu x\ a^n x.$$

then there is no finite set of typed equations of order 1 that is sound and complete for $\simeq$.

3 First order traces

The proof of Theorem 1 rests on the fact that an equation that is sound for trace equivalence can only affect the lengths of recursive loops in a rather constrained way. For example, repeated use of the equation scheme $\mu x E = \mu x E[E/x]$ can change the length of a loop only by factors of 2, e.g. for any $n \geq 1$ it can derive the 'internal' unfolding

$$\mu x\ a^n x = \mu x\ a^{2^m n} x$$

for any $m \geq 0$ but not

$$\mu x\ a^n x = \mu x\ a^{pn} x$$

for any prime $p > 2$. We show that for any finite set of sound equations there is some bound corresponding to this '2'.

We first note that any first order set of typed equations can, without loss of generality, be taken to consist of equations of the form $E = F : P$ where E and F are in T^2. In this section we characterise the first order equations that are sound for trace equivalence. We give an extended labelled transition system over T^2 (traces were initially only defined over T^1) and hence an extended notion of trace congruence $=_{\mathrm{t}}$ over T^2. After showing some basic properties of the extended transition system we show that $E = F : P$ is sound iff $E =_{\mathrm{t}} F$. In §4 we define the 'loops' of a term in T^2 and show that they are, in a certain sense, preserved by reasoning from any finite first order set of sound equations. Theorem 1 then follows immediately.

Notation We let E, F, G, A range over T^2, m range over the natural numbers, n, p, q range over the non-zero natural numbers. For $n \geq 1$ the type P^n is defined by $P^1 = P$ and $P^{n+1} = P \to P^n$. We let w, x, y range over variables of type P and

z range over variables of type P^{m+1} or P^{n+1}. We assume that all expressions are reduced to long $\beta\eta$ normal form. The terms in T^2 can be described explicitly as those of the grammar

$$E ::= 0 \;\Big|\; aE \;\Big|\; E+F \;\Big|\; \mathit{fix}\ \lambda x{:}P.\, E \;\Big|\; zE_1 \ldots E_m$$

where $a \in \mathit{Act}$, $x : P$, $m \geq 0$ and $z : P^{m+1}$. We write $+$ infix and $\vec{E}$ for $E_1 \ldots E_m$.

In order to extend the labelled transition system semantics of closed μ-expressions to T^2 two new cases must be considered — x where $x : P$ and $z\vec{E}$ where $z : P^{n+1}$ for some $n \geq 1$. The former can be dealt with using a judgment $E \rhd x$, pronounced 'E sees x', as in the definition of bisimulation of open μ-expressions of Milner [Mil84]. For the latter we introduce new labelled transitions as below, with labels that are pairs of a variable z and an $i \in 1..n$. The pair $\langle z, i\rangle$ will usually be written zi.

Definition We take labels $\mathit{Lab} \stackrel{\mathit{def}}{=} \mathit{Act} \uplus \{\, zi \mid \exists n \geq 1 \,.\, z : P^{n+1} \wedge i \in 1..n \,\}$. We let a range over Act and u range over Lab. We define binary relations $\xrightarrow{u}$ for $u \in \mathit{Lab}$ and $\rhd$ over T^2 as the least such that

$$\frac{}{aE \xrightarrow{a} E} \qquad \frac{z : P^{n+1} \quad i \in 1..n}{z\vec{E} \xrightarrow{zi} E_i} \qquad\qquad \frac{}{E \rhd E}$$

$$\frac{E \xrightarrow{u} E'}{E+F \xrightarrow{u} E'} \text{ and sym.} \qquad\qquad \frac{E \rhd E'}{E+F \rhd E'} \text{ and sym.}$$

$$\frac{E \xrightarrow{u} E'}{\mathit{fix}\ \lambda x{:}P.\, E \xrightarrow{u} E'[\mathit{fix}\ \lambda x{:}P.\, E\,/x]} \quad \frac{E \rhd E'}{\mathit{fix}\ \lambda x{:}P.\, E \rhd E'[\mathit{fix}\ \lambda x{:}P.\, E\,/x]}$$

These relations satisfy the following basic properties.

Lemma 1 *For all $E, F, G \in T^2$, $x, y : P$, $z : P^{n+1}$ and substitutions ρ:*

1. *If $E \xrightarrow{u} F$ and $\neg\exists z \in \mathrm{dom}(\rho), i \,.\, u = zi$ then $E\rho \xrightarrow{u} F\rho$.*
2. *If $E \rhd F$ then $E\rho \rhd F\rho$.*
3. *If $E[F/x] \xrightarrow{u} G$ then either $E \rhd x \wedge F \xrightarrow{u} G$ or $\exists E' \,.\, E \xrightarrow{u} E' \wedge E'[F/x] = G$.*
4. *If $E[F/x] \rhd G$ then either $E \rhd x \wedge F \rhd G$ or $\exists E' \,.\, E \rhd E' \wedge E'[F/x] = G$.*
5. *If $E \rhd F \xrightarrow{u} G$ then $E \xrightarrow{u} G$.*
6. *If $E \rhd F \rhd G$ then $E \rhd G$.*
7. *If $E \xrightarrow{u} F$ then $\mathrm{fv}(F) \subseteq \mathrm{fv}(E)$.*
8. *If $E \rhd F$ then $\mathrm{fv}(F) \subseteq \mathrm{fv}(E)$.*

9. *If $E[F/x] \rhd y$ then either $E \rhd y \neq x$ or $E \rhd x \wedge F \rhd y$.*

10. *If $E \xrightarrow{zi} F$ then $\exists \vec{E} \,.\, E \rhd z\vec{E} \wedge E_i = F$.*

PROOF Straightforward inductions on the derivations of the judgments. □

Notation If S is a set we write S^* and S^+ for the sets of sequences and non-empty sequences over S. We write the empty sequence as ϵ and sequence concatenation with juxtaposition, or occasionally with $\_\cdot\_$. We let h, k, l, t range over Lab^* and write l^n for the n-ary concatenation $l \ldots l$. If R is a binary relation we write its transitive closure as R^+ and its reflexive transitive closure as R^*. We write $\longrightarrow$ for $\cup_{u \in Lab} \xrightarrow{u}$. If $l = u_1 \ldots u_m$ we write $\xrightarrow{l}$ for the relational composition $\xrightarrow{u_1} \ldots \xrightarrow{u_m}$.

Definition The *trace set* and *extended trace set* of an $E \in T^2$ are the subsets of Lab^* and $Lab^* \times \{\, x \mid x : P \in \mathit{Var} \,\}$

$$\begin{array}{rcl} \mathrm{tr}(E) & \stackrel{\mathit{def}}{=} & \{\, l \mid \exists F \,.\, E \xrightarrow{l} F \,\} \\ \mathrm{et}(E) & \stackrel{\mathit{def}}{=} & \{\, l, x \mid \exists F \,.\, E \xrightarrow{l} F \rhd x \,\}. \end{array}$$

Two members E,F of T^2 are *trace congruent*, written $E =_{\mathrm{t}} F$, if they have the same traces and extended traces.

Lemma 2 *Trace equivalence ($=_{\mathrm{tr}}$) of closed μ-expressions coincides with trace congruence ($=_{\mathrm{t}}$) over T^1.*

PROOF The relations $\xrightarrow{a}$ for $a \in Act$ over the closed μ-expressions and T^1 agree and the relations $\xrightarrow{zi}$ restricted to $T^1 \times T^2$ are empty. Moreover for $E \in T^1$ and $x : P$ it is clear that $\neg(E \rhd x)$. □

Elements of T^2 are finite state in the following sense.

Lemma 3 *For any E the set $\{\, F \mid E \longrightarrow^* F \,\}$ is finite. We write $|E|$ for the size of this set.*

PROOF Letting $\mathrm{der}(E) \stackrel{\mathit{def}}{=} \{\, F \mid E \longrightarrow^+ F \,\}$ it is straightforward to show the following.

$$\begin{array}{rcll} \mathrm{der}(0) & = & \{\} & \\ \mathrm{der}(aE) & = & \{E\} \cup \mathrm{der}(E) & \\ \mathrm{der}(E+F) & = & \mathrm{der}(E) \cup \mathrm{der}(F) & \\ \mathrm{der}(z\vec{E}) & = & \cup_{i \in 1..m}(\{E_i\} \cup \mathrm{der}(E_i)) & \quad \text{for } z : P^{m+1} \text{ and } m \geq 0 \\ \mathrm{der}(\mathit{fix}\ \lambda x{:}P.\, E) & = & \{\, F[\mathit{fix}\ \lambda x{:}P.\, E\,/x] \mid F \in \mathrm{der}(E) \,\} & \end{array}$$

(The only interesting case is the inclusion $\subseteq$ for the *fix* $\lambda x{:}P.\,E$ case, which follows from Lemma 1, part 3.) The result follows by induction on E. $\Box$

Lemma 4 *If* $E =_{\mathrm{t}} F$ *then* $\mathrm{fv}(E) = \mathrm{fv}(F)$.

PROOF This follows from the observations $x : P \in \mathrm{fv}(E) \Rightarrow \exists l\ .\ E \xrightarrow{l} \rhd x$ and $z : P^{n+1} \in \mathrm{fv}(E) \Rightarrow \exists l\ .\ E \xrightarrow{l} \xrightarrow{z1}$, which can be shown by induction on E. $\Box$

The remainder of this section is devoted to showing that $=_{\mathrm{t}}$ is in fact a congruence and moreover is that induced by $=_{\mathrm{tr}}$. We first show a sequence of technical results relating the transition system and substitution, Lemma 5 – Corollary 14 (which are perhaps best skimmed on a first reading). We then give characterisations of the trace sets and extended trace sets of compound expressions and hence show that $=_{\mathrm{t}}$ is a congruence. Finally, by constructing a discriminating substitution, we show that if $E = F : P$ is sound for $=_{\mathrm{tr}}$ then $E =_{\mathrm{t}} F$.

For the rest of this section we let ρ range over substitutions such that, for $m \geq 0$ and $z : P^{m+1} \in \mathrm{dom}(\rho)$, $\rho(z)$ is $\lambda x_1{:}P.\ \ldots \lambda x_m{:}P.\,H_z$ for some $H_z \in T^2$.

Definition

$$\begin{array}{rcl} \mathrm{lab}(\rho) & \stackrel{\mathit{def}}{=} & \{\, zi \mid \exists n \geq 1\ .\ z : P^{n+1} \in \mathrm{dom}(\rho) \wedge i \in 1..n \,\} \\ \mathrm{null}(\rho) & \stackrel{\mathit{def}}{=} & \{\, zi \mid \exists n \geq 1\ .\ z : P^{n+1} \in \mathrm{dom}(\rho) \wedge i \in 1..n \wedge H_z \rhd x_i \,\} \end{array}$$

We first characterise the transitions of a substituted term, generalising Lemma 1 part 3 to substitutions at first order types.

Lemma 5 *If* $E\rho \xrightarrow{u} A$ *then* $\exists j \in \mathrm{null}(\rho)^*$ *such that one of the following hold.*

1. $\exists F\ .\ E \xrightarrow{j} \xrightarrow{u} F \wedge A = F\rho \wedge u \notin \mathrm{lab}(\rho)$
2. $\exists F, z, \vec{F}, H, m \geq 0\ .\ E \xrightarrow{j} F \rhd z\vec{F} \wedge z : P^{m+1} \in \mathrm{dom}(\rho) \wedge H_z \xrightarrow{u} H \wedge A = H[\vec{F}\rho/\vec{x}]$

PROOF We show the result for ρ such that $\mathrm{dom}(\rho) \cap \mathrm{fv}(\mathrm{ran}(\rho)) = \{\}$, allowing the substitution and β reduction to be performed incrementally.

Definition For $u \in \mathit{Lab}$ let $\xrightarrow{u}_{\rho} \subseteq T^2 \times T^2$ be the least relation such that

1. $E \xrightarrow{u} F \wedge u \notin \mathrm{lab}(\rho) \Rightarrow E \xrightarrow{u}_{\rho} F$
2. $E \rhd z\vec{F} \wedge z : P^{m+1} \in \mathrm{dom}(\rho) \wedge \rho(z)\vec{x} \xrightarrow{u} H \wedge \vec{x} \cap \mathrm{fv}(\mathrm{ran}(\rho)) = \{\} \Rightarrow E \xrightarrow{u}_{\rho} H[\vec{F}/\vec{x}]$
3. $E \xrightarrow{u'} \xrightarrow{u}_{\rho} F \wedge u' \in \mathrm{null}(\rho) \Rightarrow E \xrightarrow{u}_{\rho} F$

Definition Let the relation $\longrightarrow_{\beta(\rho)} \subseteq T^2 \times T^2$ be the least relation such that

1. For any $z\!:\!P^{m+1} \in \mathrm{dom}(\rho)$ $z\vec{E} \longrightarrow_{\beta(\rho)} \rho(z)\vec{E}$
2. If $E \longrightarrow_{\beta(\rho)} F$ and $w\!:\!P \notin \mathrm{dom}(\rho)$ then $\mathit{fix}\ \lambda w{:}P.\, E \longrightarrow_{\beta(\rho)} \mathit{fix}\ \lambda w{:}P.\, F$
3. For any $n \geq 1$, variable or constant $c\!:\!P^{n+1}$ and $j \in 1..n$, if $E_j \longrightarrow_{\beta(\rho)} E'_j$ and $\forall i \in 1..n$. $i \neq j \Rightarrow E_i = E'_i$ then $c\vec{E} \longrightarrow_{\beta(\rho)} c\vec{E}'$.

This is related to $\beta\eta$ equality by the following.

Lemma 6 *For all E there is some F such that $E \longrightarrow_{\beta(\rho)}{}^* F$ and $\mathrm{fv}(F) \cap \mathrm{dom}(\rho) = \{\}$.*

PROOF One can show that otherwise $E\rho$ has an infinite sequence of β reductions. □

Lemma 7 *If $E \longrightarrow_{\beta(\rho)} F$ then $E\rho = F\rho$.*

PROOF By induction on $E \longrightarrow_{\beta(\rho)} F$. □

Lemma 8 *If $E \longrightarrow_{\beta(\rho)} \xrightarrow{u}_{\rho} F$ then $E \xrightarrow{u}_{\rho} \longrightarrow_{\beta(\rho)}{}^* F$.*

PROOF By induction on derivations of $\longrightarrow_{\beta(\rho)}$. □

Now suppose $E\rho \xrightarrow{u} A$. By Lemmas 6 and 7 there is an E' such that $E \longrightarrow_{\beta(\rho)}{}^* E'$ and $E\rho = E' \xrightarrow{u} A$. By the definition of $\xrightarrow{u}_{\rho}$ we have $E' \xrightarrow{u}_{\rho} A$ so using Lemma 8 we have $E \xrightarrow{u}_{\rho} E'' \longrightarrow_{\beta(\rho)}{}^* A$ for some E''. Finally by Lemma 7 we have $E''\rho = A$. □

Lemma 5 can be lifted from single actions to sequences of actions. To state the result a pseudo-substitution on traces is required:

Definition If $(u_1 \ldots u_m) \in \mathit{Lab}^*$ and $T \subseteq \mathit{Lab}^*$ then

$$\begin{array}{rcl} (u_1 \ldots u_m)\{\rho\} & \overset{\mathit{def}}{=} & \{\, l_1 \ldots l_m \mid \forall j \in 1..m \ . \ \ \text{if } u_j = zi \in \mathrm{lab}(\rho) \text{ then } H_z \xrightarrow{l_j} \rhd x_i \\ & & \qquad\qquad\qquad\qquad\qquad\quad \text{else } l_j = u_j \,\} \\ T\{\rho\} & \overset{\mathit{def}}{=} & \bigcup_{l \in T} l\{\rho\}. \end{array}$$

Note that if $t \in l\{\rho\}$ and $t' \in l'\{\rho\}$ then $tt' \in ll'\{\rho\}$ and that if that $l \in \mathrm{null}(\rho)^*$ then $\epsilon \in l\{\rho\}$.

Lemma 9 *If $E\rho \xrightarrow{l} A$ then $\exists k \in \mathit{Lab}^*$ such that one of the following hold.*

1. $\exists F$. $E \xrightarrow{k} F \wedge A = F\rho \wedge l \in k\{\rho\}$

2. $\exists F, z, \vec{F}, H, h, m \geq 0 \;.\; E \stackrel{k}{\longrightarrow} F \rhd z\vec{F} \,\wedge\, z : P^{m+1} \in \mathrm{dom}(\rho) \,\wedge\, H_z \stackrel{h}{\longrightarrow} H \,\wedge\, A = H[\vec{F}\rho/\vec{x}] \wedge l \in k\{\rho\} \cdot h$

PROOF By induction on l using Lemma 5. □

This has an approximate converse:

Lemma 10 *If $E \stackrel{l}{\longrightarrow} \rhd F$ and $t \in l\{\rho\}$ then $E\rho \stackrel{t}{\longrightarrow} \rhd F\rho$.*

PROOF By induction on l. □

The analogue of Lemma 5 for $\rhd$ is as follows.

Lemma 11 *If $E\rho \rhd A$ then $\exists j \in \mathrm{null}(\rho)^*$ such that one of the following hold.*

1. $\exists F \,.\, E \stackrel{j}{\longrightarrow} \rhd F \wedge A = F\rho$
2. $\exists F, z, \vec{F}, H, m \geq 0 \;.\; E \stackrel{j}{\longrightarrow} F \rhd z\vec{F} \,\wedge\, z : P^{m+1} \in \mathrm{dom}(\rho) \,\wedge\, H_z \rhd H \,\wedge\, A = H[\vec{F}\rho/\vec{x}]$

PROOF Again, we show the result for ρ such that $\mathrm{dom}(\rho) \cap \mathrm{fv}(\mathrm{ran}(\rho)) = \{\}$.

Definition Let $\rhd_\rho \subseteq T^2 \times T^2$ be the least relation such that

1. $E \rhd F \Rightarrow E \rhd_\rho F$
2. $E \rhd z\vec{F} \,\wedge\, z : P^{m+1} \in \mathrm{dom}(\rho) \,\wedge\, \rho(z)\vec{x} \rhd H \,\wedge\, \vec{x} \cap \mathrm{fv}(\mathrm{ran}(\rho)) = \{\} \Rightarrow E \rhd_\rho H[\vec{F}/\vec{x}]$
3. $E \stackrel{u}{\longrightarrow} \rhd_\rho F \wedge u \in \mathrm{null}(\rho) \Rightarrow E \rhd_\rho F$

Lemma 12 *If $E \longrightarrow_{\beta(\rho)} \rhd_\rho F$ then $E \rhd_\rho \longrightarrow_{\beta(\rho)}{}^* F$.*

PROOF By induction on derivations of $\longrightarrow_{\beta(\rho)}$. □

Now suppose $E\rho \rhd A$. By Lemmas 6 and 7 there is an E' such that $E \longrightarrow_{\beta(\rho)}{}^* E'$ and $E\rho = E' \rhd A$. By the definition of $\rhd_\rho$ we have $E' \rhd_\rho A$ so using Lemma 12 we have $E \rhd_\rho E'' \longrightarrow_{\beta(\rho)}{}^* A$ for some E''. Finally by Lemma 7 we have $E''\rho = A$. □

Corollary 13 *If $E\rho \rhd x$ then $\exists j \in \mathrm{null}(\rho)^*$ such that one of the following hold.*

1. $E \stackrel{j}{\longrightarrow} \rhd x \wedge x \notin \mathrm{dom}(\rho)$
2. $\exists F, z, \vec{F}, m \geq 0 \,.\, E \stackrel{j}{\longrightarrow} F \rhd z\vec{F} \wedge z : P^{m+1} \in \mathrm{dom}(\rho) \wedge H_z \rhd x$

PROOF This follows from Lemma 11 and the result for $E\rho = x$, which can be shown by considering $E \longrightarrow_{\beta(\rho)}{}^* x$. □

Corollary 14 *If $E\rho \rhd$ fix M then $\exists j \in \mathrm{null}(\rho)^*$ such that one of the following hold.*

1. $\exists M' \,.\, E \xrightarrow{j} \rhd \mathit{fix}\, M' \wedge M'\rho = M$

2. $\exists z, \vec{F}, M', m \geq 0 \;.\; E \xrightarrow{j} \rhd z\vec{F} \;\wedge\; z : P^{m+1} \in \mathrm{dom}(\rho) \;\wedge\; H_z \rhd \mathit{fix}\, M' \;\wedge\; M'[\vec{F}\rho/\vec{x}] = M$

PROOF This follows from Lemma 11 and the result for $E\rho = \mathit{fix}\, M$, which can be shown by considering $E \longrightarrow_{\beta(\rho)}{}^{*} \mathit{fix}\, M$. □

The effects on trace sets and extended trace sets of the various operators can now be characterised.

Lemma 15 *If $z : P^{n+1}$ for some $n \geq 1$ and $\vec{x}$ are new then:*

$$
\begin{array}{rcl}
\mathrm{tr}(aE) & = & \{\, al \mid l \in \mathrm{tr}(E) \,\} \cup \{\epsilon\} \\
\mathrm{et}(aE) & = & \{\, al, x \mid l, x \in \mathrm{et}(E) \,\} \\
\mathrm{tr}(E+F) & = & \mathrm{tr}(E) \cup \mathrm{tr}(F) \\
\mathrm{et}(E+F) & = & \mathrm{et}(E) \cup \mathrm{et}(F) \\
\mathrm{tr}(\mathit{fix}\, \lambda x{:}P.\, E) & = & \{\, l_1 \ldots l_m l_{m+1} \mid \\
 & & \quad m \geq 0 \wedge l_{m+1} \in \mathrm{tr}(E) \wedge \forall i \in 1..m \,.\, l_i, x \in \mathrm{et}(E) \,\} \\
\mathrm{et}(\mathit{fix}\, \lambda x{:}P.\, E) & = & \{\, l_1 \ldots l_m l_{m+1}, y \mid \\
 & & m \geq 0 \wedge l_{m+1}, y \in \mathrm{et}(E) \wedge y \neq x \wedge \forall i \in 1..m \,.\, l_i, x \in \mathrm{et}(E) \,\} \\
\mathrm{tr}(z\vec{E}) & = & \{\, \langle z, i\rangle l \mid l \in \mathrm{tr}(E_i) \wedge i \in 1..n \,\} \cup \{\epsilon\} \\
\mathrm{et}(z\vec{E}) & = & \{\, \langle z, i\rangle l, x \mid l, x \in \mathrm{et}(E_i) \wedge i \in 1..n \,\} \\
\mathrm{tr}(E[F/x]) & = & \mathrm{tr}(E) \cup \{\, lt \mid l, x \in \mathrm{et}(E) \wedge t \in \mathrm{tr}(F) \,\} \\
\mathrm{et}(E[F/x]) & = & \{\, l, y \mid l, y \in \mathrm{et}(E) \wedge y \neq x \,\} \\
 & & \cup \{\, ll', y \mid l, x \in \mathrm{et}(E) \wedge l', y \in \mathrm{et}(F) \,\} \\
\mathrm{tr}(E[H/z]) & = & \mathrm{tr}(E)\{H/z\} \\
 & & \cup \{\, lt \mid \exists l' \,.\, \exists i \,.\, l'\langle z, i\rangle \in \mathrm{tr}(E) \wedge l \in l'\{H/z\} \wedge t \in \mathrm{tr}(H\vec{x}) \,\} \\
\mathrm{et}(E[H/z]) & = & \{\, l', y \mid \exists l \,.\, l, y \in \mathrm{et}(E) \wedge l' \in l\{H/z\} \,\} \\
 & & \cup \{\, l'l'', y \mid \exists l \,.\, \exists i \,.\, l\langle z, i\rangle \in \mathrm{tr}(E) \wedge l' \in l\{H/z\} \wedge l'', y \in \mathrm{et}(H\vec{x}) \wedge y \notin \vec{x} \,\}
\end{array}
$$

PROOF We show the result for $\mathit{fix}\, \lambda x{:}P.\, E$ and $E[H/z]$. For the former, and the inclusion $\supseteq$, suppose that $m \geq 0$, $\forall i \in 1..m \,.\, l_i, x \in \mathrm{et}(E)$ and $l_{m+1} \in \mathrm{tr}(E)$. We can assume without loss of generality that $\forall i \,.\, l_i \neq \epsilon$. By the definitions of $\mathrm{tr}(\_), \mathrm{et}(\_)$ there exist F_i for $i \in 1..m+1$ such that

$$\forall i \in 1..m \,.\, E \xrightarrow{l_i} F_i \rhd x \qquad \text{and} \qquad E \xrightarrow{l_{m+1}} F_{m+1}.$$

By the structured operational semantics of §3 (henceforth 'the SOS') and Lemma 1 part 1

$$\forall i \in 1..m+1 \,.\, \mathit{fix}\, \lambda x{:}P.\, E \xrightarrow{l_i} F_i[\mathit{fix}\, \lambda x{:}P.\, E/x].$$

By Lemma 1 part 5

$$\forall i \in 1..m, j \in 1..m+1 \,.\, F_i[\mathit{fix}\, \lambda x{:}P.\, E/x] \xrightarrow{l_j} F_j[\mathit{fix}\, \lambda x{:}P.\, E/x]$$

so $\mathit{fix}\ \lambda x{:}P.\ E \xrightarrow{l_1 \ldots l_{m+1}} F_{m+1}[\mathit{fix}\ \lambda x{:}P.\ E/x]$ and $l_1 \ldots l_{m+1} \in \mathrm{tr}(\mathit{fix}\ \lambda x{:}P.\ E)$. If in addition $F_{m+1} \rhd y \neq x$ then by Lemma 1 part 2 $F_{m+1}[\mathit{fix}\ \lambda x{:}P.\ E/x] \rhd y$ so $l_1 \ldots l_{m+1}, y \in \mathrm{et}(\mathit{fix}\ \lambda x{:}P.\ E)$.

For the inclusion $\mathrm{tr}(\mathit{fix}\ \lambda x{:}P.\ E) \subseteq \ldots$, suppose that $\mathit{fix}\ \lambda x{:}P.\ E \xrightarrow{u_1} F_1 \ldots \xrightarrow{u_p} F_p$ for some $p \geq 1$. By the SOS there is an E_1 such that $E \xrightarrow{u_1} E_1$ and $E_1[\mathit{fix}\ \lambda x{:}P.\ E/x] = F_1$. By Lemma 1 part 3 for all $i \in 1..p-1$ there exists E_{i+1} such that

$$E_{i+1}[\mathit{fix}\ \lambda x{:}P.\ E/x] = F_{i+1} \quad \text{and} \quad (E_i \xrightarrow{u_{i+1}} E_{i+1} \vee (E_i \rhd x \wedge E \xrightarrow{u_{i+1}} E_{i+1})).$$

The sequence $u_1 \ldots u_p$ can then be partitioned into $l_1 \ldots l_m l_{m+1}$ as required, taking $m \geq 0$ to be the number of occurrences of the second disjunct. For the inclusion $\mathrm{et}(\mathit{fix}\ \lambda x{:}P.\ E) \subseteq \ldots$, suppose also that $F_p \rhd y \neq x$. By Lemma 1 part 9 and the SOS either $E_p \rhd y$ or $E_p \rhd x \wedge E \rhd y$. In either case the sequence $u_1 \ldots u_p$ can be partitioned into $l_1 \ldots l_m l_{m+1}$ as before — in the second taking $l_{m+1} = \epsilon$.

For $E[H/z]$ the inclusions $\supseteq$ follow from Lemmas 10 and 1. The inclusion $\mathrm{tr}(E[H/z]) \subseteq \ldots$ is immediate from Lemma 9. The inclusion $\mathrm{et}(E[H/z]) \subseteq \ldots$ follows from Lemma 9 and Corollary 13. □

Definition An equivalence relation $\simeq$ over T^2 is a congruence if it is closed under $\vdash$, i.e. if $\{\, M = N : P \mid M \simeq N \,\} \vdash E = F : P$ implies that $E \simeq F$.

Lemma 16 *An equivalence relation $\simeq$ over T^2 is a congruence iff for all $x : P$, $m \geq 0$, $z : P^{m+1}$ and $H : P^{m+1}$, if $\forall i\ .\ E_i \simeq F_i$ then*

$$\begin{aligned} aE_1 &\simeq aF_1 \\ E_1 + E_2 &\simeq F_1 + F_2 \\ \mathit{fix}\ \lambda x{:}P.\ E_1 &\simeq \mathit{fix}\ \lambda x{:}P.\ F_1 \\ zE_1 \ldots E_m &\simeq zF_1 \ldots F_m \\ E_1[H/z] &\simeq F_1[H/z] \end{aligned}$$

PROOF The left-to-right implication is straightforward. The other can be shown by induction on proofs of $\{\, M = N : P \mid M \simeq N \,\} \vdash E = F : P$ that are suitably normalised. □

Corollary 17 *$=_{\mathrm{t}}$ is a congruence.*

PROOF By inspection of Lemma 15 $=_{\mathrm{t}}$ satisfies the properties of Lemma 16. □

Lemma 18 *If $\{\, E_i = F_i : P \mid i \in I \,\}$ is sound for trace equivalence ($=_{\mathrm{tr}}$) then $\forall i \in I\ .\ E_i =_{\mathrm{t}} F_i$.*

PROOF Consider an equation $E_i = F_i : P$. By soundness, for all closing substitutions ρ we have $E_i\rho =_{\mathrm{tr}} F_i\rho$. Taking $\mathcal{V} = \mathrm{fv}(E_i) \cup \mathrm{fv}(F_i)$ we construct a discriminating substitution ρ with domain $\mathcal{V}$ as follows. Let $\mathcal{A}$ be the set of actions occurring in E_i or F_i. We take distinct actions a_x for each $x : P \in \mathcal{V}$ and a_{zi} for each $z : P^{n+1} \in \mathcal{V}$, $n \geq 1$ and $i \in 1..n$, ensuring that they are not in $\mathcal{A}$. Then

$$\begin{array}{rcll} \rho(x) & \stackrel{\mathit{def}}{=} & a_x 0 & \text{for } x : P \in \mathcal{V} \\ \rho(z) & \stackrel{\mathit{def}}{=} & \lambda y_1{:}P.\, \ldots \lambda y_n{:}P.\, a_{z1} y_1 + \ldots a_{zn} y_n & \text{for } z : P^{n+1} \in \mathcal{V}. \end{array}$$

Consider the subset of T^2 with free variables contained in $\mathcal{V}$ and actions contained in $\mathcal{A}$. This is closed under transitions. Letting E, F range over it, by Lemmas 1 and 5:

1. $\forall x : P \in \mathcal{V} \,.\, E\rho \xrightarrow{a_x} 0 \iff E \rhd x$
2. $\forall z : P^{n+1} \in \mathcal{V}, i \in 1..n \,.\, E\rho \xrightarrow{a_{zi}} A \iff \exists F \,.\, E \xrightarrow{zi} F \wedge F\rho = A$
3. $\forall a \in \mathcal{A} \,.\, E\rho \xrightarrow{a} A \iff \exists F \,.\, E \xrightarrow{a} F \wedge F\rho = A$

These imply that $E_i =_{\mathrm{t}} F_i$. $\square$

Remark The fact that Act is infinite is required for this result. If, for example, $Act = \{a_1, \ldots, a_n\}$ and $E \stackrel{\mathit{def}}{=} \mathit{fix}\, \lambda x{:}P.\, a_1 x + \ldots a_n x$ then $E = y + E : P$ is sound for $=_{\mathrm{tr}}$ but $E \neq_{\mathrm{t}} y + E$. This contrasts with the analogous result for bisimulation [Sew95, Theorem 7] which requires only nonempty Act.

4 Loop properties

To show the main nonaxiomatisability result (Theorem 1) we need, for any finite set $\mathcal{E}$ of sound equations, to exhibit an $n \geq 1$ such that $\mathit{fix}\, \lambda x{:}P.\, ax = \mathit{fix}\, \lambda x{:}P.\, a^n x$ is not provable from $\mathcal{E}$. This is done by constructing a family of congruences over T^2, each of which does not contain some of these equalities, such that any $\mathcal{E}$ lies within one of the family. We first define a rather intensional property of elements of T^2, their sets of *loops*, and characterise the loops of a compound expression in terms of the loops, traces and extended traces of its subexpressions. We then define relations $=_N$ over T^2, indexed by sets N of non-zero natural numbers containing 1, show that if N is multiplication-closed then each $=_{\mathrm{t}} \cap =_N$ is a congruence and prove the theorem.

Notation We let U range over subsets of Act and write $\xrightarrow{U}$ for $(\cup_{u \in U} \xrightarrow{u})^*$.

Definition $\mathrm{loops}_U\, E \stackrel{\mathit{def}}{=} \{\, l \mid l \in U^+ \wedge \exists F \,.\, E \xrightarrow{U} F \xrightarrow{l} F \,\}$

Remark This definition is intensional in that it refers to equality of terms in T^2. In general it gives a proper subset of the 'semantic U-loops' $\{ l \mid l \in U^+ \wedge \forall n \geq 1 . \exists F . E \xrightarrow{U} \xrightarrow{l^n} F \}$ of E.

The set $\mathrm{loops}_U\, E$ is clearly closed under cyclic permutation, where $l =_{\mathrm{rot}} l' \iff \exists l_1, l_2 . l = l_1 l_2 \wedge l' = l_2 l_1$ and for $T \subseteq Lab^*$ the cyclic permutation closure of T is $T^{\mathrm{rot}} \stackrel{\mathit{def}}{=} \{ l \mid \exists l' \in T . l =_{\mathrm{rot}} l' \}$.

We now characterise the effects of the various operators on loop sets, in Lemmas 19, 22 and 23. The proofs of these are essentially a refinement of the trace part of Lemma 15.

Lemma 19 *If* $z : P^{n+1}$ *for some* $n \geq 1$ *then*

$$\begin{array}{rcll}
\mathrm{loops}_U\, aE & = & \mathrm{loops}_U\, E & \textit{if } a \in U \\
 & & \{\} & \textit{otherwise} \\
\mathrm{loops}_U\, E + F & = & \mathrm{loops}_U\, E \cup \mathrm{loops}_U\, F & \\
\mathrm{loops}_U\, \mathit{fix}\, \lambda x{:}P.\, E & = & \mathrm{loops}_U\, E \cup \left\{ l_1 \ldots l_q \,\middle|\, q \geq 1 \wedge \forall i \in 1..q \,.\, E \xrightarrow{l_i} \rhd x \wedge l_i \in U^+ \right\}^{\mathrm{rot}} & \\
\mathrm{loops}_U\, zE_1 \ldots E_n & = & \bigcup \{ \mathrm{loops}_U\, E_i \mid i \in 1..n \wedge zi \in U \} &
\end{array}$$

Proof We show the inclusion $\subseteq$ for $\mathit{fix}\, \lambda x{:}P.\, E$. The following fact, allowing certain subexpressions to be 'pulled back' along transitions, is required.

> **Lemma 20** *If* $E \xrightarrow{l} F = C[G/y]$, $x \in \mathrm{fv}(F)$, $x \notin \mathrm{fv}(G)$ *and* $y \in \mathrm{fv}(C)$ *then there is some* D *such that* $E = D[G/y]$ *and* $y \in \mathrm{fv}(D)$.
>
> Proof By induction on l, with the base case $l = u$ by induction on the derivation of $E \xrightarrow{u} F$. □

Now consider a loop $l = u_{p+1} \ldots u_{p+q} \in \mathrm{loops}_U\, \mathit{fix}\, \lambda x{:}P.\, E$ due to the transitions

$$\mathit{fix}\, \lambda x{:}P.\, E \xrightarrow{u_1} F_1 \ldots \xrightarrow{u_p} F_p \xrightarrow{u_{p+1}} \ldots \xrightarrow{u_{p+q}} F_{p+q} = F_p$$

for some $p, q \geq 1$ and all $u_i \in U$. By the SOS there is an E_1 such that $E \xrightarrow{u_1} E_1$ and $E_1[\mathit{fix}\, \lambda x{:}P.\, E/x] = F_1$. By Lemma 1 part 3 $\forall i \in 1..p+q-1 . \exists E_{i+1} . E_{i+1}[\mathit{fix}\, \lambda x{:}P.\, E/x] = F_{i+1} \wedge (E_i \xrightarrow{u_{i+1}} E_{i+1} \vee (E_i \rhd x \wedge E \xrightarrow{u_{i+1}} E_{i+1}))$. If $x \notin \mathrm{fv}(E_{p+q})$ then $E \xrightarrow{U} E_{p+q} = F_p \xrightarrow{l} F_p$ so $l \in \mathrm{loops}_U\, E$. If instead $x \in \mathrm{fv}(E_{p+q})$ then by Lemma 1 parts 7, 8 $x \in \mathrm{fv}(E_p)$. We now show that $E_p = E_{p+q}$. Suppose not, then as $E_p[\mathit{fix}\, \lambda x{:}P.\, E/x] = E_{p+q}[\mathit{fix}\, \lambda x{:}P.\, E/x]$ there must be a subexpression $\mathit{fix}\, \lambda x{:}P.\, E$ of at least one of E_p and E_{p+q}. By Lemma 20 this must also be a subexpression of E, which is a contradiction, so $E_p = E_{p+q}$. Now if $\forall i \in p..p+q-1 . E_i \xrightarrow{u_{i+1}} E_{i+1}$ then $l \in \mathrm{loops}_U\, E$. Otherwise there exists some $i \in p..p+q-1$ such that $E_i \rhd x \wedge$

$E\xrightarrow{u_{i+1}}E_{i+1}$. The sequence $u_{i+1}\dots u_{p+q}u_{p+1}\dots u_i$ can then be partitioned into $l_1\dots l_q$ such that $\forall i \in 1..q\ .\ E\xrightarrow{l_i}\rhd x$ as required. □

Remark Lemma 20 is required as the operation of applying the substitution $[\mathit{fix}\,\lambda x{:}P.\,E/x]$ does not have a strong inverse property, even on the derivatives of E. For example consider $E \stackrel{\mathit{def}}{=} ax + \mathit{fix}\,\lambda y{:}P.\,a\mathit{fix}\,\lambda x{:}P.\,ax + y$, which has transitions $E\xrightarrow{a}x$ and $E\xrightarrow{a}\mathit{fix}\,\lambda x{:}P.\,E$. We have $x[\mathit{fix}\,\lambda x{:}P.\,E/x] = (\mathit{fix}\,\lambda x{:}P.\,E)[\mathit{fix}\,\lambda x{:}P.\,E/x]$ but $x \neq \mathit{fix}\,\lambda x{:}P.\,E$.

We now characterise the loops of a substituted term, first for a substitution at type P and then for a substitution at type P^{n+1}. The following lemma is required.

Lemma 21 *If $l \in \mathrm{loops}_U\,E$ then there exist a term $\mathit{fix}\,\lambda x{:}P.\,F$, a $q \geq 1$ and $l_i \in U^+$ for $i \in 1..q$ such that $E\xrightarrow{U}\rhd\,\mathit{fix}\,\lambda x{:}P.\,F$, $l =_{\mathrm{rot}} l_1\dots l_q$ and $\forall i\ .\ F\xrightarrow{l_i}\rhd x$.*

Proof Induction on E using Lemma 19. □

Lemma 22 *For $y : P$ if $E\xrightarrow{U}\rhd y$ then $\mathrm{loops}_U\,E[G/y] = (\mathrm{loops}_U\,E) \cup (\mathrm{loops}_U\,G)$ else $\mathrm{loops}_U\,E[G/y] = \mathrm{loops}_U\,E$.*

Proof The inclusions $\supseteq$ follow from Lemma 1. For the inclusions $\subseteq$, suppose that $l \in \mathrm{loops}_U\,E[G/y]$. Applying Lemma 21, $E[G/y]\xrightarrow{U}\rhd\,\mathit{fix}\,\lambda x{:}P.\,F$. By Lemma 1 part 3 either $E\xrightarrow{U}\rhd E' \wedge E'[G/y]\rhd\mathit{fix}\,\lambda x{:}P.\,F$ or $E\xrightarrow{U}\rhd y \wedge G\xrightarrow{U}\rhd\,\mathit{fix}\,\lambda x{:}P.\,F$. In the latter case $l \in \mathrm{loops}_U\,G$. In the former then by Corollary 14 either $E' \rhd \mathit{fix}\,\lambda x{:}P.\,E'' \wedge E''[G/y] = F$ or $E' \rhd y \wedge G \rhd \mathit{fix}\,\lambda x{:}P.\,F$. Again, in the latter case $l \in \mathrm{loops}_U\,G$. In the former $\forall i \in 1..q\ .\ E''\xrightarrow{l_i}\rhd x$ (as we can ensure by alpha conversion that $x \notin \mathrm{fv}(G)$) so $l \in \mathrm{loops}_U\,E$. □

Lemma 23 *If $z : P^{n+1} \in \mathrm{fv}(E)$, $H : P^{n+1}$, $n \geq 1$ and $\vec{x}$ are distinct variables of type P not in $\mathrm{fv}(H)$ then if $E\xrightarrow{U}\xrightarrow{z1}$*

$$\mathrm{loops}_U\,E[H/z] \quad = \quad \mathrm{loops}_U\,H\vec{x} \cup \bigcup\{\,U^* \cap l\{H/z\}^{\mathrm{rot}} \mid l \in \mathrm{loops}_{U'}\,E\,\}$$

otherwise

$$\mathrm{loops}_U\,E[H/z] \quad = \quad \bigcup\{\,U^* \cap l\{H/z\}^{\mathrm{rot}} \mid l \in \mathrm{loops}_{U'}\,E\,\}$$

where $U' \stackrel{\mathit{def}}{=} (U - \{z1,\dots,zn\}) \cup \{\,zi \mid \exists t \in U^*\ .\ H\vec{x}\xrightarrow{t}\rhd x_i\,\}$.

Proof $\subseteq$: As in Lemma 22, we show that any loop of $E[H/z]$ arises from an occurrence of *fix* in E (case 1.1 below) or H (cases 1.2 and 2.1 below). Suppose $l \in \mathrm{loops}_U\,E[H/z]$. By Lemma 21 there exist $\mathit{fix}\,\lambda x{:}P.\,F$, $t \in U^*$, $q \geq 1$ and $l_i \in U^+$ for $i \in 1..q$ such that $E\xrightarrow{t}\rhd\,\mathit{fix}\,\lambda x{:}P.\,F$, $l =_{\mathrm{rot}} l_1\dots l_q$ and $\forall i\ .\ F\xrightarrow{l_i}\rhd x$. By

Lemma 9 and Corollary 14 there exist $k, h \in Lab^*$, $j \in \mathrm{null}(\rho)^*$, E', H', $\vec{E}$, i such that one of the following hold.

1.1 $E \xrightarrow{k} \xrightarrow{j} \rhd \mathit{fix}\, \lambda x{:}P.\, E' \wedge t \in k\{H/z\} \wedge E'[H/z] = F$

1.2 $E \xrightarrow{k} \xrightarrow{j} \rhd z\vec{E} \wedge t \in k\{H/z\} \wedge H\vec{x} \rhd \mathit{fix}\, \lambda x{:}P.\, H' \wedge H'[\vec{E}[H/z]/\vec{x}] = F$

2.1 $E \xrightarrow{k} \rhd z\vec{E} \wedge t \in k\{H/z\} \cdot h \wedge H\vec{x} \xrightarrow{h} \rhd \mathit{fix}\, \lambda x{:}P.\, H' \wedge H'[\vec{E}[H/z]/\vec{x}] = F$

2.2 $E \xrightarrow{k} \rhd z\vec{E} \wedge t \in k\{H/z\} \cdot h \wedge H\vec{x} \xrightarrow{h} \rhd x_i \wedge E_i[H/z] \rhd \mathit{fix}\, \lambda x{:}P.\, F$

Case 2.2 reduces to case 1.1 or 1.2, as $E \xrightarrow{k} \xrightarrow{zi} E_i$ and, as $H\vec{x} \xrightarrow{h} \rhd x_i$, $t \in (k\langle z, i\rangle)\{H/z\}$. In cases 1.2 and 2.1 we can assume (by alpha conversion) that x is not free in H or $\vec{E}$ so $\forall i \in 1..q\ .\ H' \xrightarrow{l_i} \rhd x$ and $l \in \mathrm{loops}_U\, H\vec{x}$ (noting that as $t \in U^*$ we have $h \in U^*$). It is straightforward to check that $kj \in U'^*$ (resp. that $k \in U'^*$) so $E \xrightarrow{U} \xrightarrow{z1}$. In case 1.1 $kj \in U'^*$ similarly. By Lemma 9, as $\forall i \in 1..q\ .\ E'[H/z] \xrightarrow{l_i} \rhd x$, $\exists k_i \in Lab^*$ such that one of the following hold.

1 $\exists F_i\ .\ E' \xrightarrow{k_i} F_i \wedge F_i[H/z] \rhd x \wedge l_i \in k_i\{H/z\}$

2 $\exists F_i, \vec{F}_i, H_i, h_i\ .\ E' \xrightarrow{k_i} F_i \rhd z\vec{F}_i \wedge H\vec{x} \xrightarrow{h_i} H_i \wedge H_i[\vec{F}_i[H/z]/\vec{x}] \rhd x \wedge l_i \in k_i\{H/z\} \cdot h_i$

Case 2 reduces to 1, as by Corollary 13 $\exists p\ .\ H_i \rhd x_p \wedge F_{ip}[H/z] \rhd x$ (as $\mathrm{null}(\vec{F}_i[H/z]/\vec{x})$ is empty and $x \notin \mathrm{fv}(H)$) hence $E' \xrightarrow{k_i} \xrightarrow{zp} F_{ip}$, $H\vec{x} \xrightarrow{h_i} \rhd x_p$ and $l_i \in (k_i\langle z, p\rangle)\{H/z\}$.

Considering case 1 only, therefore, by Corollary 13 $\exists j_i \in \mathrm{null}(H/z)^*\ .\ F_i \xrightarrow{j_i} \rhd x$ (the other clause of Corollary 13 is ruled out by $x \notin \mathrm{fv}(H)$) so $\forall i\ .\ E' \xrightarrow{k_i} \xrightarrow{j_i} \rhd x$. As $(k_1 j_1 \ldots k_q j_q) \in U'^*$ we have $(k_1 j_1 \ldots k_q j_q) \in \mathrm{loops}_{U'}\, E$. Now $l_i \in (k_i j_i)\{H/z\}$ so $l_1 \ldots l_q \in (k_1 j_1 \ldots k_q j_q)\{H/z\}$ so $l \in (k_1 j_1 \ldots k_q j_q)\{H/z\}$.

$\supseteq$: Suppose $E \xrightarrow{U} \xrightarrow{z1}$ and $l \in \mathrm{loops}_U\, H\vec{x}$, i.e. $\exists t, t' \in U^*, G\ .\ E \xrightarrow{t} \xrightarrow{z1} \wedge H\vec{x} \xrightarrow{t'} G \xrightarrow{l} G$. We can assume without loss of generality that t does not contain any zi, then by Lemma 1 part 10 $\exists \vec{E}\ .\ E \xrightarrow{t} \rhd z\vec{E}$ and by parts 1,2 $E[H/z] \xrightarrow{t} \rhd (H\vec{x})[\vec{E}[H/z]/\vec{x}]$. By Lemma 1 part 1 $(H\vec{x})[\vec{E}[H/z]/\vec{x}] \xrightarrow{t'} \xrightarrow{l} G[\vec{E}[H/z]/\vec{x}] \xrightarrow{l} G[\vec{E}[H/z]/\vec{x}]$ so by Lemma 1 part 5 $\exists E'\ .\ E[H/z] \xrightarrow{tt'l} E' \xrightarrow{l} E'$ and as $t, t', l \in U^*$ we have $l \in \mathrm{loops}_U\, E[H/z]$.

Suppose $l' =_{\mathrm{rot}} l_1 \ldots l_q \in (u_1 \ldots u_q)\{H/z\}$, $l' \in U^*$ and $u_1 \ldots u_q \in \mathrm{loops}_{U'}\, E$. From the latter we have $\exists t \in U'^*, F\ .\ E \xrightarrow{t} F \xrightarrow{u_1 \ldots u_q} F$ and $u_1 \ldots u_q \in U'^*$. The definition of U' ensures that $t\{H/z\} \cap U^*$ is nonempty — say it contains t'. By Lemma 10 $E[H/z] \xrightarrow{t'} \rhd F[H/z]$ and $\exists G\ .\ F[H/z] \xrightarrow{l_1 \ldots l_q} G \rhd F[H/z]$ so by Lemma 1 part 5 $l' \in \mathrm{loops}_U\, E[H/z]$. $\square$

Definition If N is a set of non-zero natural numbers containing 1 then

$$E \leq_N F \iff \forall U \subseteq \mathit{Lab} \,.\, \forall l \in \mathrm{loops}_U\, E \,.\, \exists n \in N \,.\, l^n \in \mathrm{loops}_U\, F.$$

Note that if $N \subseteq N'$ then $\leq_N \subseteq \leq_{N'}$ and that if N is closed under multiplication then $\leq_N$ is a preorder. We then write $=_N$ for the equivalence $\leq_N \cap \leq_N^{-1}$.

Lemma 24 *If $x : P$, $m \geq 0$, $z : P^{m+1}$, $H : P^{m+1}$, $\forall i \,.\, E_i =_{\mathrm{t}} F_i$ and $\forall i \,.\, E_i \leq_N F_i$ then*

$$\begin{aligned} aE_1 &\leq_N aF_1 \\ E_1 + E_2 &\leq_N F_1 + F_2 \\ \mathit{fix}\,\lambda x{:}P.\, E_1 &\leq_N \mathit{fix}\,\lambda x{:}P.\, F_1 \\ zE_1 \ldots E_m &\leq_N zF_1 \ldots F_m \\ E_1[H/z] &\leq_N F_1[H/z] \end{aligned}$$

PROOF The result for $a\_$, $\_+\_$ and $z\_$ follows from Lemma 19. For $\mathit{fix}\,\lambda x{:}P.\,\_$, suppose $l \in \mathrm{loops}_U\, \mathit{fix}\,\lambda x{:}P.\, E_1$. By Lemma 19 either $l \in \mathrm{loops}_U\, E_1$ or $l =_{\mathrm{rot}} l_1 \ldots l_q \wedge \forall i \in 1..q \,.\, E_1 \xrightarrow{l_i} \rhd x$. In the first case, as $E_1 \leq_N F_1$, there is $n \in N$ such that $l^n \in \mathrm{loops}_U\, F_1$ and by Lemma 19 $l^n \in \mathrm{loops}_U\, \mathit{fix}\,\lambda x{:}P.\, F_1$. In the second case, as $\mathrm{et}(E_1) = \mathrm{et}(F_1)$, $\forall i \,.\, F_1 \xrightarrow{l_i} \rhd x$ so by Lemma 19 $l \in \mathrm{loops}_U\, \mathit{fix}\,\lambda x{:}P.\, F_1$. This suffices as by assumption $1 \in N$.

For the $\_[H/z]$ case, by Lemma 4 $z \in \mathrm{fv}(E_1) \iff z \in \mathrm{fv}(F_1)$. If $z \notin \mathrm{fv}(E_1)$ the result is trivial. Suppose otherwise and consider $l \in \mathrm{loops}_U\, E[H/z]$. If $z : P$ then by Lemma 22 either $l \in \mathrm{loops}_U\, E$ or $l \in \mathrm{loops}_U\, H \wedge E_1 \xrightarrow{U} \rhd z$. In the first case, as $E_1 \leq_N F_1$, there is $n \in N$ such that $l^n \in \mathrm{loops}_U\, F_1$ and by Lemma 22 $l^n \in \mathrm{loops}_U\, F_1[H/z]$. In the second case, as $\mathrm{et}(E_1) = \mathrm{et}(F_1)$, $F_1 \xrightarrow{U} \rhd z$ so by Lemma 22 $l^1 \in \mathrm{loops}_U\, F_1[H/z]$.

If $z : P^{n+1}$ for some $n \geq 1$ then by Lemma 23 either $l \in \mathrm{loops}_U\, H\vec{x} \wedge E_1 \xrightarrow{U}\xrightarrow{z1}$ or $l =_{\mathrm{rot}} l_1 \ldots l_q \in (u_1 \ldots u_q)\{H/z\}$ for some $u_1 \ldots u_q \in \mathrm{loops}_{U'}\, E_1$. In the first case, as $\mathrm{tr}(E_1) = \mathrm{tr}(F_1)$, $F_1 \xrightarrow{U}\xrightarrow{z1}$ so by Lemma 23 $l \in \mathrm{loops}_U\, F_1[H/z]$. In the second, as $E_1 \leq_N F_1$, there is $n \in N$ such that $(u_1 \ldots u_q)^n \in \mathrm{loops}_{U'}\, F_1$. As $l^n =_{\mathrm{rot}} (l_1 \ldots l_q)^n \in (u_1 \ldots u_q)^n\{H/z\}$ it follows that $l^n \in \mathrm{loops}_U\, F_1[H/z]$. $\square$

Corollary 25 *If N is closed under multiplication then $=_{\mathrm{t}} \cap =_N$ is a congruence.*

PROOF Immediate from Corollary 17, Lemma 24 and Lemma 16. $\square$

Any sound equation lies within $\leq_N$ for a finite N:

Lemma 26 *If $E =_{\mathrm{t}} F$ then $E \leq_{\{1,\ldots,|F|\}} F$.*

PROOF If $l \in \text{loops}_U E$ then there are $t \in U^*$ and E' such that $E \xrightarrow{t} E' \xrightarrow{l} E'$, hence for all $q \geq 1$ we have $tl^q \in \text{tr}(E)$. Putting $q = |F|$ this implies that $tl^{|F|} \in \text{tr}(F)$, so there exist F_i for $i \in 0..|F|$ such that $F \xrightarrow{t} F_0 \xrightarrow{l} F_1 \xrightarrow{l} F_2 \ldots \xrightarrow{l} F_{|F|}$. At least two of the F_i for $i \in 0..|F|$ must be equal, so for some $n \in 1..|F|$ we have $l^n \in \text{loops}_U F$. $\square$

The main theorem can now be proved.

PROOF (of Theorem 1) Suppose $\simeq$ is an equivalence over the closed μ-expressions that is finer than (or identical to) trace equivalence and $\mathcal{E} = \{ E_i = F_i : P \mid i \in I \}$ is a finite set of typed equations with $E_i, F_i \in T^2$ that is sound for $\simeq$. It follows that $\mathcal{E}$ is sound for trace equivalence ($=_{\text{tr}}$), so by Lemma 18 $\forall i \in I \ . \ E_i =_{\text{t}} F_i$. Let $n = \max \cup_{i \in I} \{|E_i|, |F_i|\}$, let N be the multiplication-closure of $\{1, \ldots, n\}$ and p the smallest prime strictly greater than n. By Lemma 26 $\forall i \in I \ . \ E_i =_N F_i$ and by Corollary 25, if $\mathcal{E} \vdash E = F : P$ then $E =_N F$.

Now N contains no multiples of p so $\mathit{fix}\ \lambda x{:}P.\, ax \neq_N \mathit{fix}\ \lambda x{:}P.\, a^p x$, hence if for all $q \geq 1$ $\mu x\ ax \simeq \mu x\ a^q x$ then $\mathcal{E}$ cannot be complete for $\simeq$. $\square$

5 Star expressions

Finite state systems have also been described using calculi with a unary or binary iteration operator in place of explicit recursion, such as the $*$-*expressions* given by

$$E ::= c \;\big|\; 0 \;\big|\; 1 \;\big|\; E + E \;\big|\; E \cdot E \;\big|\; E^* \;\big|\; E^\star E$$

where c ranges over some set $\mathcal{A}$ of actions. We include both the binary iteration $E^\star F$ of Kleene [Kle56], representing zero or more iterations of E followed by one of F, and the unary iteration E^* introduced in [CEW58], representing zero or more iterations of E.

The results of §3,4 can be applied to give simple proofs of non-finite-axiomatisability of a range of equivalences over a range of subcalculi of the $*$-expressions. We first recall some standard definitions, defining bisimulation, a trace congruence and language equivalence over the $*$-expressions via a labelled transition system equipped with a 'successful termination' predicate.

Definition The relations $\xrightarrow{c}$ for $c \in \mathcal{A}$ and predicate $\surd$ are the least over the $*$-

expressions such that

$$\frac{}{c \xrightarrow{c} 1} \qquad\qquad \frac{}{1\surd}$$

$$\frac{E \xrightarrow{c} E'}{E+F \xrightarrow{c} E'} \text{ and sym.} \qquad\qquad \frac{E\surd}{E+F\surd} \text{ and sym.}$$

$$\frac{E \xrightarrow{c} E'}{E \cdot F \xrightarrow{c} E' \cdot F} \qquad \frac{E\surd \quad F \xrightarrow{c} F'}{E \cdot F \xrightarrow{c} F'} \qquad \frac{E\surd \quad F\surd}{E \cdot F\surd}$$

$$\frac{E \xrightarrow{c} E'}{E^* \xrightarrow{c} E' \cdot E^*} \qquad\qquad \frac{}{E^*\surd}$$

$$\frac{E \xrightarrow{c} E'}{E{\star}F \xrightarrow{c} E' \cdot (E{\star}F)} \qquad \frac{F \xrightarrow{c} F'}{E{\star}F \xrightarrow{c} F'} \qquad \frac{F\surd}{E{\star}F\surd}.$$

Note that there are no rules for 0. We s, t range over $\mathcal{A}^*$. For $n \geq 1$ we define $c^{n+1} = c \cdot (c^n)$ and $c^1 = c$.

Definition Bisimulation, written $\sim$, is the largest relation over the $*$-expressions such that if $E \sim F$ then for all $c \in \mathcal{A}$

- If $E \xrightarrow{c} E'$ then $\exists F' \,.\, F \xrightarrow{c} F' \wedge E' \sim F'$.
- If $F \xrightarrow{c} F'$ then $\exists E' \,.\, E \xrightarrow{c} E' \wedge E' \sim F'$.
- $E\surd \iff F\surd$.

Definition The *trace set* and *terminated trace set* of a $*$-expression E are the subsets of $\mathcal{A}^*$

$$\begin{aligned} \mathrm{tr}(E) &\overset{def}{=} \{\, s \mid \exists F \,.\, E \xrightarrow{s} F \,\} \\ \mathrm{tt}(E) &\overset{def}{=} \{\, s \mid \exists F \,.\, E \xrightarrow{s} F\surd \,\}. \end{aligned}$$

Two $*$-expressions E,F are *trace congruent*, written $E =_{\mathrm{t}} F$, if they have the same traces and terminated traces.

Definition Two $*$-expressions E,F are *language equivalent*, written $E =_{\mathrm{l}} F$, if they have the same traces.

A variety of subcalculi of $*$-expressions have been discussed in the literature with differing notation. For reference we include a little table:

c	0	1	$+$	$\cdot$	$\_^*$	$\_^\star\_$	
c	Λ		$\vee$	$\cdot$	$\_^*$		[CEW58]
c	0	1	$+$	$\cdot$	$\_^*$		[Con71, Koz94]
c	ϕ		$+$	$\cdot$	$\_^*$		[Sal66]
c		ϵ	$+$	$\cdot$			BPA$^\epsilon$ as in [Mol89]
c			$+$	$\cdot$		$\_^*\_$	BPA* as in [BBP94, FZ94]
c	δ		$+$	$\cdot$		$\_^*\_$	BPA$^*_\delta$ as in [BBP94, FZ94, Fok94]

The cited work is variously concerned with algebras satisfying certain axioms or with particular models. We therefore need to state carefully exactly what the above correspondences are. For the first three lines the common expressions denote the same language in the standard interpretation (except that in [CEW58] E^* does not necessarily contain the empty word) as follows.

Definition The language denoted by a $*$-expression E is $\mathrm{lang}(E)$, where

$$\begin{aligned}
\mathrm{lang}(c) &\stackrel{\mathit{def}}{=} \{c\} \\
\mathrm{lang}(0) &\stackrel{\mathit{def}}{=} \{\} \\
\mathrm{lang}(1) &\stackrel{\mathit{def}}{=} \{\epsilon\} \\
\mathrm{lang}(E+F) &\stackrel{\mathit{def}}{=} \mathrm{lang}(E) \cup \mathrm{lang}(F) \\
\mathrm{lang}(E \cdot F) &\stackrel{\mathit{def}}{=} \{\, st \mid s \in \mathrm{lang}(E) \wedge t \in \mathrm{lang}(F) \,\} \\
\mathrm{lang}(E^*) &\stackrel{\mathit{def}}{=} \{\, s_1 \ldots s_m \mid m \geq 0 \wedge \forall i \in 1..m \;.\; s_i \in \mathrm{lang}(E) \,\} \\
\mathrm{lang}(E^\star F) &\stackrel{\mathit{def}}{=} \{\, s_1 \ldots s_m t \mid t \in \mathrm{lang}(F) \wedge m \geq 0 \wedge \forall i \in 1..m \;.\; s_i \in \mathrm{lang}(E) \,\}.
\end{aligned}$$

Lemma 27 $\mathrm{lang}(E) = \mathrm{tt}(E)$.

PROOF Straightforward. □

For the last three lines bisimulation as defined below agrees with the definitions in the cited work, as follows. For terms of $1, c, +, \cdot$ the transition system and bisimulation coincide with the transition system and bisimulation $\cong$ of [Mol89, §6.3.1] for BPA$^\epsilon$ (identifying 1 and ϵ). As discussed there it differs from the original BPA$^\epsilon$ semantics of [Vra86]. The transition system differs from the semantics of [BBP94, FZ94] for terms of $0, c, +, \cdot, {}^\star$, where predicates $\xrightarrow{a}\surd$ are used instead of $\surd$. However, bisimulation coincides with the bisimulation $\leftrightarrow$ over BPA$^*_\delta$ (identifying 0 and δ) defined therein.

Proposition 28 $\sim \subseteq =_{\mathrm{t}} \subseteq =_{\mathrm{l}}$

PROOF Straightforward. □

An equation over the $*$-expressions is simply a pair of $*$-expressions. If $\mathcal{E}$ is a set of equations we write $\mathcal{E} \vdash E = F$ if $E = F$ is derivable using the rules in Figure 3 augmented with the rule

$$\frac{(E = F) \in \mathcal{E}}{E = F}\ ax \quad .$$

Note that $\vdash$ allows substitution of terms for actions, as is usual when dealing with regular expressions but in contrast to the situation for μ-expressions.

Definition A relation over the $*$-expressions is a congruence if it is closed under $\vdash$.

Proposition 29 *Bisimulation ($\sim$), trace congruence ($=_\mathrm{t}$) and language equivalence ($=_\mathrm{l}$) are all congruences.*

PROOF Straightforward. □

To apply the results of §3,4 to show non-finite-axiomatisability over subcalculi of the $*$-expressions we first note that the $*$-expressions can be faithfully embedded into our lambda calculus, encoding sequential composition using function composition at type $P \to P$.

Definition We identify $\mathcal{A}$ with $\{\, c \mid c{:}P \to P \in \mathit{Var} \,\}$ and take the map $[\![\_]\!]$ from $*$-expressions to lambda calculus terms of type $P \to P$ to be

$$\begin{aligned}
[\![c]\!] &\overset{\mathit{def}}{=} c \\
[\![0]\!] &\overset{\mathit{def}}{=} \lambda x{:}P.\, 0 \\
[\![1]\!] &\overset{\mathit{def}}{=} \lambda x{:}P.\, x \\
[\![E + F]\!] &\overset{\mathit{def}}{=} \lambda x{:}P.\, ([\![E]\!]x) + ([\![F]\!]x)
\end{aligned}$$

$$\frac{}{E = E}\ \mathit{ref} \qquad \frac{E = F}{E[G/c] = F[G/c]}\ \mathit{sub}$$

$$\frac{E = F}{F = E}\ \mathit{sym} \qquad \frac{E = F \quad F = G}{E = G}\ \mathit{tran}$$

$$\frac{E = F \quad E' = F'}{E + E' = F + F'}\ +\mathit{cong} \qquad \frac{E = F \quad E' = F'}{E \cdot E' = F \cdot F'}\ \cdot\,\mathit{cong}$$

$$\frac{E = F}{E^* = F^*}\ *\mathit{cong} \qquad \frac{E = F \quad E' = F'}{E^\star E' = F^\star F'}\ {}^\star\mathit{cong}$$

Figure 3: Congruence rules for $*$-expressions

$$
\begin{aligned}
[\![E \cdot F]\!] &\stackrel{\text{def}}{=} \lambda x{:}P.\, [\![E]\!]([\![F]\!]x) \\
[\![E^*]\!] &\stackrel{\text{def}}{=} \lambda x{:}P.\, \mathit{fix}\ \lambda y{:}P.\, x + ([\![E]\!]y) \\
[\![E \star F]\!] &\stackrel{\text{def}}{=} \lambda x{:}P.\, \mathit{fix}\ \lambda y{:}P.\, ([\![F]\!]x) + ([\![E]\!]y)
\end{aligned}
$$

Trace congruence of $*$-expressions coincides with that defined over T^2, as follows. Fixing some $x : P \in \mathit{Var}$:

Lemma 30 *$E =_{\mathrm{t}} F$ iff $[\![E]\!]x =_{\mathrm{t}} [\![F]\!]x$.*

PROOF The following can be shown by routine inductions, using Lemma 1.

1. $E\surd \iff [\![E]\!]x \rhd x$
2. $E \xrightarrow{c} E' \Rightarrow [\![E]\!]x \xrightarrow{c1} [\![E']\!]x$
3. $[\![E]\!]x \xrightarrow{c1} A \Rightarrow \exists E' \,.\, E \xrightarrow{c} E' \wedge [\![E']\!]x = A.$

These imply that $c_1 \ldots c_m \in \mathrm{tr}(E) \iff \langle c_1, 1\rangle \ldots \langle c_m, 1\rangle \in \mathrm{tr}([\![E]\!]x)$ and that $c_1 \ldots c_m \in \mathrm{tt}(E) \iff \langle c_1, 1\rangle \ldots \langle c_m, 1\rangle, x \in \mathrm{et}([\![E]\!]x)$. $\square$

Embedding a set of equations by

$$[\![\{\, E_i = F_i \mid i \in I \,\}]\!] \stackrel{\text{def}}{=} \{\, [\![E_i]\!]x = [\![F_i]\!]x : P \mid i \in I \,\},$$

the embedding respects provability.

Lemma 31 *If $\mathcal{E} \vdash E = F$ then $[\![\mathcal{E}]\!] \vdash [\![E]\!]x = [\![F]\!]x : P$*

PROOF By induction on proofs, using the fact that $[\![E[F/c]]\!] = [\![E]\!][[\![F]\!]/c]$ in the sub case. $\square$

Lemma 32 *If $\mathcal{E} = \{\, E_i = F_i \mid i \in I \,\}$ is a finite set of equations between $*$-expressions with $\forall i \in I \,.\, E_i =_{\mathrm{t}} F_i$ then there is some $N \subseteq \mathbb{N}$, closed under multiplication and containing 1, such that $\forall E, F \,.\, (\mathcal{E} \vdash E = F) \Rightarrow [\![E]\!]x =_N [\![F]\!]x$ and there is some $p \geq 1$ that is not a factor of any $n \in N$.*

PROOF Let $n = \max \cup_{i \in I} \{ |[\![E_i]\!]x|, |[\![F_i]\!]x| \}$, let N be the multiplication-closure of $\{1, \ldots, n\}$ and p the smallest prime strictly greater than n. By Lemma 30 $\forall i \in I \,.\, [\![E_i]\!]x =_{\mathrm{t}} [\![F_i]\!]x$ so by Lemma 26 $\forall i \in I \,.\, [\![E_i]\!]x =_N [\![F_i]\!]x$. Now suppose $\mathcal{E} \vdash E = F$. By Lemma 31 $[\![\mathcal{E}]\!] \vdash [\![E]\!]x = [\![F]\!]x : P$ so by Corollary 25 $[\![E]\!]x =_N [\![F]\!]x$. $\square$

Theorem 2 *If $\simeq$ is an equivalence over a subcalculus of $*$-expressions that is closed under 0, c, $\cdot$ and either $*$ or $\star$ and $\simeq$ lies between trace congruence and bisimulation then there is no finite axiomatisation for $\simeq$.*

PROOF Consider a finite set $\mathcal{E}$ of equations that is sound for $\simeq$ (and hence sound for $=_{\rm t}$). Take N and p as given by Lemma 32 and consider the relevant pair of terms below.

$$E_1 = (c^p)^* \cdot 0 \qquad F_1 = c^* \cdot 0$$
$$E_2 = (c^p)^\star 0 \qquad F_2 = c^\star 0$$

We have $E_1 \sim F_1$ and $E_2 \sim F_2$, hence $E_1 \simeq F_1$ (resp. $E_2 \simeq F_2$). Now $[\![E_1]\!]x = [\![E_2]\!]x = \mathit{fix}\ \lambda y{:}P.\, 0 + c^p y$ and $[\![F_1]\!]x = [\![F_2]\!]x = \mathit{fix}\ \lambda y{:}P.\, 0 + cy$. These do not lie in $=_N$ so by Lemma 32 $E_1 = F_1$ (resp. $E_2 = F_2$) is not provable from $\mathcal{E}$. □

Theorem 3 *There is no finite axiomatisation for trace congruence over any subcalculus of $*$-expressions that is closed under c, $+$, $\cdot$ and either $*$ or *.*

PROOF Consider a finite set $\mathcal{E}$ of equations that is sound for $=_{\rm t}$. Take N and p as given by Lemma 32 and consider the relevant pair of terms below.

$$E_3 = (c^p)^* \cdot (c + \ldots + c^{p-1}) \quad F_3 = c^* \cdot c$$
$$E_4 = (c^p)^\star (c + \ldots + c^{p-1}) \quad F_4 = c^\star c$$

We have $E_3 =_{\rm t} F_3$ and $E_4 =_{\rm t} F_4$. Now $[\![E_3]\!]x = [\![E_4]\!]x = \mathit{fix}\ \lambda y{:}P.\, (cx + \ldots + c^{p-1}x) + c^p y$ and $[\![F_3]\!]x = [\![F_4]\!]x = \mathit{fix}\ \lambda y{:}P.\, cx + cy$. These do not lie in $=_N$ so by Lemma 32 $E_3 = F_3$ (resp. $E_4 = F_4$) is not provable from $\mathcal{E}$. □

Theorem 2 implies that there is no finite axiomatisation for bisimulation over BPA$^*_\delta$, in sharp contrast to the following positive result of Fokkink and Zantema.

Theorem 4 (Fokkink and Zantema [FZ94]) *The axioms below are sound and complete for bisimulation over BPA*, i.e. over expressions of $c, +, \cdot, \star$.*

$$\begin{array}{rcl}
c + d &=& d + c \\
(c + d) + e &=& c + (d + e) \\
c + c &=& c \\
(c + d) \cdot e &=& c \cdot e + d \cdot e \\
(c \cdot d) \cdot e &=& c \cdot (d \cdot e) \\
c \cdot (c^\star d) + d &=& c^\star d \\
c^\star (d \cdot e) &=& (c^\star d) \cdot e \\
c^\star (d \cdot ((c + d)^\star e) + e) &=& (c + d)^\star e
\end{array}$$

6 Discussion

In this section we give a brief overview of some previous work and mention some possible generalisations. The overview is far from exhaustive, in particular excluding work using infinitary rules (such as the Approximation Induction Principle of ACP and ω-induction), work on the axiomatisation of partial orders, on equivalences strictly between trace congruence and bisimulation, on calculi with parallel composition or on infinite state calculi. This leaves a substantial literature dealing with axiomatisation of equivalences over calculi denoting finite state machines. A part of it is summarised in Figure 4, classified by the equivalence, calculus and strength of logic addressed and labelled $\surd$ (resp. $\times$) if finite complete systems are given (resp. shown not to exist). Care must be taken when interpreting the figure as there are differing definitions, in particular of the calculi of $*$-expressions and of language and trace equivalences. Results without citations are those of this paper. Results labelled [Sew95] were also announced in [Sew94]. The figure is not intended to imply that all vertices have equal interest.

The first negative result, that language equivalence of $*$-expressions is not finitely equationally axiomatisable, was apparently given in an incomplete form by Redko [Red64] and Salomaa and later completed by Pilling. Three proofs are given by Conway [Con71]. Salomaa gave a finite impure Horn clause axiomatisation in [Sal66],

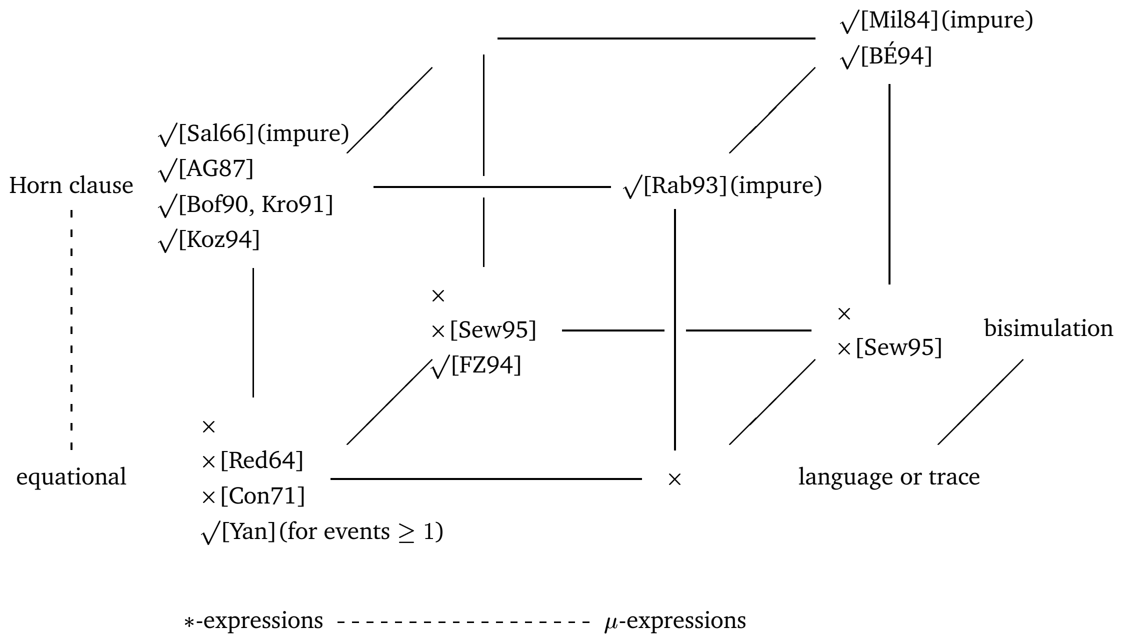

Figure 4: Finite axiomatisability results

using the implication

$$E = E \cdot F + G \wedge \epsilon \notin \text{lang}(F) \rightarrow E = G \cdot F^*$$

which asserts the uniqueness of certain fixed points. Similar axiomatisations have been given for a number of equivalences over μ-expressions. The figure shows that of Milner for bisimulation [Mil84], using an implication reproduced in §1, and that of Rabinovich for trace congruence [Rab93]; there are also results by Milner for weak bisimulation congruence [Mil89] and van Glabbeek for branching bisimulation congruence [Gla93a] and divergence bisimulation [Gla93b].

Finite pure Horn clause axiomatisations have been given for language equivalence of $*$-expressions by Arkhangelskii and Gorshkov [AG87], Boffa [Bof90], Krob [Kro91] and Kozen[Koz94]. A finite pure Horn clause axiomatisation for bisimulation of μ-expressions has been given by Bloom and Ésik [BÉ94], using an implication reproduced in §1.

Finite equational axiomatisations have been given by Yanov for language equivalence of the $*$-expressions whose languages contain the empty word [Yan] and by Fokkink and Zantema for bisimulation of the subcalculus of $*$-expressions without zero, unit or unary $*$ [FZ94].

The nonexistence of finite equational axiomatisations for bisimulation was shown by the author for μ-expressions and for subcalculi of $*$-expressions containing zero [Sew95, Sew94].

Various infinite but simple equational axiomatisations have been given, e.g. for language equivalence of $*$-expressions by Conway [Con71] and Krob [Kro91], in the general setting of iteration theories by Bloom, Ésik and Taubner [BÉ93a, BÉ93b, BÉT93] and for bisimulation of μ-expression by the author [Sew95].

6.1 Other signatures

Our nonaxiomatisability result for μ-expressions (Theorem 1) is weaker than might be desired, in that the typed equations considered do not contain variables ranging over actions. The signature of the lambda calculus used could be modified slightly, adding a base type A of actions and taking constants

$$\begin{aligned}
&0 : P \\
&a : A \quad \text{for each } a \in Act \\
&. : A \rightarrow P \rightarrow P \\
&+ : P \rightarrow P \rightarrow P \\
&\mathit{fix} : (P \rightarrow P) \rightarrow P.
\end{aligned}$$

We conjecture that the proof of Theorem 1 could be adapted to this signature without essential difficulty. This signature also allows the statement of nonaxiomatisability results about equivalences that abstract from a distinguished action $\tau \in Act$, such as the weak bisimulation congruence of [Mil89]. We conjecture that the proof could be adapted to these at the cost of some uninteresting complications.

More generally, one might consider an arbitrary signature of first order constants together with $\mathit{fix} : (B \to B) \to B$ for some base types B. The first order transition system of §3 could be adapted by treating constants in the same way as variables, e.g. by replacing the rules for prefix and sum by

$$\frac{c : P^{n+1} \in Con \quad i \in 1..n}{c\vec{E} \xrightarrow{ci} E_i} \ .$$

This would simplify the technical results of §3. For the signature of §2 the original transition relations can be recovered from the new, with e.g. the original $\xrightarrow{a}$ equal to the new $(\xrightarrow{+1} \cup \xrightarrow{+2})^* \xrightarrow{a1}$.

6.2 Relative axiomatisability

Questions of axiomatisability can be sharpened by considering whether one equivalence is finitely equationally axiomatisable relative to another, i.e. whether, for equivalences $\simeq_1$ and $\simeq_2$, there is a finite set of equations that together with the implication

$$E \simeq_1 F \to E = F$$

are sound and complete for $\simeq_2$. The author showed in [Sew95] that for the μ-expressions bisimulation is axiomatisable relative to infinite term equality (the equality induced by unwinding recursions to give infinite trees), with the equations

$$\begin{aligned} E + (F + G) &= (E + F) + G \\ E + F &= F + E \\ E + 0 &= E \\ E + E &= E \end{aligned}$$

and that weak bisimulation congruence is axiomatisable relative to bisimulation, with the equations

$$\begin{aligned} \mu x\ (E + ay)[\mu y\ F + \tau G/y] &= \mu x\ (E + ay + aG)[\mu y\ F + \tau G/y] \\ \mu x\ (E + \tau y)[\mu y\ F + G/y] &= \mu x\ (E + \tau y + G)[\mu y\ F + G/y] \\ a\mu x\ E &= a\mu x\ E + \tau x. \end{aligned}$$

These are presented as schemas over μ-expressions, but are expressible as typed equations in the signature of §6.1.

Whether trace congruence or language equivalence are axiomatisable relative to bisimulation remains open.

6.3 Equational axiomatisability over $*$-expressions

The results for finite equational axiomatisability over subcalculi of $*$-expressions show a delicate interaction between the equivalence and the expressiveness of the subcalculus. This is depicted in Figure 5, in which each vertex is labelled with a subset of $\{0, 1, \star, *\}$ and denotes the subcalculus of $*$-expressions closed under those operators and also under $\cdot$, $+$ and c for $c \in \mathcal{A}$. Some of the operators $\{0, 1, \star, *\}$ are interdefinable (up to bisimulation), in particular $E^\star F \sim E^* \cdot F$, $E^* \sim E^\star 1$ and $1 \sim 0^*$. This is indicated by double lines joining the equivalent subcalculi. The finite equational axiomatisability results of each subcalculus are shown to the right of its vertex. The results shown are consequences of Theorem 2 (for all equivalences between trace congruence and bisimulation), Theorem 3 (for trace congruence), the theorem of [Con71, page 106] (for language equivalence) and the positive result of [FZ94] (for bisimulation) reproduced as Theorem 4.

The figure does not show positive results by Yanov [Yan] for language equivalence of the $*$-expressions whose languages contain the empty word, Fokkink [Fok94] for

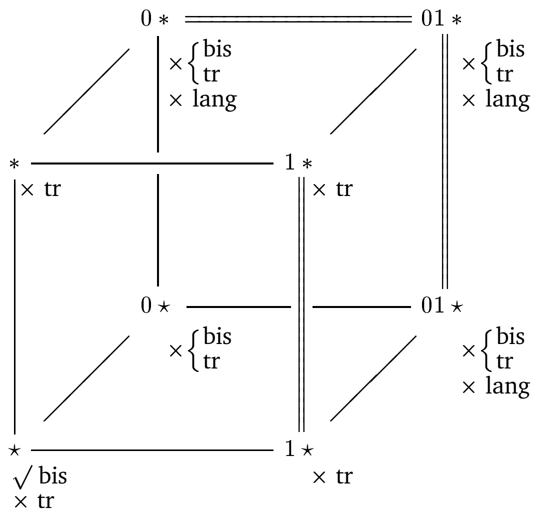

Figure 5: Finite equational axiomatisability over subcalculi of $*$-expressions

bisimulation over MPA_δ^*, i.e. the subcalculus

$$E ::= 0 \Bigm| c \cdot E \Bigm| E + E \Bigm| c^\star E,$$

by Aceto and Ingólfsdóttir [AI95] for weak bisimulation congruence over MPA_δ^* and by Aceto, Fokkink, van Glabbeek, and Ingólfsdóttir [AFvGI96] for a number of congruences that abstract from internal actions. Finally, in [AFI96] Aceto, Fokkink, and Ingólfsdóttir have shown that equivalences between ready simulation and completed trace equivalence are not finitely axiomatisable over BPA^*.

The most interesting open problem seems to be that of finding a single nonaxiomatisability proof for all equivalences between language equivalence and bisimulation, for the back face of the cube. A possible approach might be to consider the *normed U-loops* of $E \in T^2$, i.e. $\{\, l \mid \exists F, x \;.\; E \xrightarrow{U} F \xrightarrow{l} F \xrightarrow{U} \rhd x \,\}$.

Acknowledgements I would like to thank Zoltan Ésik, Wan Fokkink, Ole Jensen and Robin Milner for discussions on this work. I acknowledge support from SERC studentship 90311819, ESPRIT BRA 6454 'CONFER' and the EPSRC grant GR/K 38403 'Action Structures and the Pi Calculus'. Paul Taylor's diagram macros were used.

References

[AFI96] Luca Aceto, Willem Jan Fokkink, and Anna Ingólfsdóttir. A menagerie of non-finitely based process semantics over BPA^* : From ready simulation semantics to completed traces. Research Report RS–96–23, BRICS, Department of Mathematics and Computer Science, Aalborg University, July 1996.

[AFvGI96] Luca Aceto, Wan Fokkink, Rob van Glabbeek, and Anna Ingólfsdóttir. Axiomatizing prefix iteration with silent steps. *Information and Computation*, 127(1):26–40, 1996.

[AG87] K. B. Arkhangelskii and P. V. Gorshkov. Implicational axioms for the algebra of regular languages. *Doklady Akad. Nauk, USSR, ser A.*, 10:67–69, 1987. (in Russian).

[AI95] L. Aceto and A. Ingólfsdóttir. A complete equational axiomatization for prefix iteration with silent steps. Research Report RS–95–5, BRICS (Basic Research in Computer Science, Centre of the Danish Research Foundation), Department of Mathematics and Computer Science, Aalborg University, January 1995.

[BBP94] J. A. Bergstra, I. Bethke, and A. Ponse. Process algebra with iteration and nesting. *The Computer Journal*, 37(4):243–258, 1994. Also as University of Amsterdam Programming Research Group report P9314.

[BÉ93a] Stephen L. Bloom and Zoltán Ésik. Equational axioms for regular sets. *Math. Struct. in Comp. Science*, 3:1–24, 1993.

[BÉ93b] Stephen L. Bloom and Zoltán Ésik. *Iteration Theories: The Equational Logic of Iterative Processes*. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1993.

[BÉ94] Stephen L. Bloom and Zoltán Ésik. Iteration algebras of finite state process behaviors. Draft, February 1994.

[BÉT93] Stephen L. Bloom, Zoltán Ésik, and Dirk Taubner. Iteration theories of synchronization trees. *Information and Computation*, 102(1), January 1993.

[Bof90] M. Boffa. Une remarque sur les systèmes complets d'identités rationelles. *Theoret. Inform. Applic.*, 24(4):419–423, 1990.

[CEW58] Irving M. Copi, Calvin C. Elgot, and Jesse B. Wright. Realization of events by logical nets. *Journal of the ACM*, 5(2):181–196, April 1958.

[Con71] J. H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, 1971.

[Fok94] W. J. Fokkink. A complete equational axiomatisation for prefix iteration. *Information Processing Letters*, 52(6):333–337, December 1994. Also as CWI report CS-R9415.

[FZ94] Wan Fokkink and Hans Zantema. Basic process algebra with iteration: Completeness of its equational axioms. *The Computer Journal*, 37(4):259–267, 1994. Also as CWI report CS–R9368.

[Gla90] R. J. van Glabbeek. The linear time – branching time spectrum (extended abstract). In J.C.M. Baeten and J.W. Klop, editors, Proceedings *CONCUR '90, Theories of Concurrency: Unification and Extension,* Amsterdam, August 1990, volume 458 of *Lecture Notes in Computer Science*, pages 278–297. Springer-Verlag, 1990.

[Gla93a] R. J. van Glabbeek. A complete axiomatization for branching bisimulation congruence of finite-state behaviours. In A.M. Borzyszkowski and S. Sokołowski, editors, Proceedings 18^{th} International Symposium on *Mathematical Foundations of Computer Science,* MFCS '93, Gdansk, Poland, August/September 1993, volume 711 of *Lecture Notes in Computer Science*, pages 473–484. Springer-Verlag, 1993.

[Gla93b] R. J. van Glabeek. Divergence bisimulation. Personal communication, 1993.

[Hoa85] C. A. R. Hoare. *Communicating Sequential Processes*. Series in Computer Science. Prentice-Hall International, 1985.

[Kle56] S. C. Kleene. Representation of events in nerve nets and finite automata. In C. E. Shannon and J. McCarthy, editors, *Automata Studies*, pages 3–41. Princeton University Press, 1956. Annals of Mathematics Studies 34.

[Koz94] Dexter Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110:366–390, 1994. Also in LICS '91.

[Kro91] Daniel Krob. Complete systems of B-rational identities. *Theoretical Computer Science*, 89:207–343, 1991.

[Mil84] Robin Milner. A complete inference system for a class of regular behaviours. *Journal of Computer and System Sciences*, 28(3):439–466, 1984.

[Mil89] Robin Milner. A complete axiomatisation for observational congruence of finite state behaviours. *Information and Computation*, 81:227–247, 1989.

[Mol89] Faron Moller. *Axioms for Concurrency*. PhD thesis, University of Edinburgh, 1989.

[Nie89] Flemming Nielson. The typed λ-calculus with first-class processes. In *Proc. PARLE '89, LNCS 366*. Springer-Verlag, 1989.

[Par81] D. M. R. Park. Concurrency and automata on infinite sequences. In *Proc. 5th G.I. Conference, LNCS 104*. Springer-Verlag, 1981.

[Rab93] Alexander Rabinovich. A complete axiomatisation for trace congruence of finite state behaviors. In S. Brookes, M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of Mathematical Foundations of Programming Semantics (IX), LNCS 802*, pages 530–543, 1993.

[Red64] V. N. Redko. On defining relations for the algebra of regular events. *Ukrain. Mat. Zh.*, 16:120–126, 1964. (in Russian).

[Sal66] Arto Salomaa. Two complete axiom systems for the algebra of regular events. *Journal of the ACM*, 13(1):158–169, January 1966.

[San93] Davide Sangiorgi. *Expressing Mobility in Process Algebras: First-Order and Higher-Order Paradigms*. PhD thesis, University of Edinburgh, 1993.

[Sew94] Peter M. Sewell. Bisimulation is not finitely (first-order) equationally axiomatisable. In *Proc. 9th IEEE Symposium on Logic in Computer Science*, pages 62–70. IEEE, 1994.

[Sew95] Peter Michael Sewell. *The Algebra of Finite State Processes*. PhD thesis, University of Edinburgh, October 1995. Dept. of Computer Science technical report CST-118-95, also published as LFCS-95-328.

[Vra86] J. L. M. Vrancken. The algebra of communicating processes with empty process. Technical Report FVI 86-01, University of Amsterdam, Department of Computer Science, 1986. A later version is to appear in *Theoretical Computer Science* 177(2):287-328.

[Yan] Yanov. See [Con71, p. 108].