Bisimulation is not Finitely (First Order) Equationally Axiomatisable

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Abstract

This paper considers the existence of finite equational axiomatisations of bisimulation over a calculus of finite state processes. To express even simple properties such as $\mu XE = \mu XE[E/X]$ equationally it is necessary to use some notation for substitutions. Accordingly the calculus is embedded in a simply typed lambda calculus, allowing axioms such as the above to be written as equations of higher type rather than as equation schemes. Notions of higher order transition system and bisimulation are then defined and using them the nonexistence of finite axiomatisations containing at most first order variables is shown.

The same technique is then applied to calculi of star expressions containing a zero process — in contrast to the positive result given in [FZ93] for BPA, which differs only in that it does not contain a zero.*

1 Introduction

In this paper we consider the existence of finite equational axiomatisations for bisimulation over finite state processes. Such questions of axiomatisability of intended models for process calculi have been widely studied, with several motivations. Firstly completeness or nonexistence results, and particularly the proofs thereof, provide insight into the nature and expressiveness of the equivalence and operators involved. This insight may be useful in the design of less ad hoc calculi. It is dependent on the metalanguage used to express axioms — if this is very strong (e.g. a higher order logic) then the definition of the equivalence can be written down directly, whereas if it is weak then no interesting properties can be captured. Secondly, complete axiomatisations permit a comparison with the alternative view that takes a set of axioms as primary. Lastly, any sound axioms may be useful for human or machine manipulation, especially but not necessarily if complete.

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Our starting point is [Mil84] in which a simple calculus of μ-expressions is introduced, given by

$$E ::= 0 \mid X \mid aE \mid E + E \mid \mu XE$$

where X and a are drawn from some sets $\mathrm{Var}, \mathrm{Act}$ of variables and action constants. A complete inference system is given for a suitable notion of bisimulation over these consisting of rules for equivalence and congruence, a number of equation schemes and the inference rule scheme

$$\frac{X \text{ guarded in } F \quad E = F[E/X]}{E = \mu X\ F}$$

expressing the unique existence of certain fixed points. Axiomatisations for some other congruences such as branching bisimulation congruence [Gla93a], divergence bisimulation [Gla93b], observational congruence [Mil89] and trace congruence [Rab93] use a similar scheme each with a suitable ad hoc definition of 'guarded'. Here we investigate whether the full power of an inference rule scheme is required.

To express even simple properties of fixed points such as

$$\begin{array}{rcl} \mu X\ E &=& \mu X\ E[E/X] \\ \mu X\ E &=& E[\mu X\ E/X] \\ \mu X\ E[X,X,X,X] &=& \\ \multicolumn{3}{r}{\mu X\ E[X,X,\mu Y\ E[X,Y,X,Y],\mu Y\ E[X,Y,X,Y]]} \end{array}$$

some notation for substitution is required. We will embed the μ-expressions in a simply typed lambda calculus in which axioms such as the above can be written as equations of higher type rather than as equation schemes. The main theorem, stated in §2, asserts the nonexistence of finite axiomatisations containing at most first order variables. The proof rests on the fact that finite axiomatisations only provide bisimulations of certain 'widths', which we illustrate for the first axiom above. Writing a^n for $\overbrace{a \cdots a}^{n\ times}$, repeated use of it can derive the 'internal' unfolding

$$\mu X\ a^n X = \mu X\ a^{2^k n} X$$

for any $k \geq 0$ but not

$$\mu X\ a^n X = \mu X\ a^{3n} X.$$

The details are somewhat lengthy (occupying §2–5) but may be of some independent interest. In particular, notions of higher order transition system and bisimulation are given which might be interesting when considering richer higher order calculi.

The intermediate results can be applied to give an easy proof of the nonexistence of finite axiomatisations of bisimulation over calculi of star expressions containing a zero process. This is done in §6, where it is related to the positive result of [FZ93] for BPA* (which has no zero). Finally §7 discusses the relationship with some other previous and possible future work. Proofs will largely be omitted but may be found in the author's forthcoming thesis [Sew].

2 Basic definitions

From now on we shall be considering terms of a simply typed lambda calculus with a single base type P of processes and the following constants:

$$\begin{array}{l} 0 : P \\ a : P \rightarrow P \quad \text{for each } a \in \text{Act} \\ + : P \rightarrow P \rightarrow P \\ \mathit{fix} : (P \rightarrow P) \rightarrow P \end{array}$$

Notation and definitions will be taken from [Mit90]. In particular we write typability and $\beta\eta$ provable equality as $\Gamma \triangleright E : \sigma$ and $\mathcal{E} \vdash E = F : \sigma$ where E, F are terms; σ is a type; Γ a context and $\mathcal{E}$ a set of well typed equations. The order of a type is as usual:

$$\begin{array}{rcl} \text{order}(P) & = & 0 \\ \text{order}(\sigma \rightarrow \tau) & = & \max\{1 + \text{order}(\sigma), \text{order}(\tau)\}. \end{array}$$

We take some type assignment $\mathcal{K}$ with a countable infinity of variables at each type and for $k \in \mathbb{N} \cup \{\omega\}$ write T_k^σ for the set of terms E for which there is some type σ and context $\Gamma \subseteq \mathcal{K}$, containing only variables of order $\leq k$, such that $\Gamma \triangleright E : \sigma$. There is an obvious bijection between the μ-expressions and the long $\beta\eta$ normal forms in T_0^P with for example

$$\mu X\ aY + X \leftrightarrow \mathit{fix}\ \lambda x : P.\ +\ (ay)\ (x)$$

For any equivalence over μ-expressions this induces an equivalence over T_0^P, closing under $\beta\eta$ equality.

Definition An axiomatisation for an equivalence $\sim$ over μ-expressions is a set $\mathcal{E}$ of typed equalities that is sound, *i.e.*

$$\forall E, F \in T_0^P\ \mathcal{E} \vdash E = F : P \Rightarrow E \sim F$$

and complete,

$$\forall E, F \in T_0^P\ \mathcal{E} \vdash E = F : P \Leftarrow E \sim F.$$

Note that if $\sim$ is not both a congruence for $a, +, \mu X$ and substitutive (*i.e.* $E \sim F \Rightarrow E[G/x] \sim F[G/x]$) then there can be no axiomatisation in this sense. Further, bisimulation is a substitutive congruence.

Some sample axioms (the first three corresponding to the axiom schemes given earlier) are below for $\Gamma = \{x : P \rightarrow P \rightarrow P, y : P \rightarrow P, z : P \rightarrow P \rightarrow P \rightarrow P \rightarrow P\}$. They can all be shown sound for bisimulation by using theorem 4 below.

$$\begin{array}{rcl} \Gamma \triangleright \mathit{fix}\ y & = & \mathit{fix}\ \lambda x : P.\ y(yx) : P \\ \Gamma \triangleright \mathit{fix}\ y & = & y(\mathit{fix}\ y) : P \\ \Gamma \triangleright \mathit{fix}\ \lambda x : P.\ zxxxx & = & \mathit{fix}\ \lambda x : P. \\ & & zxx(\mathit{fix}\ \lambda y : P.\ zxyxy)(\mathit{fix}\ \lambda y : P.\ zxyxy) : P \\ \Gamma \triangleright \mathit{fix}\ \lambda y : P.\ xyy & = & \mathit{fix}\ \lambda y : P. \\ & & xy(\mathit{fix}\ \lambda z : P.\ xzz) : P \end{array}$$

We define the order of a finite set of typed equalities to be the maximum order of any variable (free or bound) therein. For example the axioms above all have order 1. The main theorem can now be stated.

Theorem 1 *If* Act *is non-empty there is no finite axiomatisation of order ≤ 1 for any substitutive congruence $\sim$ finer than (or identical to) bisimulation that for all $n \geq 1$ satisfies*

$$\mu X\ aX \sim \mu X\ a^n X.$$

The proof of this is in several steps. In the rest of this section we define an extensional equivalence $\sim_{\text{ext}}{}^\sigma$ over open terms that contains all sound equations. In

§3 higher order transition systems and higher order bisimulation are defined and we show that all processes are 'finite state' in the appropriate sense. In §4 higher order bisimulation and ${\sim_{\mathrm{ext}}}^P$ are shown to coincide over the T_1^P terms — hence all sound axioms lie within finite higher order bisimulations. Finally in §5 we construct from these an equivalence over terms which is preserved by all proofs and pick out bisimilar but non equivalent terms.

First note that axiomatisations can without loss of generality be assumed to have free variables only of specified types. In §4 there will be technical difficulties with variables of order ≥ 2 so for convenience we shall assume that there are no free variables of these types.

Definition Let $\mathcal{T}$ be the typed applicative structure of all terms in the T_1^σ with $\mathrm{App}^{\sigma \to \tau} E\ F \stackrel{def}{=} EF$ and $\mathrm{const}(c) \stackrel{def}{=} c$, as in [Mit90, §2.4.2].

Notation We write '…substitution ρ for E, F' where E, F are some terms in T_ω^P for substitutions ρ whose (finite) domain includes the free variables of E and F such that for all $x : \sigma \quad \rho(x) \in T_0^\sigma$.

Definition If $\sim$ is a substitutive congruence over μ-expressions the family of typed relations ${\sim_{\mathrm{ext}}}^\sigma$ over $\mathcal{T}$ is given by

- $E \ {\sim_{\mathrm{ext}}}^P \ F$ iff for all substitutions ρ for E, F $E\rho \sim F\rho$.
- $E \ {\sim_{\mathrm{ext}}}^{\sigma \to \tau} \ F$ iff for all $G \in T_1^\sigma \quad EG \ {\sim_{\mathrm{ext}}}^\tau \ FG$.

Theorem 2 *If $\sim$ is a substitutive congruence over μ-expressions then a set of typed equations $\mathcal{E}$ over $\mathcal{T}$ is sound for $\sim$ iff $\mathcal{E} \subseteq \sim_{\mathrm{ext}}$.*

Proof The left to right implication is a simple induction on types. For the right to left one can define a logical relation over $\mathcal{T}$ and show it coincides with $\sim_{\mathrm{ext}}$. □

3 Higher order bisimulation

In this section a more intensional definition of an equivalence is given, in two steps. Firstly a notion of higher order transition system is given and then higher order bisimulation is defined over it. The restriction to low order variables is not yet needed so this is all over T_ω^P. We generalise a definition in [Mil84]. There (where variables are all of 'type P') an extended transition system is defined over μ-expressions consisting of the usual labelled transitions together with predicates $E \rhd X$ (pronounced 'E sees X' or 'X is visible in E'). An extended bisimulation then requires matching of visibilities at each state as well as transitions and can be shown equal to the relevant special case of $\sim_{\mathrm{ext}}$. Here variables may be of higher type and so applied to arguments (which may themselves be of higher type) so we need a more sophisticated visibility predicate.

Notation From now on we take all terms mentioned to be in normal form unless stated otherwise and work up to α equivalence when convenient. We write $+$ infix except when emphasising the distinction between lambda calculus terms and others. We write $\tilde{E}$ for a tuple $E_1 \cdots E_n$ and $\longrightarrow$ for $\cup_{a \in \mathrm{Act}} \stackrel{a}{\longrightarrow}$. Bisimulation over T_0^P will be written $\sim$ and ${\sim_{\mathrm{ext}}}^P$ as just $\sim_{\mathrm{ext}}$.

Definition Take the relations $\stackrel{a}{\longrightarrow}|_{a \in \mathrm{Act}}$ and $\rhd$ to be the least over normal forms in T_ω^P such that

$$\frac{}{aE \stackrel{a}{\longrightarrow} E} \qquad \frac{}{x\tilde{E} \rhd x\tilde{E}}$$

$$\frac{E \stackrel{a}{\longrightarrow} E'}{E + F \stackrel{a}{\longrightarrow} E'} \qquad \frac{E \rhd E'}{E + F \rhd E'} \quad \text{and sym.}$$

$$\frac{E \stackrel{a}{\longrightarrow} E'}{fix\ \lambda y : P.\ E \stackrel{a}{\longrightarrow} E'[fix\ \lambda y : P.\ E/y]}$$

$$\frac{E \rhd x\tilde{E} \quad x \neq y}{fix\ \lambda y : P.\ E \rhd x\tilde{E}[fix\ \lambda y : P.\ E/y]}$$

For example consider the term

$$E \stackrel{def}{=} fix\ \lambda y : P.\ by + x(\lambda z : (P \to P) \to P.\ (zx + y))$$

with $\{x : P \to P\} \rhd E : P$. We have

$$\begin{array}{ll} & E \stackrel{b}{\longrightarrow} E \\ \text{and} & E \rhd x(\lambda z : (P \to P) \to P.(zx + E)). \end{array}$$

Higher order bisimulation is defined using this transition system:

Definition If R is a relation on T_ω^P then the typed relations $\hat{R}^\sigma$ over T_ω^σ are given by

- $\hat{R}^P = R$
- $E\ \hat{R}^{\sigma \to \tau}\ F$ iff for all $x : \sigma \in \mathcal{K}$ that are not free in $E, F \quad Ex\ \hat{R}^\tau\ Fx$.

Note that we are writing Ex for its normal form, according to the convention above.

Definition Such an R is a higher order bisimulation if $E\ R\ F$ implies

- If $E \xrightarrow{a} E'$ then $\exists F'\ F \xrightarrow{a} F'$ and $E'\ R\ F'$
- If $E \rhd x\tilde{E}$ then $\exists \tilde{F}\ F \rhd x\tilde{F}$ and $\forall i\ E_i\ \hat{R}^{\sigma_i}\ F_i$

and vice versa.

For example if

$$\begin{aligned} E &\stackrel{def}{=} \mathit{fix}\, \lambda x : P.\ y(yx) \\ F &\stackrel{def}{=} \mathit{fix}\, \lambda x : P.\ yx \end{aligned}$$

then $E \sim_{\mathrm{ho}} F$ is shown by taking a relation $\{\langle E, F\rangle, \langle yE, F\rangle\}$, the only visibilities or transitions being

$$\begin{array}{ll} E \rhd y(y(E)) & F \rhd yF \\ yE \rhd yE & F \rhd yF. \end{array}$$

As we are working with a mild generalization of finite state processes it is to be expected that all higher order bisimulations between them are in some sense finitely generated. This is reasonably straightforward. Here we need and state a result only for terms in T_1^P.

Definition The derivatives of a term $E \in T_1^P$ are $\mathrm{der}(E) \stackrel{def}{=} \{F \in T_1^P \mid E \rightsquigarrow^* F\}$ where $\rightsquigarrow$ is the least relation over T_1^P such that

$$\begin{aligned} E \longrightarrow E' &\quad\Rightarrow\quad E \rightsquigarrow E' \\ E \rhd x\tilde{E} &\quad\Rightarrow\quad \forall i\ E \rightsquigarrow E_i \end{aligned}$$

Lemma 1 *If $E \in T_1^P$ then $\mathrm{der}(E)$ is finite. Further if $E \sim_{\mathrm{ho}} F$ then there is a higher order bisimulation contained in $\mathrm{der}(E) \times \mathrm{der}(F)$ relating them.*

Proof Straightforward. $\square$

The terms in the T_ω^σ can be viewed as a small fragment of the higher order π calculus of [San93], taking a single object sort $s \mapsto ()$ and the agents with no parallel composition, infinitary sum, matching, variables of sorts containing s or infinitely many/higher order defined constants. We conjecture that $\sim_{\mathrm{ho}}$ coincides with 'normal bisimulation' over these.

4 $\sim_{\mathbf{ext}} = \sim_{\mathbf{ho}}$

We now show that over T_1^P the equivalences $\sim_{\mathrm{ext}}$ and $\sim_{\mathrm{ho}}$ coincide. First we consider the transitions of a substituted term $E\rho$. The transition and visibility predicates are related by the following.

Lemma 2 *If $E \in T_\omega^P$, ρ is a substitution for E and $E\rho \xrightarrow{a} A$ then either $E \xrightarrow{a} E'$ and $E'\rho = A$ or $E \rhd x\tilde{E}$ and $\rho(x)\tilde{E}\rho \xrightarrow{a} A$.*

Proof Induction on the derivation of $E\rho \xrightarrow{a} A$. $\square$

In general there will be a complex pattern of β reduction involved in reducing the $\rho(x)\tilde{E}\rho$ term appearing above to normal form. If $E \in T_1^P$ and $E, \rho(x)$ are in normal form, however, it is simple and a direct inductive characterization of the transitions of $E\rho$ can be given. For brevity it is not reproduced here but is important — the lack of such a result for arbitrary E is problematic when attempting to prove a more general nonexistence result.

Notation From now on we let E, F range over T_1^P.

Theorem 3 *If Act is nonempty and $E \sim_{\mathrm{ext}} F$ then $E \sim_{\mathrm{ho}} F$.*

Proof Suppose there is some action $a \in \mathrm{Act}$. By lemma 1 there is a largest N such that some derivative of E or F is higher order bisimilar to $a^N 0$ (take $N = 0$ if there are none such). Using this we construct a substitution ρ for E, F. For $y : \underbrace{P \to \cdots \to P}_{m\ times} \to P$ put

$$\begin{aligned} \rho(y) &\stackrel{def}{=} \lambda z_1 : P. \cdots \lambda z_m : P.\ aA_y \\ A_y &\stackrel{def}{=} a0 + a^{N+3+y}0 + \sum_{i \in 1..m} aB_i \\ B_i &\stackrel{def}{=} a^{2i}0 + a^{2i+1}z_i \end{aligned}$$

(eliding some injective function from variables to the naturals). One can then check that

$$\{E', F' \mid E' \in \mathrm{der}(E), F' \in \mathrm{der}(F) \text{ and } E'\rho \sim_{\mathrm{ho}} F'\rho\}$$

is a higher order bisimulation. $\square$

Theorem 4 *If $E \sim_{\mathrm{ho}} F$ then $E \sim_{\mathrm{ext}} F$.*

Proof Given the premise we must show for all substitutions ρ for E, F that $E\rho \sim_{\mathrm{ho}} F\rho$. This is done indirectly. We construct below transition systems $E{\star}\rho, F{\star}\rho$ and show in the next two lemmas that

$$E\rho \sim_{\mathrm{ho}} E{\star}\rho \sim_{\mathrm{ho}} F{\star}\rho \sim_{\mathrm{ho}} E\rho.$$

In the next section information relating the 'loop structure' of $E\rho$ and $F\rho$ is extracted from these bisimulations. $\square$

The transition system $E{\star}\rho$ differs from $E\rho$ in that states that might be identified by the non-injectivity of ρ are split apart. For example if

$$\begin{aligned} E &\stackrel{def}{=} y + aE' \\ E' &\stackrel{def}{=} \mathit{fix}\, \lambda x : P.\ a(z + ax) \\ \rho(y) \stackrel{def}{=} \rho(z) &\stackrel{def}{=} 0 \end{aligned}$$

then $E\rho \xrightarrow{a} E'\rho \xrightarrow{a} E\rho$ whereas $E{\star}\rho$ is isomorphic to

$$\bullet \xrightarrow{a} \bullet \underset{a}{\overset{a}{\rightleftarrows}} \bullet$$

To define $E{\star}\rho$ we first fix some notation. Given ρ a substitution for E, F we suppose that for all $y : \overbrace{P \to \cdots \to P}^{n\ times} \to P \in \mathrm{dom}(\rho)$ that $\rho(y)$ is of the form $\lambda z_1 : P. \ \cdots \lambda z_n : P. \ H_y$ with each z_i not free in E, F or occuring in $\mathrm{dom}(\rho)$ and y not free in any $H_{y'}$. The inference system of §3 is extended to one for inferring transitions labelled by non empty finite sequences of actions, with the rules

$$\frac{E \xrightarrow{a} F}{E \xrightarrow{a::nil} F} \qquad \frac{E \xrightarrow{a} F \quad F \xrightarrow{l} G}{E \xrightarrow{a::l} F}$$

If d is an inference tree of this system with conclusion $E \xrightarrow{l} F$ we write $d : E \xrightarrow{l} F$.

Definition Given ρ a substitution for E the transition system $E{\star}\rho$ has states

$$\begin{array}{rcl} S & \stackrel{def}{=} & \{\langle E' \rangle \mid E' \in \mathrm{der}(E)\} \\ & \uplus & \{\langle H'[\tilde{E}/\tilde{z}], y\tilde{E}, d\rangle \mid d : H_y \xrightarrow{l} H' \\ & & \quad \text{and } \exists E' \in \mathrm{der}(E) \ E' \rhd y\tilde{E}\} \end{array}$$

with root $\langle E \rangle$. The definitions of the transitions and visibilities are omitted.

Lemma 3 *If ρ is a substitution for E then $E\rho \sim_{\mathrm{ho}} E{\star}\rho$.*

Proof One can check that

$$\begin{array}{rl} & \{A\rho, \langle A \rangle \mid \langle A \rangle \text{ is a state of } E{\star}\rho\} \\ \cup & \{A\rho, \langle A, F, d\rangle \mid \langle A, F, d\rangle \text{ is a state of } E{\star}\rho\} \end{array}$$

is a higher order bisimulation, using induction on the transition derivations. □

Lemma 4 *If $E \sim_{\mathrm{ho}} F$ and ρ is a substitution for E, F then $E{\star}\rho \sim_{\mathrm{ho}} F{\star}\rho$.*

Proof By lemma 1 there is a finite higher order bisimulation R with $E \ R \ F$. Let the relation Q between the states of $E{\star}\rho$ and $F{\star}\rho$ be

$$\begin{array}{rcl} Q & \stackrel{def}{=} & \{\langle E' \rangle, \langle F' \rangle \mid E' \ R \ F'\} \\ & \cup & \{\langle H'[\tilde{E}/\tilde{z}], y\tilde{E}, d\rangle, \langle H'[\tilde{F}/\tilde{z}], y\tilde{F}, d\rangle \mid \\ & & \exists E' \in \mathrm{der}(E), F' \in \mathrm{der}(F) \text{ such that} \\ & & E' \rhd y\tilde{E}, F' \rhd y\tilde{F}, \\ & & \forall j \ E_j \ R \ F_j \text{ and } d : H_y \xrightarrow{l} H'\} \end{array}$$

One can check that Q is a higher order bisimulation betwixt $E{\star}\rho$ and $F{\star}\rho$, using induction on the transition derivations. □

5 Loop properties

The instantiations $E\rho, F\rho$ of a higher order bisimilar E, F are uniform in a sense captured by the following definition and theorem.

Definition For $u \geq 1$ take the predicate L_u and equivalence relation $\equiv_u$ over states in a transition system to be

- $L_u s$ iff s has a loop with no prime factor $\geq u$, *i.e.* $s \longrightarrow^* s' \longrightarrow^n s'$ for some s' and n with no prime factor $\geq u$.
- $s \equiv_u s'$ iff $\forall v \geq u \ L_v s \Leftrightarrow L_v s'$

Theorem 5 *If $E \sim_{\mathrm{ho}} F$ then there is some $u \geq 1$ such that for all substitutions ρ for E, F $E\rho \equiv_u F\rho$.*

Proof This follows from the following two lemmas. □

Lemma 5 *If ρ is a substitution for E then for all $u \geq 1$ $E\rho \equiv_u E{\star}\rho$.*

Proof This is a little intricate and is omitted. □

Definition For a finite relation $U \subseteq A \times B$ say the width of U is

$$\max\{\max_{a \in A} \#\{b \mid a \ U \ b\}, \max_{b \in B} \#\{a \mid a \ U \ b\}\}.$$

Lemma 6 *If $E \sim_{\mathrm{ho}} F$ then there is some $u \geq 1$ such that for all substitutions ρ for E, F $E{\star}\rho \equiv_u F{\star}\rho$.*

Proof Consider the R and Q in the proof of lemma 4. There is a $u \geq 1$, dependent on R but not on ρ, strictly greater than the width of Q. To see that $E{\star}\rho \equiv_u F{\star}\rho$ suppose that $s \ Q \ t, v \geq u$ and $L_v s$, *i.e.* for some s' and some $n \geq 1$ with no prime factors $\geq v$ $s \longrightarrow^* s' \longrightarrow^n s'$. As Q is a higher order bisimulation there exist $t_i \mid_{i \geq 0}$ such that

$$\begin{array}{ccccccc} & & s' & & s' & & s' \ \cdots \\ & & Q & & Q & & Q \ \cdots \\ t & \longrightarrow^* & t_0 & \longrightarrow^n & t_1 & \longrightarrow^n & t_2 \ \cdots \end{array}$$

but $\#\{t_i \mid i \geq 0\} < u$ so for some $k \in 1..u-1$ $t \longrightarrow^* t' \longrightarrow^{kn} t'$. Further, kn has no prime factors $\geq v$. □

The equivalences $\equiv_u$ have the following congruence property.

Lemma 7 *For $M, N \in T_0^P$ and $C[\_]$ a context from*

$$C ::= \_ \mid x \mid 0 \mid aC \mid C + C \mid \mathit{fix}\,\lambda a : P.\, C$$

(where $x : P \in \mathcal{K}$), if $M \sim_{\text{ho}} N$ and $M \equiv_u N$ then $C[M] \equiv_u C[N]$.

PROOF Straightforward. $\square$

Lemma 8 *If $\mathcal{E}$ is a finite set of typed equalities that is sound for bisimulation and of order ≤ 1 then there is some $u \geq 1$ such that for all $M, N \in T_0^P$ if $\mathcal{E} \vdash M = N : P$ then $M \equiv_u N$.*

PROOF We assume wlog that $\mathcal{E}$ contains only equalities at type P. By theorem 2 each equation lies within $\sim_{\text{ext}}$ and so by theorem 3 within $\sim_{\text{ho}}$. Take u to be the largest of those given by theorem 5 applied to each equation. An induction on a (suitably normalised) derivation of $\mathcal{E} \vdash M = N : P$ then suffices, using lemma 7 in the inductive steps and theorem 5 at the uses of axioms. $\square$

The main theorem follows by noting that if $\mathcal{E}$ is sound for an equivalence finer than bisimulation then it is sound for bisimulation and that, if q is the smallest prime strictly greater than the u given by the previous lemma, then

$$\mathit{fix}\,\lambda x : P.\, ax \not\equiv_u \mathit{fix}\,\lambda x : P.\, a^q x.$$

$\square$(of theorem 1)

6 Star expressions

Finite state systems have also been described using calculi with a unary or binary iteration operator in place of explicit recursion, such as the following star expressions

$$E ::= x \mid a \mid 0 \mid 1 \mid E + E \mid E \cdot E \mid E^* \mid E^\star E$$

where $a \in \text{Act}$. This is as expressive as the μ-expressions, up to language equivalence, however it expresses fewer bisimulation classes ([Mil84]). The results of the previous sections can be applied to give an easy proof of the nonaxiomatisability of bisimulation over these. First we make precise the transition system and bisimulation semantics used.

Definition Take the relations $\xrightarrow{a}|_{a \in \text{Act}}$ and predicate $\surd$ to be the least over closed star expressions such that

$$\begin{array}{lll}
\dfrac{}{a \xrightarrow{a} 1} & & \dfrac{}{1\surd} \\[2ex]
\dfrac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} & \text{and sym.} & \dfrac{E\surd}{E + F\surd} \\[2ex]
\dfrac{E \xrightarrow{a} E'}{E \cdot F \xrightarrow{a} E' \cdot F} & \dfrac{E\surd \quad F \xrightarrow{a} F'}{E \cdot F \xrightarrow{a} F'} & \dfrac{E\surd \quad F\surd}{E \cdot F\surd} \\[2ex]
\dfrac{E \xrightarrow{a} E'}{E^* \xrightarrow{a} E' \cdot E^*} & & \dfrac{}{E^*\surd} \\[2ex]
\dfrac{E \xrightarrow{a} E'}{E^\star F \xrightarrow{a} E' \cdot (E^\star F)} & \dfrac{F \xrightarrow{a} F'}{E^\star F \xrightarrow{a} F'} & \dfrac{F\surd}{E^\star F\surd}
\end{array}$$

Definition Say closed star expressions E, F are bisimilar, written $\sim_{\surd}$, if there is a symmetric relation R such that $E\ R\ F$ and for all $E'\ R\ F'$

- If $E' \xrightarrow{a} E''$ then $\exists F''\ F' \xrightarrow{a} F''$ and $E''\ R\ F''$
- If $E'\surd$ then $F'\surd$.

This differs from the semantics of [BBP93, FZ93] in that to give semantics to 1 a judgement $\surd$ is used instead of $\xrightarrow{a} \surd$. For terms of $\{1, a, +, \cdot\}$ it is that of [Mol89, §6.3.1] for BPA$^\epsilon$ (identifying 1 and ϵ).

Proposition 1 *For terms of $\{0, a, +, \cdot, {}^\star\}$ bisimulation as defined above coincides with bisimulation over $BPA_\delta^\star$ (identifying 0 and δ) as defined in [BBP93].*

PROOF Straightforward. $\square$

Theorem 6 *There is no finite axiomatisation for bisimulation over any set of star expressions closed under $\{0, a, +, \cdot\}$ and one of $\{\_^*, \_^\star\_\}$.*

This is shown below. It contrasts nicely with the following.

Theorem 7 (Fokkink and Zantema) *The axioms below are sound and complete for bisimulation over expressions of $\{a, +, \cdot, \_^\star\_\}$.*

$$\begin{array}{rcl}
x + y &=& y + x \\
(x + y) + z &=& x + (y + z) \\
x + x &=& x \\
(x + y) \cdot z &=& x \cdot z + y \cdot z \\
(x \cdot y) \cdot z &=& x \cdot (y \cdot z) \\
x \cdot (x^\star y) + y &=& x^\star y \\
x^\star (y \cdot z) &=& (x^\star y) \cdot z
\end{array}$$

$$x^\star(y\cdot((x+y)^\star z)+z) \quad = \quad (x+y)^\star z$$

Proof This is immediate from the result of [FZ93] and proposition 1 above. □

The star expressions can be faithfully embedded into our lambda calculus:

Definition Take the map $[\![\_]\!]$ from star expressions to lambda calculus terms of type $P \to P$ to be

$$\begin{aligned}
[\![x]\!] &= x \text{ where we suppose } x : P \to P \in \mathcal{K} \\
[\![a]\!] &= a \\
[\![0]\!] &= \lambda y : P.\, 0 \\
[\![1]\!] &= \lambda y : P.\, y \\
[\![E+F]\!] &= \lambda y : P.\, +([\![E]\!]y)([\![F]\!]y) \\
[\![E\cdot F]\!] &= \lambda y : P.\, [\![E]\!]([\![F]\!]y) \\
[\![E^*]\!] &= \lambda y : P.\, fix\, \lambda z : P.\, +(y)([\![E]\!]z) \\
[\![E^\star F]\!] &= \lambda y : P.\, fix\, \lambda z : P.\, +([\![F]\!]y)([\![E]\!]z)
\end{aligned}$$

Lemma 9 *If $y : P \in \mathcal{K}$ then*

$$E \sim_{\surd} F \quad \Longleftrightarrow \quad [\![E]\!]y \sim_{\mathrm{ho}} [\![F]\!]y.$$

Proof Straightforward. □

To show theorem 6 consider a sound finite set $\mathcal{A}$ of axioms. The encodings of these must be higher order bisimilar (as the terms constructed in the proof of theorem 3 are all expressible) so $[\![\mathcal{A}]\!]$ is a sound set in our earlier sense. By lemma 8 there is some u such that if $M, N \in T_0^P$ and $[\![\mathcal{A}]\!] \vdash M = N : P$ then $M \equiv_u N$. Now consider the relevant pair of terms below, where q is the smallest prime strictly greater than u.

$$\begin{aligned}
E &\stackrel{def}{=} a^* \cdot 0 & \qquad E &\stackrel{def}{=} a^\star 0 \\
F &\stackrel{def}{=} (\overbrace{a\cdot \ldots \cdot a}^{q\ times})^* \cdot 0 & \qquad F &\stackrel{def}{=} (\overbrace{a\cdot \ldots \cdot a}^{q\ times})^\star 0
\end{aligned}$$

In each case $[\![E]\!]y \not\equiv_u [\![F]\!]y$ so $[\![\mathcal{A}]\!] \not\vdash [\![E]\!]y = [\![F]\!]y : P$ so $\mathcal{A} \not\vdash E = F$ but $E \sim_{\surd} F$. □(of theorem 6)

The addition of other operators does not affect the result, so long as their semantics is expressible by an encoding into lambda terms as above.

7 Discussion

There is a substantial literature dealing with axiomatisation of equivalences over calculi denoting finite state systems. Some of this is referred to in the diagram below, classified by the equivalence, calculus and strength of logic addressed and labelled $\surd$ (resp. $\times$) if finite complete systems are given (resp. shown not to exist). It is far from exhaustive. In particular no mention is made of calculi with parallel operators or proof systems with infinitary rules such as the approximation induction principle of ACP or the ω rule arising from domain theoretic models [Hen88]. Care must be taken when comparing the results as there are differing definitions, for example of the star expressions.

Looking first at the equational face, it was shown by Redko [Red64, completed by Pilling] and Conway [Con71] that no finite system exists for language equality of star expressions. The latter and the present work can both be seen as stemming from the impossibility of equationally expressing arbitrary 'internal' unfoldings of iteration/recursion. It would be nice to have a single proof capturing this. Yanov gave a finite system for language equality of the star expressions whose languages contain the empty word [Yan] and we have already seen the finite system of Fokkink and Zantema for bisimulation of star expressions without a zero process [FZ93].

Turning to the implicational face, for language equivalence of star expressions several finite systems have been given, of which we recall just a couple of rules. Salomaa gave an impure system in [Sal66] using the rule

$$\frac{E = E\cdot F + G \text{ the empty word not in language}(F)}{E = G\cdot F^*}$$

which has a side condition that is not preserved by substitution. Pure systems have been given by Arkhangelskii and Gorshkov [AG87], Boffa and Krob [Bof90, Kro91] and Kozen [Koz91]. Sample rules from these are

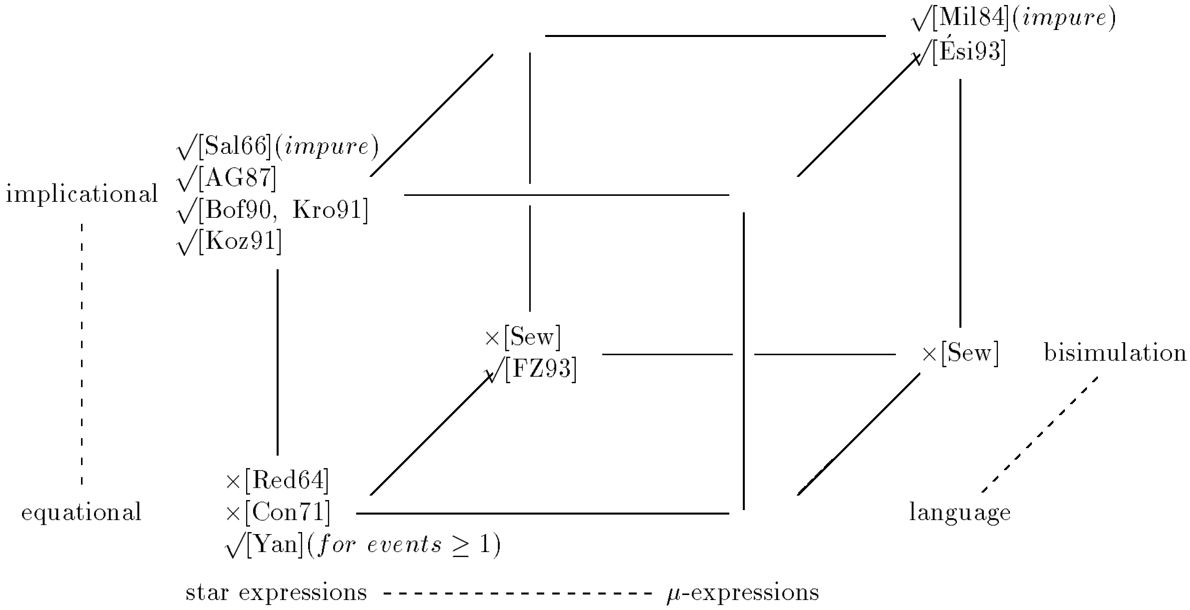
$$\frac{(a_1+a_2)^*\cdot a_3 = (b_1+b_2)^*\cdot b_3}{(a_1+a_2)^*\cdot a_3 = (b_2+b_1\cdot a_1^*\cdot a_2)^*\cdot(b_3+b_1\cdot a_1^*\cdot a_3)}$$

$$\frac{x\cdot x = x}{x^* = 1+x}$$

$$\frac{x\cdot y + y = y}{x^*\cdot y + y = y}$$

For bisimulation of μ-expressions a finite implicational system has been given by Milner [Mil84] with the impure rule we saw in §1. A finite pure system has been given by Ésik [Ési93] using the rule

$$\frac{fix\,\lambda u : P.\, tuu = fix\,\lambda u : P.\, t'uu}{fix\,\lambda u : P.\, tuu = fix\,\lambda v : P.\, t(fix\,\lambda u : P.\, t'uv)v}$$



derived from work on iteration theories.

Finally we should mention that various illuminating infinite but simple equational systems exist. Discussion of these may be found for example for language equivalence of star expressions in [Kro91], in the general setting of iteration theories in [BE93a, BE93b, BET93] and for bisimulation of μ-expressions in [Sew].

As to future work, several directions spring to mind. Firstly one could attempt to remove the restriction in the present work to axioms of order one. This requires an improved understanding of the interaction between β reduction and the transition relation — probably involving a richer notion of higher order transition system — that might be interesting for its own sake. Secondly one could consider coarser equivalences and richer signatures, for example with a base type A of actions and a constant $. : A \to P \to P$. This would be closer to the higher order π calculus, with it's distinction between names and variables.

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