Surrey Mathematics Lectures 2010

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Abstract

This is a course of 20 lectures for first year undergraduates. The content may be quickly inferred from the Section headings.

1 Introduction

This is a course (EE1.ma.A1) for about 70 first year students in the Faculty of Engineering and Physical Sciences at the University of Surrey, at Guildford, in the Autumn Term of 2010. They have a variety of backgrounds with different nationalities and differing bodies of previous knowledge. The lecture material which I have written below, and which I have delivered in 20 lectures, is supported by discussions of unassessed work with tutors, and by associated homework marked by graduate students. Previous lecturers for this course were Dr. Anne Skeldon (before 2003), Professor Ian Roulstone (2003 - 2007) and Dr. Henk Bruin (2008 - 2009). The notes below follow closely the topics treated by Dr. Skeldon, but the material has been completely rewritten by me. The students were each given a copy of a 118-page booklet containing another version of the course notes, and 8 Assignment sheets.

2 Books

No textbook is uniquely recommended. Many are available, often very large, with more than 1000 pages, and therefore covering much more material than is described here.

The best book that I know which covers the area sometimes called Mathematical Methods is

Calculus, by J.Marsden and A.Weinstein, Benjamin/Cummings Publishing Co., Menlo Park, California. 1980. 1012 pp.

More recent books are

Engineering Mathematics, by K.A.Stroud and D.J. Booth, Palgrave/Macmillan. 2007. 1258 pp., and

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Modern Engineering Mathematics, by G.James, Pearson/Prentice Hall, 2001. 978 pp.

Other books with similar scope include

Calculus, by R.A.Adams, Addison-Wesley. 1999. 1117 pp., and

Calculus with Analytic Geometry, by C.H.Edwards and D.E.Penney, Prentice Hall. 1998. 1120 pp.

3 Elementary operations

The operations of addition, subtraction, multiplication and division, and the rules of precedence for the order of performing these operations are mentioned. For example, brackets are removed from inside to out, and brackets are used to indicate the order of operations.

E.g. in arithmetic where we deal only with numbers, we write

6 x (8 - (4 - 5)) = 6 x (8 - (-1)) = 6 x (8 + 1) = 6 x 9 = 54.

6 x 8 - (4 - 5) = 48 - (-1) = 48 + 1 = 49.

Also in algebra, where letters represent numbers,

$4a$ x $6b$ means (4 x a) x (6 x b) = 4 x a x 6 x b = 4 x 6 x a x b = 24 x a x b = 24 ab.

Where there is no ambiguity we can omit the x sign and agree that writing the symbols next to each other means multiplication.

$3a + 6a = 9a$ is thus simplified, but $3a + b$ cannot be simplified further.

Care is required not to confuse the letter x with the multiplication symbol x. Brackets can avoid use of the latter. For example, we multiply sequentially

$(x - 2)(x^2 - 4x + 2) = x(x^2 - 4x + 2) - 2(x^2 - 4x + 2)$

$= x^3 - 4x^2 + 2x - 2x^2 + 8x - 4 = x^3 - 6x^2 + 10x - 4.$

Don't confuse z with 2. Write numbers before letters, e.g. $2x$ not $x2$. Note the ordering is that of descending powers.

4 Factorizing

This means identifying terms which can be coupled together, thus:

$y^2 + 6y + 8 = y^2 + (4 + 2)y$ + (4 x 2) = $(y + 4)(y + 2)$,

$y^2 - 6y + 8 = y^2 - (4 + 2)y$ + (-4) x (-2) = $(y - 4)(y - 2)$,

$y^2 + 2y - 8 = y^2 + (4 - 2)y$ + 4 x (-2) = $(y + 4)(y - 2)$.

5 Quadratic equations

"Quadratic" *means* the highest power present is *two*, i.e. *squared*. For example

$x^2 + 6x + 8 = 0$ is a *quadratic* equation for x, because the highest power of the *unknown* x present is 2.

How do we find x? By *factorizing*, as follows.

$x^2 + 6x + 8 = 0$ implies $(x + 4)(x + 2) = 0$,

so either $x + 4 = 0$ or $x + 2 = 0$ or both.

Thus $x = -4$ and $x = -2$ are two (and the only two) solutions of the quadratic equation.

There is a *graphical* representation. We imagine the graph

$y = x^2 - 6x + 8 = (x - 4)(x - 2)$

[Diagram]

plotted in the x, y plane. It is a *parabola* (highest power 2) which cuts the x-axis $y = 0$ at $x = 2$ and $x = 4$.

6 Solution of a general quadratic equation

Solve $ax^2 + bx + c = 0$ where x is the unknown, and a, b, c are given constants.

The method is to "complete the square", i.e. to rewrite the equation as

$a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a} = 0$

[Check: $a(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a} = ax^2 + bx + c$]

so $(x + \frac{b}{2a})^2 = \frac{1}{a}(\frac{b^2}{4a} - c)$

$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$

$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2-4ac)}}{2a}$.

So general solution for *any* choice of a, b, c is

$x = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a}$.

7 Example with real roots

A solution is sometimes called a root.

$x^2 + 6x + 8 = 0$.

[Diagram]

This is an example of $ax^2 + bx + c = 0$ with $a = 1, b = 6, c = 8$ and we can either

quote the formula to give $x = \frac{-6 \pm \sqrt{(36-32)}}{2} = 3 \pm 1 = -2$ or -4

or factorise the equation giving $(x + 4)(x + 2) = 0$ and therefore

$x + 4 = 0$ or $x + 2 = 0$ with the same result.

So this example $x^2 + 6x + 8 = 0$ has *real* roots - 2 and - 4, and a graph of $y = x^2 + 6x + 8$ can be shown.

In examples which do not look easy we can always quote the formula, but it is better first to look to see if it is easy to factorise the equation with two linear factors.

8 Example with complex roots

Solve $x^2 + x + 1 = 0$. This is an example of $ax^2 + bx + c = 0$ with $a = b = c = 1$, so in the formula $b^2 - 4ac = 1 - 4 = -3$, which is negative. Quoting the formula gives

$x = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a} = \frac{-1 \pm \sqrt{(-3)}}{2}$.

There are no *real* solutions, and this is because the graph of $y = x^2 + x + 1$ does *not* cross the x-axis.

To plot it, find the turning point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1 = 0$ at $x = -\frac{1}{2}$, $y = \frac{3}{4}$. Note that $y = 1$ at $x = 0$.

[Diagram]

9 Complex numbers

Real numbers, whether they be

integers - 93, - 11, 0, 1, 5, 1003,

fractions - $\frac{22}{7}$, - $\frac{1}{3}$, $\frac{1}{2}$, (rational)

non-fractions 3.14159... $= \pi$, 2.71828...$= e$, (irrational)

can each be represented by a point somewhere on the *real line*.

[Diagram]

Complex numbers are different animals which are defined by introducing a brand new idea.

$\sqrt{(-1)} = i$ which is *defined* to be a solution of the equation $z^2 + 1 = 0$, so that $z^2 = -1$ is satisfied by *two* solutions $z = \pm\sqrt{(-1)} = \pm i$.

Sometimes j is used instead of i.

We can next use $i = \sqrt{(-1)}$ to define a new idea

$x + iy$ which is called a *complex number*, in which x and y are *real* numbers. We often write

$z = x + iy$ for a complex number having a real part x and an imaginary part iy.

It may seem surprising to introduce an imaginary number $\sqrt{-1} = i$ but it turns out to be useful.

For example, we can then say that *every* quadratic equation $ax^2 + bx + c = 0$ has *two* solutions

$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ which will be *real* if $b^2 > 4ac$ and *complex* if $b^2 < 4ac$.

For example $x^2 + x + 1 = 0$, in which $a = b = c = 1$, has two solutions

$x = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$.

Thus *both* $x = \frac{-1+i\sqrt{3}}{2}$ and $x = \frac{-1-i\sqrt{3}}{2}$ satisfy $x^2 + x + 1 = 0$. But these solutions are complex numbers and *not* real numbers.

10 Definition of equality

Two complex numbers $x + iy$ and $a + ib$ are equal if and only if their *real* parts are equal ($x = a$) *and* their *imaginary* parts are equal ($y = b$). We must have *both*.

Example: $a + ib = 3 + 4i$ means that $a = 3$ and $b = 4$.

Next we must discuss, by examples, addition, subtraction and multiplication of complex numbers. Division is treated later.

Addition and Subtraction. Define $w = 2 + i, z = 3 + 5i$.

Then $w + z = 2 + i + 3 + 5i = 5 + 6i, w - z = 2 + i - (3 + 5i) = -1 - 4i$.

We add the real parts, and add the imaginary parts, *separately.* And we subtract the real parts, and subtract the imaginary parts, also *separately.*

Multiplication. The *product* of $w = 2 + i$ and $z = 3 + 5i$ is $wz = (2 + i)(3 + 5i) = 6 + 10i + 3i + 5i^2 = 6 + 13i - 5 = 1 + 13i$ using $i^2 = -1$.

11 Complex conjugate

The complex conjugate of $z = a + ib$ with real a and b is *defined* to be $a - ib$, and is often written $\bar{z}$ or sometimes z^*.

Examples of complex conjugate are $z = 3+4i$ with $\bar{z} = 3-4i$; $z = i$ with $\bar{z} = -i$; $z = 3$ with $\bar{z} = 3$.

12 Some properties of complex conjugates

The product $z\bar{z} = (3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2 = 9 - 16i^2 = 9 + 16 + 25$ is *real.*

In *general* the product of z and $\bar{z}$ is always *real* and *positive*.

$z = a + ib$ with $\bar{z} = a - ib$ implies

$z\bar{z} = (a + ib)(a - ib) = a^2 - iab + iab - i^2b^2 = a^2 + b^2$

which is real and positive for any z and for any real a, b.

Also *any* z which satisfies $z = \bar{z}$ must be real because $z = a + ib$ implies $\bar{z} = a - ib$, so that $a + ib = a - ib$. This implies $a - a + i(b + b) = 0$, so that $2ib = 0$, and therefore $b = 0$ and $z = a = \bar{z}$.

13 Modulus or amplitude of a complex number

Because $z = a + ib$ with $\bar{z} = a - ib$ implies $z\bar{z} = a^2 + b^2$ which is always *positive*, the square root $\sqrt{(a^2 + b^2)}$ is called the *modulus* of z, and is written $\mid z \mid = \sqrt{(a^2 + b^2)}$.

We also see that $z = a + ib$ and $\bar{z} = a - ib$ imply $z + \bar{z} = 2a$ and $z - \bar{z} = 2ib$. Thus

$a = \frac{1}{2}(z + \bar{z})$ is the *real* part $= Re(z)$ of z.

$ib = \frac{1}{2}(z - \bar{z})$ is the *imaginary* part $= Im(z)$ of z, with $b = -\frac{i}{2}(z - \bar{z})$.

14 Division of complex numbers

Example: given $z = 2 + 3i$ and $w = 4 + 5i$, find $\frac{z}{w}$ in the form $a + ib$.

Method: multiply numerator and denominator by the complex conjugate of the denominator.

$\frac{z}{w} = \frac{2+3i}{4+5i} = \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)} = \frac{8+12i-10i-15i^2}{6+20i-20i-25i^2} = \frac{8+15+2i}{16+25} = \frac{23+2i}{41} = \frac{23}{41} + i\frac{2}{41}$,

so that the real part of $\frac{z}{w}$ is $\frac{23}{41}$, and the imaginary part of $\frac{z}{w}$ is $\frac{2i}{41}$.

15 Geometry of complex numbers

Any complex number, such as $z = x + iy$ with real x and y, and $i = \sqrt{(-1)}$, can be represented by a point on the x, y plane.

[Diagram]

This picture is called the *Argand diagram* after the Frenchman Jean-Robert Argand who invented it in 1806.

The magnitude x of the *real* part is plotted on the *horizontal* axis.

The magnitude y of the *imaginary* part is plotted on the *vertical* axis.

To discuss the *geometry* of complex numbers $z = x + iy$ we need the *magnitude* $r = \sqrt{(x^2 + y^2)}$ ($r^2 = x^2 + y^2$ by Pythagoras), also called the *modulus* $\mid z \mid = r$, and the *amplitude* $\theta = \tan^{-1}\frac{y}{x}$ which is also called the *argument*, so that $\tan\theta = \frac{y}{x}$.

[Diagram]

We can also write $\frac{x}{r} = \cos\theta$ or $x = r\cos\theta$, and $\frac{y}{r} = \sin\theta$ or $y = r\sin\theta$.

Because $\theta = \theta + 2\pi = \theta + 2n\pi$ for n revolutions of the radius (e.g. 50° is the same as 50° + 360° = 410°) it is conventional to *restrict* θ to the range $-\pi < \theta \leq \pi$ of *principal* values.

[Diagram]

If we use the complex conjugate $\bar{z} = x - iy$ of $z = x + iy$ we have $z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$, so we can also write the modulus as $r = \sqrt{(x^2 + y^2)} = \sqrt{(z\bar{z})}$.

Examples: find the modulus (magnitude) and argument (amplitude) of $1 + i\sqrt{3}$. This is $r(\cos\,\theta + \mathrm{i}\sin\,\theta)$ if

$\tan\,\theta = \frac{r sin\theta}{r cos\theta} = \frac{\sqrt{3}}{1} = \sqrt{3}$, so

$\theta = tan^{-1}\sqrt{3} = 60° = \frac{\pi}{3}$ radians,

and if $r^2 = r^2(cos^2\theta + sin^2\theta) = 1 + (\sqrt{3})^2 = 4$, so r= 2.

Particular points which can be plotted on the Argand diagram are $i, 1+i, 1-i, 1$.

[Diagram]

16 Polar coordinates

These are radius (r) and angle (θ) alternatives to cartesian x, y coordinates given by

$x = r\cos\,\theta$, $y = r\sin\,\theta$ so that $r = \sqrt{(x^2 + y^2)}$, $\theta = \tan^{-1}\frac{y}{x}$.

[Diagram]

For example, a circle of radius a may be described as $x = a\cos\theta$ with $y = a\sin\theta$, or $r = a$ for all θ.

17 Exponentials

Any two numbers a and b can be used to construct two more numbers a^b and b^a. For example, 4 and 3 deliver $4^3 = 64$ and $3^4 = 81$. The "power" 3 in the first case, or 4 in the second, is called the "exponent".

When the exponent is allowed to be a variable, called x say, then c^x is called an *exponential* function with base c. If c is *any* positive number, and x is *real*, then $c^x \to \infty$ as $x \to \infty$, $c^0 = 1$ (a definition), and $c^x \to 0$ as $x \to -\infty$. The graph can be sketched.

[Diagram]

There is a very special value of c, namely 2.7182818285... which is always denoted by e, for which the gradient $\frac{\mathrm{d}e^x}{\mathrm{d}x}$ of e^x at every x is the value of e^x there, i.e.

$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$.

This particular and famous function e^x is called *the* exponential function.

18 Imaginary and complex exponents

Exponents, like the real x in e^x above, can *also* be imaginary (like iy with real y and $i = \sqrt{(}-1))$ or complex (like $x + iy$).

It can be proved that, for the special number $e = 2.71828...$, $e^{i\theta} = \cos\theta + \mathrm{i}\sin\theta$ with any real θ (Euler 1748). No elementary proof exists, so we shall treat this result as an axiom.

We shall use it with polar coordinates r, θ and cartesian coordinates x, y to construct alternative versions of any complex number as follows.

$z = x + iy = r\mathrm{cos}\theta + ir\mathrm{sin}\theta = r(\mathrm{cos}\theta + i\mathrm{sin}\theta) = re^{i\theta}$.

[Diagram]

This follows because $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{y}$.

Examples are $1 + i\sqrt{3} = 2\mathrm{e}^{\frac{i\pi}{3}}$ with $\theta = 60° = \frac{\pi}{3}$ radians;

$i = e^{\frac{i\pi}{2}}$ with $\theta = 90° = \frac{\pi}{2}$ radians;

$1 + i = \sqrt{2}e^{\frac{i\pi}{4}}$ with $\theta = 45° = \frac{\pi}{4}$ radians;

$1 - i = \sqrt{2}e^{\frac{-i\pi}{4}}$ with $\theta = -45° = -\frac{\pi}{4}$ radians;

$1 = 1e^{i0}$ with $e^0 = 1$.

[Diagram]

19 A famous example

of $z = x + iy = re^{i\theta}$

[Diagram]

is the case when $r = 1$ and $\theta = \pi$ so that $x = -1$ and $y = 0$ giving

$e^{i\pi} = -1$

which some people think is *the most famous formula in mathematics.*

This is because it relates three different but basic things e, π, i in a very simple way.

20 Examples of multiplication and division via the exponential form

For any two complex numbers $z = re^{i\theta}$ and $w = se^{i\phi}$

the product is $zw = re^{i\theta}se^{i\phi} = rse^{i(\theta+\phi)}$

so we multiply the magnitudes (moduli) and add the angles (arguments);

and the quotient is $\frac{z}{w} = \frac{re^{i\theta}}{se^{i\phi}} = \frac{r}{s}e^{i(\theta-\phi)}$

so we divide the moduli and subtract the angles.

Particular cases include i and $e^{i\phi}$ whose product is $ie^{i\phi} = e^{i(\phi+\frac{\pi}{2})}$

because $i =$cos$\frac{\pi}{2} + i$sin$\frac{\pi}{2} = e^{i\frac{\pi}{2}}$ (using cos$\frac{\pi}{2} = 0$

and sin$\frac{\pi}{2} = 1$).

So multiplying by i rotates the complex number by $\frac{\pi}{2}$.

Also $z = 5e^{3i}$ and $w = 3e^{2i}$ have product $zw = 5e^{3i}.3e^{2i} = 15e^{15i}$

and quotient $\frac{z}{w} = \frac{5e^{3i}}{3e^{2i}} = \frac{5}{3}e^{i}$.

Another type of example is that $z = 5e^{i\frac{\pi}{3}}$ implies

$z^{10} = (5e^{i\frac{\pi}{3}})^{10} = 5^{10}e^{i\frac{10\pi}{3}} = 5^{10}(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3})$.

21 Addition and subtraction of fractions

This is done by changing each fraction so that they have the same bottom (then called the *common denominator*), and then adding or subtracting the *numerators* (tops).

$\frac{1}{2} + \frac{3}{2} = \frac{1+3}{2} = \frac{4}{2} = \frac{2}{1} = 2.$

$\frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{2+3}{6} = \frac{5}{6}.$

The above is *arithmetic*, but the same method works in *algebra* as follows.

$\frac{w}{5} + \frac{3w}{4} = \frac{4w}{20} + \frac{15w}{20} = \frac{4w+15w}{20} = \frac{19w}{20}$.

$\frac{y}{2} + \frac{y}{8} = \frac{4y}{8} + \frac{y}{8} = \frac{4y+y}{8} = \frac{5y}{8}$.

$\frac{1}{w} + \frac{1}{w-1} = \frac{w-1}{w(w-1)} + \frac{w}{w(w-1)} = \frac{w-1+w}{w(w-1)} = \frac{2w-1}{w(w-1)}$.

$\frac{2}{y-1} + \frac{3}{y+1} + \frac{4}{(y+1)^2}$

$= \frac{2(y+1)^2+3(y-1)(y+1)+4(y-1)}{(y-1)(y+1)^2}$

$= \frac{2(y^2+2y+1)+3(y^2-1)+4(y-1)}{(y-1)(y+1)^2}$

$= \frac{5y^2+8y-5}{(y-1)(y+1)^2}$.

22 Partial fractions

This is the name sometimes used for the *reverse* procedure in which we write a given function as a sum of simpler fractions such as (the previous example reversed)

$\frac{5y^2+8y-5}{(y-1)(y+1)^2} = \frac{2}{y-1} + \frac{3}{y+1} + \frac{4}{(y+1)^2}$.

This can make it easier to work with the expression, e.g. to integrate it (later in the course).

The method is as follows.

1. Factorise the denominator (as far as possible) into, e.g., linear [ax + b] or quadratic [$(ax + b)^2$] terms.

2. Use these factors to construct terms like

$\frac{A}{ax+b}$ or $\frac{B}{(ax+b)^2}$ for some *constants* A, B, a, b.

3. Add to recover the original expression.

4. The purpose is, for example, to help integration.

Example.

Write $\frac{8y+1}{2y^2-y-1}$ in partial fractions.

Step 1. Factorise the denominator $2y^2 - y - 1 = (2y + 1)(y - 1)$.

Step 2. Use these factors to write

$\frac{8y+1}{2y^2-y-1} = \frac{8y+1}{(2y+1)(y-1)} = \frac{A}{2y+1} + \frac{B}{y-1} = \frac{A(y-1)+B(2y+1)}{2y^2-y-1} = \frac{(A+2B)y+B-A}{2y^2-y-1}$

with A and B to be found.

Step 3. Equate coefficients, so we must have

$A + 2B = 8$ and $B - A = 1$,

so $A + 2(A + 1) = 8$ giving $3A = 6$ and therefore $A = 2, B = 3$.

Step 4. We conclude that

$\frac{8y+1}{2y^2-y-1} = \frac{2}{2y+1} + \frac{3}{y-1}$.

The method works in this example because

(i) the numerator is a lower degree polynomial than the denominator ($\frac{linear}{quadratic}$);

(ii) there are no repeated factors;

(iii) all factors are linear.

Example (with *repeated* factors , and *nonlinear* factors).

Express $\frac{3x^2+4x+6}{(x-1)^2(x^2+3x+1)}$ in partial fractions.

Try $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+3x+1} = \frac{A(x-1)(x^2+3x+1)+B(x^2+3x+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+3x+1)}$

so that the numerator must be

$3x^2 + 4x + 6 = A(x^3 + 3x^2 + x - x^2 - 3x - 1) + B(x^2 + 3x + 1) + (Cx + D)(x^2 - 2x + 1)$

$= A(x^3 + 2x^2 - 2x - 1) + B(x^2 + 3x + 1) + C(x^3 - 2x^2 + x) + D(x^2 - 2x + 1)$

$= x^3(A + C) + x^2(2A + B - 2C + D) + x(-2A + 3B + C - 2D) - A + B + D$

so by comparing coefficients on the left and right sides we need

$A + C = 0, 2A + B - 2C + D = 3, -2A + 3B - 2D = 4, -A + B + D = 6.$

Thus $C = -A$ implies $4A + B + D = 3, -3A + 3B - 2D = 4, -A + B + D = 6.$

This leaves three equations for three unknowns, and substituting $A = B + D - 6$ from the last into the middle two gives

$4(B+D)-24+B+D=3$, $-3(B+D)+18+3B-2D=4$ and thus

$5B+5D=27$ and $-5D=-14$ so that

$D=\frac{14}{5}$, $5B=27-14=13$ giving $B=\frac{13}{5}$, $A=\frac{27}{5}-6=-\frac{3}{5}$, $C=\frac{3}{5}$

and therefore finally

$\frac{3x^2+4x+6}{(x-1)^2(x^2+3x+1)}=-\frac{3}{5(x-1)}+\frac{13}{5(x-1)^2}+\frac{3x+14}{5(x^2+3x+1)}$.

23 Solution of linear simultaneous equations

Example: find x, y which satisfy *both* of $x+y=2$ and $2x+3y=5$.

The first gives $y=2-x$ which can be substituted into the second to give $2x+3(2-x)=5$, and therefore $2x-3x=5-6$ or $-x=-1$. Hence $x=1$ and $y=2-1=1$. Substituting back into the starting equations confirms that $x=y=1$ is the solution.

A general notation: find x, y which satisfy *both* of

$ax+by=p$ and $cx+dy+q$ for known a, b, c, d, p, q.

These can be written in *matrix* notation, using a row-on-column definition of matrix multiplication, in the form

$A\mathbf{x}=\mathbf{p}$

where A is a 2x2 square matrix of the coefficients a, b, c, d, $\mathbf{x}$ is a 2x1 column matrix x, y of the unknowns, and $\mathbf{p}$ is a 2x1 column matrix p, q of the known right hand side.

24 A general solution method

The following method is sometimes called *Gaussian elimination* with "back substitution" after Carl Friedrich Gauss (1777 - 1855).

We wish to solve

$ax+by=p$ simultaneously with $cx+dy=q$ (which pair could be written in matrix form $A\mathbf{x}=\mathbf{p}$), for the unknowns x, y where a, b, c, d, p, q are known.

Rewriting the first equation as $x=\frac{p}{a}-\frac{by}{a}$ and substituting into the second gives

$\frac{cp}{a}-\frac{cby}{a}+dy=q$ and hence $y=\frac{aq-cp}{ad-cb}$ provided $ad-cb\neq 0$.

"Back substitution" of this y into the first equation above then gives

$x = \frac{p}{a} - \frac{b(aq-cp)}{q(ad-cb)}$.

This method always works *provided* the *determinant* of coefficients $ad - bc \neq 0$.

25 Example

$x + y = 2$ with $2x + 3y = 5$ can be displayed in matrix form.

Substituting $x = 2 - y$ from the first into the second gives

$2(2 - y) + 3y = 5$ and therefore $y = 5 - 4 = 1$.

Back substitution then gives $x = 2 - y = 1$.

We can check this answer as $1 + 1 = 2$ with $2 + 3 = 5$. Notice that the determinant of coefficients is $3 - 2 = 1 \neq 0$.

26 Another example

This time we have three unknowns x, y, z instead of two as above. Solve

$x + 2y + 5z = 5$, $2x + y + z = 4$, $x - y + z = 4$.

We can rewrite the first of these as $x = 5 - 2y - 5z$, and substituting in the second and third gives

$-3y - 9z = -6$ and $-3y - 4z = -1$.

Rewriting the first of these as $3y = 6 - 9z$ and substituting in the second gives $-6 + 9z - 4z = -1$ so that $z = 1$.

Back substituting gives $-3y - 9 = -6$ and therefore $y = -1$.

Returning to the first equation then gives $x = 5 - 2y - 5z = 5 + 2 - 5 = 2$.

Substituting the completed solution $x = 2, y = -1, z = 1$ into the original system of three equations verifies that all three are satisfied.

27 Alternative notation

For some purposes *suffix notation* is a useful alternative. For example, the problem

$x + y = 2$, $2x + 3y = 5$

can be rewritten $x_1 + x_2 = 2$, $2x_1 + 3x_2 = 5$.

This can also be rewritten in *matrix notation* using row-on-column matrix multiplication, and square and rectangular arrays. These permit an alternative version of systems of equations like

$a_{11}x_1 + a_{12}x_2 = b_1$ with $a_{21}x_1 + a_{22}x_2 = b_2$.

This system can be written in the matrix format as $A\mathbf{x} = \mathbf{b}$, where A is a square 2 x 2 matrix, and $\mathbf{x}$ and $\mathbf{b}$ are 2 x 1 column matrices representing the unknown and known variables respectively.

28 Functions

A function is a procedure (or a recipe, or a rule) for converting one number, or set of numbers, into another number or set of numbers. We can represent this procedure by a box which pictures

output y = function $f(x)$ of input x, which means $y = f(x)$.

[Diagram]

E.g. if $f(x) = x^2$, then an input $x = -2$ implies an output $y = (-2)^2 = 4$.

The "function" describes the operation of converting x into y.

A "graph" is a "picture" of a function, e.g. it is the set of points in the x, y plane such that $y = f(x)$.

29 Example

The relation between centigrade (C) and Fahrenheit (F) measures of temperature is a *straight* line passing through

the freezing point of water $C = 0$ or $F = 32$ degrees, and

the boiling point of water $C = 100$ or $F = 212$ degrees,

so it has the *gradient* or *slope* $\frac{212-32}{100-0} = 1.8$ and the equation of the straight line is

$F - 32 = 1.8C$ or $F = 1.8C + 32$.

[Diagram]

If we rewrite $F = y$ and $C = x$ we have $y = 1.8x + 32$.

As an example of the use of these equations, the question arise of whether there is a temperature which is the *same* on the C and F scales, i.e which has $C = F$.

Putting $F = C$ or $y = x$ in the function gives $x = 1.8x + 32$ so that $0.8x = -32$ and therefore $x = -\frac{32}{0.8} = -40$.

[Diagram]

Thus $F = C$ at -40 degrees. The graphical solution is where the two straight lines $F = 1.8C + 32$ and $F = C$ cross.

30 Circular graph

A circle of radius r has equation $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$ and $y = \pm\sqrt{1 - x^2}$.

[Diagram]

31 Parabola

This is the graph of the function $y = f(x)$ when $f(x) = x^2$, i.e. $y = x^2$.

[Diagram]

32 Cubic

Plotting graphs can involve the following particular questions.

1. Find where they cross the axes.

For example $y = x^3 - x$ crosses the x-axis at $y = 0$ where $x(x^2 - 1) = 0$, i.e. where $x = 0$ and $x = \pm 1$.

2. Find what happens at large x, e.g. in this example $x \to +\infty$ implies $y \to +\infty$ and $x \to -\infty$ implies $y \to -\infty$.

[Diagram]

3. Find the possible turning points, where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Here $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1 = 0$ at $x^2 = \frac{1}{3}$, i.e. where $x = \pm\frac{1}{\sqrt{3}}$, $y = \pm\frac{1}{3\sqrt{3}} \mp \frac{1}{\sqrt{3}}$.

There is a *local* maximum if $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing, i.e. if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ ("minus mountain").

There is a *local* minimum if $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing, i.e. if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ ("plus plate").

In this example $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$, so there is a local maximum at $x = -\frac{1}{\sqrt{3}}$ ("mountain"), and a local minimum at $x = +\frac{1}{\sqrt{3}}$ ("plate").

33 Asymptotes

These are situations where the graph of $y = f(x)$ approaches a steady behaviour as $x \to +\infty$ or $x \to -\infty$.

For example $y = \frac{1}{x-1}$ has the following properties.

[Diagram]

As $x \to +1$ from $x < 1$, $y \to -\infty$.

As $x \to +1$ from $x > 1$, $y \to +\infty$.

As $x \to +\infty$, $y \to 0$ from above ($y \to +0$).

As $x \to -\infty$, $y \to 0$ from below ($y \to -0$).

The graph is called a *rectangular hyperbola*.

34 Composition, or combination, of functions

This is about the application of a sequence of functions, rather than about just one function. The input to a function $f()$ delivers an output which can be put into another function $g[]$, which then delivers a combined output $g[f()]$.

[Diagram]

The picture illustrates such a combination. Another notation which is sometimes used (but not recommended by me) is

$$g \circ f = g(f(x)).$$

This means that we work out $f(x)$, and then work out $g(f)$ which will depend on x.

An example is $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ with $g(f) = x^3 + 3$. For this the *composition* $g(f) = g \circ f = f^3 + 3 = x^{\frac{3}{2}} + 3$, which is *not* the same as $f(g) = f \circ g = g^{\frac{1}{2}} = (x^3 + 3)^{\frac{1}{2}}$.

In general $g \circ f \neq f \circ g$, i.e. $g(f) \neq f(g)$, i.e the composition operation is *not* commutative. The order in which we do the two operations *does* matter.

35 Inversion of functions

We may be given $output = f(input)$ or $y = f(x)$ or $y = y(x)$, and we wish to know $x = x(y)$, i.e. how the input depends on the output.

An example is when we are given the *exponential* function $y = e^x$ where

$e = 2.718281828459045235...$

The inverse is the *log* function $x = \ln y$ where ln stands for "natural logarithm" which is associated with John Napier, 1550 - 1617. Thus ln also stands for "Napierian logarithm".

[Diagram]

In electrical engineering we find current I and voltage V related by

$I = I_0 \exp(\frac{qV}{kT} - 1)$ with constants I_0, q, k, T

which has inverse

$V = \frac{kT}{q}[1 + ln\frac{I}{I_0}]$

which allows us to calculate V when I is known.

The mathematical notation for the inverse of $x = f(y)$ might seem to be $y = f^{-1}(x)$, but it better not to use this because $f^{-1}(x)$ might be confused with $\frac{1}{f(x)}$.

Instead we say that $x = f(y)$ implies $y = g(x)$ where $f[g(x)] = x$.

Another explicit example is that $y = x^3 + 1$ implies $x^3 = y - 1$ from which we get the inverse $x = (y - 1)^{\frac{1}{3}}$.

The guideline is : make it clear that you know what you are doing.

Another example is that $y = x^2$ has inverse $x = \pm y^{\frac{1}{2}}$.

[Diagram]

36 Even and odd functions

An *even* function satisfies $y(x) = y(-x)$ for all x, so an even function is *symmetric* under $reflection$ in the vertical (y) axis.

[Diagrams - piecewise smooth, and smooth]

An *odd* function satisfies $y(x) = -y(-x)$ for all x, so an odd function is *symmetric* under *rotation* by 180° about the origin.

[Diagrams - smooth like cubic, and piecewise smooth]

An odd function must pass through the origin if it is defined there at all, because

$y(0) = -y(0)$ implies $2y(0) = 0$ which implies $y(0) = 0$.

A simple even function is $f(x) = x^2$ and a simple odd function is $f(x) = x$.

[Diagrams]

Much of mathematics gets its confidence by being expressible in *Theorem/Proof* format. An example is the following.

Theorem

Every function $f(x)$ can be expressed as the sum of an *even* function $g(x)$ and an *odd* function $h(x)$. That is, every $f(x) = g(x) + h(x)$ where $g(x) = g(-x)$ (even) and $h(x) = -h(-x)$ (odd) for every x.

Proof

We can rewrite *any* $f(x)$ as

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$= g(x) + h(x)$

where $g(x) = \frac{1}{2}[f(x) + f(-x)] = g(-x)$ is even,

and $h(x) = \frac{1}{2}[f(x) - f(-x)] =$ - $h(-x)$ is odd.

Q.E.D. = quod erat demonstrandum = which was to be proved.

37 Trigonometric examples

$y = \sin x$ is an odd function because the sine wave is *antisymmetric*. For example $\sin \frac{\pi}{2} = 1$ and $\sin(-\frac{\pi}{2}) =$ - 1, and in general $\sin x =$ - $\sin(-x)$ for all x.

[Diagram]

$y = \cos x$ is an even function because the cosine wave is *symmetric*. For example cos $\frac{3\pi}{2}$ = - 1 and cos $(-\frac{3\pi}{2})$ = - 1, and in general cos x = cos $(-x)$ for all x.

[Diagram]

38 Integration

The integral of *any odd* function over a symmetric interval is zero.

For example the integral of $\frac{x^2 sinx + tanx}{(x^2+1)^2}$ over $-1 \leq x \leq +1$ is zero, because x^2 and $(x^2+1)^2$ are even, and sin$x$ and tan$x$ are odd, and the interval $-1 \leq x \leq +1$ is symmetric.

39 Further examples

x^3 + sin x = -$(-x)^3$ - sin $(-x)$ = -[$(-x)^3$ + sin $(-x)$] is odd.

$x + x^2$ = - [$(-x) - (-x)^2$] is neither odd nor even.

In general notation a function $f(x)$ is neither odd nor even if $f(-x) \neq f(x)$ *and* $f(-x) \neq -f(x)$.

Writing the above example as $f(x) = x+x^2$ we see that $f(-x) = (-x)+(-x)^2 = -x+x^2$

and therefore

$f(-x) = f(x)$ *only* where $(-x) + (-x)^2 = x + x^2$, i.e. where $x = 0$ only and not everywhere, so $f(x)$ is not even.

Also $f(-x)$ = - $f(x)$ *only* where $(-x) + (-x)^2$ = - $(x + x^2)$, i.e. only where $x = 0$ and not everywhere, so $f(x)$ is not odd either.

40 Periodic functions

Definition: if a function $g(x)$ has the property that $g(x) = g(x + L)$ for all x and some number L, then $g(x)$ is *periodic*.

The *smallest* value of L for which the values of $g(x)$ repeat like this is called the *period*.

The most familiar examples of periodic functions are the trigonometric functions sin x and cos x, whose graphs we display.

[Diagrams]

In both cases, for any integer n,

$\sin x = \sin(x \pm 2\pi) = \sin(x \pm 2n\pi)$ for any integer n, and

$\cos x = \cos(x \pm 2\pi) = \cos(x \pm 2n\pi)$ for any integer n,

so that the *period* of each is 2π.

Frequency $= \frac{k}{period}$ for some constant k (perhaps $k = 1$) which depends on the context.

Periodic functions do not have to be *smooth* (e.g. the saw-tooth function is not smooth at isolated points) or *continuous* (e.g. the square wave function illustrates this - it has vertical discontinuities at isolated places), nor even *symmetrical*.

[Diagrams]

Functions can also have *composite* definitions such as

$f(x) = x$ for $0 \leq x \leq 1$

$f(x) = 1$ for $1 \leq x \leq 2$

$f(x) = f(x + 2n)$ for $n = 1, 2, 3, 4, ...$

To sketch this we divide the real line up into intervals of length 1. Then within the closed intervals $0 \leq x \leq 1$, $2 \leq x \leq 3$, $4 \leq x \leq 5$, etc. we have a line of slope 1

and within the half-open intervals $1 \leq x < 2$, $3 \leq x < 4$, $5 \leq x < 6$, etc. we have a horizontal line of height 1,

so this is a *periodic* function with a slightly complicated definition which is not just a *single* formula.

41 Trigonometry

Trigonometry is "triangle measuring", in a triangle with vertices (corners) at A, B, C
where there are angles labeled *alpha*, *beta*, *gamma*
whose opposite sides have lengths $a = BC, b = CA, c = AB$.

[Diagram]

Right-angled triangles are of special importance. Choosing the corner C to be the right angle, $\gamma = 90°$. The side AB opposite the right angle is called the *hypotenuse*.

[Diagram]

The angle made by a *full* circle is called 2π radians or $360°$ (degrees), so that

1 radian $= \frac{360}{2\pi} = \frac{180}{\pi} = 57.32$ degrees, because $\pi = 3.14 = \frac{22}{7}$ approximately.

42 Definitions of trigonometric functions

In a right-angled triangle with hypotenuse of length r and the other two sides of lengths x and y, the angle θ opposite y has associated functions defined by

[Diagram]

$\sin\,\theta = \frac{x}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$,

$\cos\,\theta = \frac{y}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$,

$\tan\,\theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$.

Their inverses are called

$\sec\,\theta = \frac{1}{\cos\theta}$,
$\text{cosec}\,\theta = \frac{1}{\sin\theta}$,
$\cot\,\theta = \frac{1}{\tan\theta}$.

We have already seen the graphs of $\sin\,\theta$, $\cos\,\theta$ and $\tan\,\theta = \frac{sin\theta}{cos\theta}$, which are all periodic with period 2π.

[Diagram]

43 Trigonometric identities

There are many of these, all provable from the definitions.

Example: $\sin^2\alpha + \cos^2\alpha = 1$.

This is another version of Pythagoras's Theorem. The Course Booklet contains other examples.

We previously quoted the formula (Section 17) that the exponential number e satisfies

$e^{i\theta} = \cos\theta + i\sin\theta$.

If we assume this, we can use it to prove several trigonometric identities as follows.

The definition of indices means that

$e^{\alpha}e^{\beta} = e^{(\alpha+\beta)}$ for any real α and β.

Returning to the context of imaginary and complex numbers

$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$ for any real α and β.

Thus

$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$
$= \cos\alpha\cos\beta + i^2\sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$
$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$.

Equating real and imaginary parts gives

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

44 Series expansions

Many functions have *series expansions*. To explain these we introduce another notation for derivatives

$\frac{dy}{dx} = y'(x)$, $\frac{d^2y}{dx^2} = y''(x)$, $\frac{d^3y}{dx^3} = y'''(x)$.

Thus a prime or dash is used to denote differentiation, so that, for example,

$y'(x)$ is the first derivative evaluated at general x,

$y'(0)$ is the first derivative evaluated at $x = 0$,

$y''(3)$ is the second derivative evaluated at $x = 3$,

and so on.

It can be proved (but we shall assume it to be true) that a smooth function $y(x)$ ["smooth" mean that it has a unique tangent, and therefore a derivative, at every point] can be expressed as a Maclaurin series

$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{3!}y'''(0) + ... + x^n\frac{y^(n)(0)}{n!} + ...$

in which, for example "factorial" 4 is 4! = 4 x 3 x 2 x 1.

[Colin Maclaurin, 1698 - 1746, became a professor at the age of 19].

[Diagram]

The Maclaurin series is an "expansion about the origin" $x = 0$. This gives a good "approximation" for *small* x, say, because (for example)

if $x = \frac{1}{10}$, then $x^2 = \frac{1}{100}$, $x^3 = \frac{1}{1000}$, etc.

For example, the Maclaurin expansion of $y(x) = x^2 + \sin x$ requires us to calculate

$y'(x) = 2x + \cos x$, $y''(x) = 2 - \sin x$, $y'''(x) = - \cos x$, etc. at the origin, where

$y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$, $y'''(0) = - 1$, etc.

[Diagram]

Thus the approximation near the origin (i.e. for small x) is

$y(x) = 0 + x.1 + 2\frac{x^2}{2} + (-1)\frac{x^3}{6} + ... = x + x^2 - \frac{x^3}{6}$.

This illustrates how the Maclaurin series is an "expansion about the origin". If we want an expansion about some *other* point, say $x = a$, we require a generalisation of the Maclaurin seies which is called the Taylor series

$y(x) = y(a) + (x - a)y'(a) + \frac{(x-a)^2}{2}y''(a) + \frac{(x-a)^3}{6}y'''(a) + ...$

[This was established by Brook Taylor, 1685 - 1731.]

Examples of Maclaurin series are

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$,

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$,

because, for example, when

$y(x) = \cos x$, $y'(x) = - \sin x$, $y''(x) = - \cos x$, $y'''(x) = \sin x$, ... so that

$y(0) = 1$, $y'(0) = 0$, $y''(0) = - 1$, $y'''(0) = 0$.

45 Inverse trigonometric functions

$y = \sin x$ means that y is the sine of the angle x, and the *inverse function* is sometimes written

$x = \sin^{-1} y$ which means that x is the angle whose sine is y.

[Diagram]

Thus the inverse can be pictured just by turning the graph round by 90°.

Notation sometimes used is that if $y = \sin x$, then $x = \text{arc}\ \sin y$ or $x = \sin^{-1} y$. This is not wholly satisfactory because it suggests, but does *not* mean, $x = \frac{1}{\sin y}$.

[Diagram]

[Diagram]

More graphical examples are provided by $y = \cos x$ with $x = \text{arc}\ \cos y$, and $y = \tan x$ with x with $x = \text{arc}\ \tan y$. It should be noticed that these inverse functions are not single-valued, so care is need when using calculators. These might imply restricted ranges like

- $\pi <$arc cos $y < \pi$, and - $\frac{\pi}{2} <$arc tan $y < \frac{\pi}{2}$.

Example: find *all* the angles that satisfy $\cos x = \frac{1}{2}$.

[Diagram]

The principal solution is $x = 60° = \frac{\pi}{3}$ radians. *All* solutions are $x = \pm\frac{\pi}{3} + 2n\pi$ for $n = 0, \pm1, \pm2, ...$

Example:

Find the amplitude and phase of a signal produced by the addition 3 cos t + 4 cos t.

This *means* that we write $3\cos t + 4\cos t = r\sin(t + \alpha)$, and then try to find the *amplitude* r and *phase* α which are thus implied, where $0 \leq \alpha \leq 2\pi$.

To solve we write $r\sin(t + \alpha) = r(\sin t\cos\alpha + \cos t\sin\alpha)$, which shows that we need

$r\ \sin(\alpha) = 3$ and $r\ \cos(\alpha) = 4$.

These are two simultaneous equations for two unknowns r and α. They imply

$r = 5$ and $\tan\alpha = \frac{3}{4}$.

[Diagram]

From the tan graph, or tables, $\alpha = 37$ degrees.

46 Exponential functions

These are any functions of the form $f(x) = b^x$ for some positive constant b called the *base*, and some variable x called the *exponent*.

They have the following properties.

1. $f(0) = b^0 = 1$ at $x = 0$.

2. $f(x) = b^x > 0$ for all x (because $b > 0$).

3. $f(x) \to +\infty$ as $x \to +\infty$.

4. $f(x) \to 0$ as $x \to -\infty$.

[Diagram]

5. There is a special value of b, always written $e = 2.718...$ as we have already seen, for which $\frac{\mathrm{d}f}{\mathrm{d}x} = f$, i.e. gradient = value for all x.

6. There is a series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...,$

where $n! = n(n-1)(n-2)...3.2.1$ = factorial n.

7. $f(x) = -b^x$ is the reflection of b^x in the horizontal axis.

8. $f(x) = b^{-x}$ is the reflection of b^x in the vertical axis.

[Diagram]

47 Logarithms

Logarithms are the *inverse* of exponential functions.

This means that any exponential function $y = b^x$ for any given base b with exponent x has an *inverse*, which is *called*

$x = \log_b y$ (log y to base b).

[Diagrams]

There are two particular bases for logarithms which are in common use, namely

$b = 10$ so that $y = 10^x$ has inverse $x = \log_{10} y$, which are called "logs to base 10"; and

$b = e$ so that $y = \mathrm{e}^x$ has inverse $x = \log_e y = \ln y$.

The latter alternative is called the "natural logarithm" or "Napierian logarithm".

John Napier (1550 - 1617) of Merchieston Castle, Edinburgh, was the first inventor of logarithms in 1614.

Theorem

The log of a product is the sum of the logs.

That is, using any base, $\log MN = \log M + \log N$.

Proof

Using any base b, consider the numbers defined by $M = b^m$ and $N = b^n$.

Then by definition

$m = \log_b M$ and $n = \log_b N$ for logs to the same base b.

Also $MN = b^m b^n = b^{(m+n)}$ so $m + n = \log_b MN$.

Therefore

$\log_b MN = \log_b M + \log_b N$.

Q.E.D. = Quod Erat Demonstrandum = which was to be proved.

This Theorem is the basis for the use of "log tables" to make multiplication easier:

1. look up the logs of the two numbers M and N;

2. add these logs: $\log M + \log N = \log MN$;

3. look up the "antilogs" (inverse log tables) to find MN.

48 Hyperbolic functions

We now use the very special *exponential* number $e = 2.71828...$ which has the property $\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$ to define

$\sinh x = \frac{e^x - e^{-x}}{2}$ (called the hyperbolic sine),

$\cosh x = \frac{e^x + e^{-x}}{2}$ (called the hyperbolic cosine), tanh $x = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (called the hyperbolic tangent).

These are *not* periodic functions like sin x, cos x and tan x. Instead their graphs are constructed from those of e^x and e^{-x}, and these "hyperbolic" functions satisfy different relationships to the trigonometric functions.

The graph of cosh$x$ can be shown to represent the shape of a uniform hanging rope such as a washing line, or a telephone wire.

[Diagram]

[Diagram]

General Theorem

Hyperbolic functions satisfy $\cosh^2 x$ - $\sinh^2 x = 1$, in contrast to the trigonometric relation $\cos^2 x + \sin^2 x = 1$.

Proof

$\cosh^2 x$ - $\sinh^2 x = 1$ from the definitions of cosh x and of sinh x above.

$\cos^2 x + \sin^2 x = 1$ from Pythagoras' Theorem. Q.E.D.

[Diagram]

49 Sample Theorem

The solution of 5cosh x + 3sinh $x = 4$ is $x = \ln \frac{1}{2}$.

Proof

The definitions can be used to rewrite the Theorem as

$5(\frac{e^x + e^{-x}}{2}) + 3(\frac{e^x - e^{-x}}{2}) = 4$,

$8e^x + 2e^{-x} = 8$,

$4(e^x)^2 - e^x + 1 = 0$,

$(2e^x - 1)^2 = 0$,

$e^x = \frac{1}{2}$,

Taking natural logarithms then gives

$x = \ln \frac{1}{2}$ as the solution.

50 Osborne's Rule

This is an empirical rule which says that any relation between trigonometric quantities can be converted into a valid corresponding relation between hyperbolic quantities by changing the sign of any product (or implied product) of two sines. For example

$\cos(A + B) = \cos A \cos B$ - $\sin A \sin B$

delivers

$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$.

The proof follows from

$\cosh ix = \frac{e^{ix}+e^{-ix}}{2} = \cos x$, $\sinh ix = \frac{e^{ix}-e^{-ix}}{2} = i \sin x$.

51 Limits

Some functions $y(x)$ approach a *limit* or limiting value as x approaches some particular value.

For example, the equation $xy = 1$ defines a function $y = \frac{1}{x}$ which has the properties that

x and y have the same sign (both positive or both negative), and that

as $x \to +\infty$, $y \to +0$, (tends to zero through positive values)

as $x \to -\infty$, $y \to -0$,

as $y \to +\infty$, $x \to +0$,

as $y \to -\infty$, $x \to -0$.

These all illustrate *limits*, and diagrams display them.

[Diagram]

[Diagram]

Other (easier) examples are the limits as $x \to 2$ of

$x^2 + 3x + 1$ which is 4 + 6 + 1 = 11, and of $\frac{x+1}{x+2}$ which is $\frac{3}{4}$.

But the limit of $\frac{x^2+2x-3}{x-1}$ as $x \to 1$ appears to be $\frac{1+2-3}{1-1} = \frac{0}{0}$ which is *undefined*.

52 Methods for finding limits

We have to find more information if a limit appears to be $\frac{0}{0}$ or $\frac{\infty}{\infty}$ because these are *undefined*.

We can try to factorise the numerator and denominator, and then cancel any common factors *before* going to the limit, as follows.

1. The limit of $\frac{x^2+2x-3}{x-1}$ as $x \to 1$ *appears* to be $\frac{1+2-3}{1-1} = \frac{0}{0}$ which is undefined, but

$\frac{x^2+2x-3}{x-1} = \frac{(x-1)(x+3)}{x-1} = x+3 \to 4$ as $x \to 1$,

so lim $\frac{x^2+2x-3}{x-1}$ as $x \to 1$ is 4.

2. The limit of $\frac{x^2-7x+12}{x-3}$ as $x \to 3$ *appears* to be $\frac{9-21+12}{3-3} = \frac{0}{0}$ which is undefined, but

$\frac{x^2-7x+12}{x-3} = \frac{(x-3)(x-4)}{x-3} = x-4 \to -1$ as $x \to 3$,

so lim $\frac{x^2-7x+12}{x-3}$ as $x \to 3$ is - 1.

3. The limit of $\frac{(3+x)^2-9}{x}$ as $x \to 0$ *appears* to be $\frac{3^2-9}{0} = \frac{0}{0}$ which is undefined, but

$\frac{(3+x)^2-9}{x} = \frac{9+6x+x^2-9}{x} = 6+x \to 6$ as $x \to 0$,

so lim $\frac{(3+x)^2-9}{x}$ as $x \to 0$ is 6.

4. The limit of $\frac{\sqrt{(5+x)}-\sqrt{(4+2x)}}{x-1}$ as $x \to 1$ *appears* to be $\frac{\sqrt{6}-\sqrt{6}}{0} = \frac{0}{0}$ which is undefined, but we can avoid the ambiguity by writing the fraction as

$\frac{[\sqrt{(5+x)}-\sqrt{(4+2x)}][\sqrt{(5+x)}+\sqrt{(4+2x)}]}{(x-1)[\sqrt{(5+x)}+\sqrt{(4+2x)}]}$
$= \frac{(5+x)-(4+2x)}{(x-1)[\sqrt{(5+x)}+\sqrt{(4+2x)}]}$
$= \frac{1-x}{(x-1)[\sqrt{(5+x)}+\sqrt{(4+2x)}]}$
$= \frac{-1}{[\sqrt{(5+x)}+\sqrt{(4+2x)}]}$
$\to \frac{-1}{2\sqrt{6}}$ as $x \to 1$.

5. We can sometimes use a series expansion, for example the limit as $x \to 0$ of $\frac{sinx}{x}$ appears to be $\frac{0}{0}$ which is undefined, but we know that the Maclaurin series for small x of

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ so that

$\frac{sinx}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$ which tends to 1 as $x \to 0$

so that the limit of $\frac{sinx}{x}$ is 1 as $x \to 0$.

53 More methods for finding limits

In these cases the naive approach appears to give $\frac{\infty}{\infty}$, which has to be avoided.

1. The limit as $x \to \infty$ of

$\frac{x^2+2x+3}{2x^2+x+1}$ is obtained by dividing top and bottom by x^2, which gives

$\frac{1+\frac{2}{x}+\frac{3}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}} \to \frac{1}{2}$.

2. The limit as $x \to \infty$ of

$\frac{e^x+1}{3e^x+2}$ is obtained by dividing top and bottom by e^x, which gives

$\frac{1+e^{-x}}{3+2e^{-x}} \to \frac{1}{3}$.

3. If we require the limit as $x \to -\infty$, we can use the fact that $e^x \to 0$ as $x \to -\infty$ so that
$\frac{e^x+1}{3e^x+2} \to \frac{1}{2}$ as $x \to -\infty$.

54 Differentiation

This allows us to discuss *rate* of change accurately.

For example, velocity v is rate of change of distance s with time t, i.e.

$v = \frac{\mathrm{d}s}{\mathrm{d}t}$,

and acceleration a is rate of change of velocity with time, i.e.

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}s}{\mathrm{d}t}) = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$.

In electricity, current I = rate of change of charge Q with time t, i.e.

$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$.

The gradient of the graph of a function $y(x)$ is the rate of change of $y(x)$, which is the *local* slope or gradient of the graph.

[Diagram]

When the graph is *not* a straight line, the local slope varies from place to place with x.

The *second* derivative $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{\mathrm{d}y}{\mathrm{d}x}) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

tells us how the slope (or gradient) is changing.

[Diagram]

55 The limiting process

To find the slope at A of the graph $y = y(x)$ construct a triangle ABC in which

A has coordinates $(x, y(x))$, B has coordinates $(x + h, y(x + h))$.

[Diagram]

Then the slope of the hypotenuse AB is

$\frac{BC}{AB} = \frac{y(x+h)-y(x)}{(x+h)-x} = \frac{y(x+h)-y(x)}{h}$.

[Diagram]

Now introduce a *limiting* process which allows B to move towards A so that the gradient of the hypotenuse AB approaches the gradient of the *tangent* to the curve at A. We write

limit as $h \to 0$ of $\frac{y(x+h)-y(h)}{h} = \frac{\mathrm{d}y}{\mathrm{d}x}$.

This is the *gradient* of the curve $y = y(x)$ at A.

A common alternative notation is $\frac{\mathrm{d}y}{\mathrm{d}x} = y'(x)$.

56 Examples

(a) Parabola $y(x) = x^2$.

[Diagram]

The gradient at a typical point is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \text{limit } \frac{(x+h)^2-x^2}{h}$ as $h \to 0$

$= \lim \frac{x^2+2xh+h^2-x^2}{h} = \lim \frac{2xh+h^2}{h} = \lim\ (2x + h) = 2x.$

(b) Constant $y(x) = 3$.

[Diagram]

$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim \frac{y(x+h)-y(x)}{h} = \lim \frac{3-3}{h} = \lim 0 = 0.$

Note that we evaluate the numerator *before* proceeding to the limit $h \to 0$.

(c) The general power $y = x^n$ for any fixed n (not necessarily an integer) has derivative

$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}.$

[Diagram]

(d) When $n = 3$ in (c) the cubic $y(x) = x^3$ has gradient

$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim \frac{(x+h)^3-x^3}{h} = \lim \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim \frac{3x^2h+3xh^2+h^3}{h} = \lim(3x^2 + 3xh + h^2) = 3x^2$

$= nx^{n-1}$ for $n = 3$.

[Diagram]

(e) When $n = -\frac{1}{3}$, the cube root $y(x) = x^{-\frac{1}{3}}$ can be sketched by writing

[Diagram]

$y^3 = \frac{1}{x}$ so that $x = \frac{1}{y^3}$

which is a smooth curve having two disjoint parts and such that

$y \to +\infty$ where $x \to +0$ (zero through positive values),

$y \to -\infty$ where $x \to -0$ (zero through negative values),

$x \to +\infty$ where $y \to +0$,

$x \to -\infty$ where $y \to -0$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}x^{(-\frac{1}{3}-1)} = -\frac{1}{3}x^{-\frac{4}{3}}.$

(f) The sum or difference $y(x) = f(x) \pm g(x)$ has derivative

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x}.$

For example if $y(x) = x^3 \pm \frac{1}{x} = x^3 \pm x^{-1}$,

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \mp \frac{1}{x^2}.$

(g) $y(x) = \frac{4}{x^3} = 4x^{-3}$ has $\frac{\mathrm{d}y}{\mathrm{d}x}$ = - $12x^{-4}$ = - $\frac{12}{x^4}$.

57 Differentiation of a product

The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of $y(x) = u(x)(v(x)$ is the limit as $h \to 0$ of

$\frac{y(x+h)-y(x)}{h} = \frac{u(x+h)v(x+h)-u(x)v(x)}{h}$

$=\lim \frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h}$

$=u(x)\lim\frac{v(x+h)-v(x)}{h} + v(x)\lim\frac{u(x+h)-u(x)}{h}$

so $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$.

An example is $y(x) = x^2\sin x$ whose derivative is

$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\frac{\mathrm{d}(sinx)}{\mathrm{d}x} + \frac{\mathrm{d}x^2}{\mathrm{d}x}\sin \mathrm{x} = x^2\cos x + 2x\sin x.$

This uses the facts that $\frac{\mathrm{d}sinx}{\mathrm{d}x} = \cos x$ and $\frac{\mathrm{d}cosx}{\mathrm{d}x}$ = - sin x, which we take to be axioms here.

The differentiation of a triple product $y(x) = u(x)v(x)w(x)$ works in the same way by an extension of the formula above for a double product, namely

$\frac{\mathrm{d}y}{\mathrm{d}x} = vw\frac{\mathrm{d}u}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + uv\frac{\mathrm{d}w}{\mathrm{d}x}$.

An explicit example is that the derivative of the triple product $y(x) = 3x\mathrm{e}^x\tan x$ is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}3x}{\mathrm{d}x}\mathrm{e}^x\tan x + 3x\frac{\mathrm{d}e^x}{\mathrm{d}x}\tan x + 3x\mathrm{e}^x\frac{\mathrm{d}tanx}{\mathrm{d}x}$.

Using $\frac{\mathrm{d}3x}{\mathrm{d}x} = 3$, $\frac{\mathrm{d}e^x}{\mathrm{d}x}$, $\frac{\mathrm{d}tanx}{\mathrm{d}x} = \sec^2 x = \frac{1}{\mathrm{d}cos^2}x$,

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^x\tan x + 3x\mathrm{e}^x\tan x + 3\frac{xe^x}{\mathrm{d}cos^2x}$.

58 More examples

1. Remembering that ln x is the log to base e of x, the definition $y = \mathrm{e}^x$ has inverse $x = \ln y$, and the derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ implies

$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{e^x} = \frac{1}{y}$, so that

$\frac{\mathrm{d}lny}{\mathrm{d}y} = \frac{1}{y}$.

2. $y(x) = 3\sin x\ln x$ is an example of a product

$y = u(x)v(x)$ whose derivative is $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x}v$

so the example has derivative

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos x \ln x + 3\sin x \frac{1}{x} = 3(\cos x \ln x + \frac{sinx}{x})$.

59 Chain rule

This is the statement that the derivative of a function of a function $y(x) = f[g(x)]$ is obtained by the formula

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$.

The following are two illustrations.

1. $y(x) = \cos x^2$ provides an example in which $f(g) = \cos g$ and $g(x) = x^2$.

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$ = - (sin g).$2x$ = - $2x$sin x^2.

2. $y(x) = (1 + 3x^2)^{10} = f(g(x))$ has $f(g) = g^{10}$ and $g(x) = 1 + 3x^2$.

Therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} = 10g^9(6x) = 10(1 + 3x^2)^9.6x = 60x(1 + 3x^2)^9$.

60 Quotient rule

The quotient of two functions $u(x)$ and $v(x)$ is the result of dividing one by the other, giving another function, for example $\frac{u(x)}{v(x)} = q(x)$ say.

The derivative of this is obtained by treating it as the product of $u(x)$ and $\frac{1}{v(x)}$.

This leads to $v^2\frac{\mathrm{d}}{\mathrm{d}x}(\frac{u}{v}) = v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}$.

For example,

$\frac{\mathrm{d}}{\mathrm{d}x}(\frac{cosx}{x^2}) = \frac{-xsinx-2cosx}{x^3}$.

61 Inverse functions

Any function $y(x)$ has an inverse $x(y)$ in the sense that the axes are just turned round through a right angle to describe the same curve in a different way.

[Diagrams]

If we insert the inverse into the original function we get

$y[x(y)] = y$ or alternatively $x[y(x)] = x$.

Differentiating with respect to y, or x, respectively, using the chain rule, gives

$\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}y}{\mathrm{d}y} = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}x} = 1$

so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$.

For an example we can use $y = \sin^{-1}x$, which means $x = \sin\, y$ and not $y = \frac{1}{sinx} = (sinx)^{-1}$.

Differentiating $x = siny(x)$ with respect to x gives

$1 = (cosy)\frac{\mathrm{d}y}{\mathrm{d}x}$ by the chain rule, so that

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{cosy} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$

because $x = siny$ has derivative

$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos\, y$.

62 Differentiation of implicit functions

We might need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y(x)$ is given *implicitly* but not explicitly. For example, in

$y^3 + 3y = x^2$

we do not know $y(x)$ explicitly because we have not solved the cubic, but we can still differentiate the equation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ using the chain rule, as follows.

$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

$3(y^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{3(y^2+1)}$.

A second example is to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = a^x$ for any constant a. Taking logs we find

$\mathrm{ln}y = x\mathrm{ln}a$ and differentiating this with respect to x gives

$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln\, a$ so that

$\frac{\mathrm{d}y}{\mathrm{d}x} = y\ln\, a = a^x\ln\, a$.

63 Higher derivatives

Any function $f(x)$ or curve $y = f(x)$ has a *value* y at each given x, and also a *slope* or *gradient* $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ at each x. But it also has a *curvature* or rate of change of slope

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{d}{dx}(\frac{\mathrm{d}y}{\mathrm{d}x}) = f''(x)$ and so on.

[Diagram]

For example, the parabola $y = x^2$ has slope $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ (so the slope increases linearly with x), but it also has *curvature* represented by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$.

[Diagram]

Thus this second derivative happens to be *constant* for a *parabola*, and the *third* derivative $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0$.

An example of these derivatives is provided by a particle or motor-bike in motion which travels a distance $s(t)$ in time t, so that its speed is $\frac{\mathrm{d}s}{\mathrm{d}t}$ and its acceleration is $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$.

Newton's Law (1687) says that

"$force = mass$ x $acceleration$", or $F = m\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ for a particle of mass m. This means, for example, that if a force is sustained at the value F, then the particle of mass m upon which it is acting will move with a constant acceleration $\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = \frac{F}{m}$ in the direction of the force.

But if the acceleration oscillates like $\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -\sin t$, this must mean that an oscillating force is being applied to it, and the velocity will oscillate like

$\frac{\mathrm{d}s}{\mathrm{d}t} = \cos t + k$ around a constant value k, and the distance s traveled in time t will oscillate according to the formula $s =$ - sin$t$ + kt + c, where c is another constant.

The rate of change of acceleration will be

$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}^2s}{\mathrm{d}t^2}) = \frac{\mathrm{d}^3s}{\mathrm{d}t^3} =$ - cos$t$ and so also oscillates.

[Sir Isaac Newton, P.R.S., 1642 - 1727, is one of the most famous figures in applied mathematics, and reknowned for his work on mechanics and on optics.]

64 Maxima and minima - optimisation

A curve $y = y(x)$ may have *local* maxima M_1, M_2 and *local* minima m_1, m_2 as in the diagram.

[Diagram]

At all four of these *turning* points the gradient $\frac{dy}{dx} = 0$.

At each local maximum this gradient is *decreasing* from positive to negative, so

$\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2} < 0$,

i.e. the *second* derivative is *negative* there.

At each local minimum this gradient is *increasing* from negative to positive, so

$\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2} > 0$,

i.e. the *second* derivative is *positive* there.

For an example we locate the turning points of the *cubic* function

$y(x) = 2x^3 + 3x^2 - 180x + 600$

and find out whether they are local maxima or local minima.

The turning points will have zero slope, so we need to find where

$\frac{dy}{dx} = 6x^2 + 6x - 180 = 6(x^2 + x - 30 = 6(x + 6)(x - 5)$

is zero. This happens at $x = 5$ and $x = -6$.

Which way does the function turn here? It will turn

upwards (local minimum) if $\frac{d^2y}{dx^2} > 0$, and

downwards (local maximum) if $\frac{d^2y}{dx^2} < 0$ there.

From above we find that $\frac{d^2y}{dx^2} = 12x + 6$. This is 66 at $x = 5$ so that we have a local minimum there, and it is - 66 at $x =$ - 6, so we have a local maximum there.

[Diagram]

The diagram shows this result, with y(- 6) $= 1356$, and $y(5) = 25$.

65 Parametric description of curves

It is sometimes convenient to describe a curve in the x, y plane by using a third (intermediate) parameter, say t.

For example, $(y - 2)^2 = 4a(x + 1)$ is a parabola for any constant a.

[Diagram]

This single equation can also be written as two "parametric" equations $x+1 = at^2$ with $y-2 = 2at$, because $(2at)^2 = 4a^2t^2 = (4a)(at^2)$ for every value of t.

Another example is the circle $x^2+y^2 = r^2$ with constant radius r. This can be written in parametric form $x = r\text{cos}\theta$, $y = r\text{sin}\theta$ where θ is the parameter using Pythagoras's Theorem.

[Diagram]

Gradients $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be found for *any* parametric description $y = y(t), x = x(t)$ because

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} . \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$.

For example the circle $x = \text{cos}\theta$, $y = \text{sin}\theta$ has gradient

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} . \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{cos\theta}{-sin\theta} = \text{-}\frac{1}{tan\theta}$

which is the slope of the tangent.

[Diagram]

66 Hyperbolic functions

The hyperbola $x^2 - y^2 = 1$ has *asymptotes* $x^2 - y^2 = 0$, i.e.

$(x-y)(x+y) = 0$ and therefore $x = y$ with $x = -y$.

[Diagram]

In *parametric* form this hyperbola can be written

$x = \cosh\, t = \frac{e^t+e^{-t}}{2}$ with $x = \sinh\, t = \frac{e^t-e^{-t}}{2}$.

To verify this parametric form we see that

$x^2 - y^2 = [\frac{e^t+e^{-t}}{2}]^2 - [\frac{e^t-e^{-t}}{2}]^2$

$= \frac{1}{4}[(e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t})] = 1.$

67 Functions of more than one variable

x^2 and $\text{cos}x$ are functions of *one* variable x.

A function like $f(x) = x^2$ has a graph $y = f(x)$, i.e. $y = x^2$, which is a curve on a *two*-dimensional page.

[Diagram]

Now we consider functions $f(x, y, z, ...)$ of several variables $x, y, z, ...$

A function like $f(x, y) = x^2 + y^2$ represents a *surface* $z = x^2 + y^2$ in *three*-dimensional space spanned by x, y, z.

This example is a *parabolic* bowl.

[Diagram]

A *sphere* with radius r will have equation

$x^2 + y^2 + z^2 = r^2$ which can also be written

$z^2 = r^2 - (x^2 + y^2)$ or $z = \pm\sqrt{r^2 - (x^2 + y^2)}$

or $z = f(x, y)$ where $f(x, y) = \pm\sqrt{r^2 - (x^2 + y^2)}$.

[Diagram]

68 Partial differentiation

This means that we are working with a function of *several* variables but differentiating it with respect to only *one* variable at a time, holding the others *fixed*.

A *new* symbol is used for *partial* differentiation.

The *ordinary* derivative of $f(x)$ is

$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim \frac{f(x+h)-f(x)}{h}$ as $h \to 0$.

For example, if $f(x) = x^2$,

$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim \frac{(x+h)^2-x^2}{h} = \lim \frac{2xh+h^2}{h} = \lim\ (2x + h) = 2x$ as $h \to 0$.

But the *partial* derivatives of $f(x, y)$ with respect to x and y are written

$\frac{\partial f}{\partial x} = \lim \frac{(x+h,y)-f(x,y)}{h}$ as $h \to 0$ and $\frac{\partial f}{\partial y} = \lim \frac{(x,y+k)-f(x,y)}{k}$ as $k \to 0$.

In practice, all the usual rules for differentiation with respect to *one* variable work, because we are holding all the *other* variables *fixed*.

69 Examples of first partial derivatives

$f(x,y) = x^2 + 2y^2 + 5xy$ has $\frac{\partial f}{\partial x} = 2x + 5y$ and $\frac{\partial f}{\partial y} = 4y + 5x$.

To find the partial derivative of

$f(x,y) = \frac{1}{1+x^2+3y^2} = (1 + x^2 + 3y^2)^{-1}$

we introduce an *intermediate* variable $u = 1 + x^2 + 3y^2$ so that $f = u^{-1}$ is an example of $f(x,y) = f[u(x,y)]$.

Then we use the chain rule to get

$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} = (-u^{-2})(2x) = -\ \frac{2x}{(1+x^2+3y^2)^2}.$

$\frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial y} = (-u^{-2})(6y) = -\ \frac{6y}{(1+x^2+3y^2)^2}.$

70 Higher derivatives

We can differentiate a function of *one* variable *several* times. For example, if $s = s(t)$ describes a $distance - time$ graph, its slope $\frac{\mathrm{d}s}{\mathrm{d}t}$ is the *speed* and its curvature represented by the second derivative $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ is the *acceleration*.

Likewise a function of *several* variables such as $f(x,y)$ has not only *first partial* derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, but also *second partial* derivatives

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}),$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}),$

and with commutative mixed second derivatives.

There is an alternative *suffix* notation for partial derivatives, namely

$\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$, and

$\frac{\partial^2 f}{\partial x^2} = f_{xx},\ \frac{\partial^2 f}{\partial y^2} = f_{yy},\ \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x}.$

There are also higher (for example third and fourth) partial derivatives.

71 Examples of partial derivatives

1. $f(x,y) = y\sin x + x^2y^3$ has

$\frac{\partial f}{\partial x}$ = $y$cos$x$ + $2xy^3$, $\frac{\partial f}{\partial y}$ = sin$x$ + $3x^2y^2$,

$\frac{\partial^2 f}{\partial x^2}$ = - $y$sin$x$ + $2y^3$, $\frac{\partial^2 f}{\partial y^2}$ = $6x^2y$,

$\frac{\partial^2 f}{\partial x \partial y}$ = cos$x$ + $6xy^2$ = f_{yx}.

2. $g(x, y) = \cos(x^2 + y^2)$ has

$\frac{\partial g}{\partial x}$ = -$2x\sin(x^2 + y^2)$

$\frac{\partial^2 g}{\partial x \partial y}$ = - $4xy\cos(x^2 + y^2)$.

Exercise: find the other second partial derivatives.

3. $h(x, y) = \mathrm{e}^u$ where $u = x^2 + y^2$.

$\frac{\partial h}{\partial x} = \frac{\mathrm{d}h}{\mathrm{d}u}\frac{\partial u}{\partial x} = 2xe^u$ because $\frac{\mathrm{d}e^u}{\mathrm{d}u} = \mathrm{e}^u$.

$\frac{\partial^2 h}{\partial x^2} = 2\mathrm{e}^u + 2x\frac{\partial(e^u)}{\partial x} = 2\mathrm{e}^u + 2x(2x\mathrm{e}^u) = 2(1 + x^2)\mathrm{e}^u$.

4.Find all the first and second derivatives of $f(x, y)$ =ln(xy) + xy^3.

To handle this remember that v =e^u has inverse u =ln$v$, that is, the exponential and (natural) logarithm functions are mutual inverses.

[Diagram]

Using $\frac{\mathrm{d}v}{\mathrm{d}u} = \mathrm{e}^u = v$ and $\frac{\mathrm{d}u}{\mathrm{d}v} = \mathrm{e}^{-u} = \frac{1}{v}$,

$\frac{\partial f}{\partial x} = \frac{1}{xy}\frac{\partial(xy)}{\partial x} + y^3 = \frac{1}{xy}y + y^3 = \frac{1}{x} + y^3$.

$\frac{\partial f}{\partial y} = \frac{1}{xy}\frac{\partial(xy)}{\partial x} + 3xy^2 = \frac{1}{y} + 3xy^2$.

$\frac{\partial^2 f}{\partial x^2}$ = - $\frac{1}{x^2}$, $\frac{\partial^2 f}{\partial y^2}$ = - $\frac{1}{y^2}$ + $6xy$, $\frac{\partial^2 f}{\partial x \partial y} = 3y^2$.

5. The temperature T at any point x, y, z in a rectangular block at time t varies according to the function $T(t, x, y, z)$ which satisfies the *diffusion* equation

$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$.

Verify that T = 3sin$x$sin$y$sin$z$e^{-3t} satisfies this equation.

We find $\frac{\partial T}{\partial t}$ = - $3T$ because $\frac{\partial(e^{-3t})}{\partial t}$ = -3e^{-3t}.

Also $\frac{\partial T}{\partial x}$ = 3cos$x$sin$y$sin$z$e^{-3t}

so $\frac{\partial^2 T}{\partial x^2}$ = - 3sin$x$sin$y$sin$z$e^{-3t} = -T.

Similarly $\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2}$ = -T,

so $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$ - $\frac{\partial T}{\partial t}$ = - T - T - T -(-3T) = 0

as required.

6. The volume of a circular cylinder of radius r and height h is $V = \pi r^2 h$, so $h = \frac{V}{\pi r^2}$

and small changes δr and δV would cause a small change $\delta h = \frac{\partial h}{\partial V}\delta V + \frac{\partial h}{\partial r}\delta r$ in $h = \frac{Vr^{-2}}{\pi}$.

Because $\frac{\partial h}{\partial V} = \frac{1}{\pi r^2}$ and $\frac{\partial h}{\partial r} = \frac{-2V}{\pi r^3}$ we find

$\delta h = \frac{\delta V}{\pi r^2}$ - $\frac{2V\delta r}{\pi r^3}$.

[Diagram]

Trying $r = 5$, $\delta r = 0.1$, $V = 100$, $\delta V = 5$ gives $\delta h = 0.114$.

72 Measurements of error

If a measurement $f(x, y)$ depends on x and y, and there are small changes δx, δy which represent possible errors in the measurements of x and y, then the implied *error* in f is

$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$, and we sometimes use the

fractional error = $\frac{\delta f}{f} = \frac{error}{value}$. and also the

percentage error $\frac{\delta f}{f}$ x 100.

There can be *more* variables.

An example is to find the percentage error in a measurement of

$f(a, b, c) = a^2 b^{\frac{1}{2}} c^{-3}$

where a, b, and c are known to within 1 percent, 2 percent and 5 percent respectively.

We have ln$f$ = ln a^2 + ln $b^{\frac{1}{2}}$ + ln c^{-3} = 2ln a + $\frac{1}{2}$ln b - 3ln c

because the log of a product is the sum of the logs.

Differentiating gives the relation between the fractional errors

$\frac{1}{f}\delta f = \frac{2}{a}\delta a + \frac{1}{2b}\delta b$ - $\frac{3}{c}\delta c$

so the percentage errors are

$\frac{1}{f}\delta f$ x 100 = $\frac{2}{a}\delta a$ x 100 + $\frac{1}{2b}\delta b$ x 100 - $\frac{3}{c}\delta c$ x 100

and the worst percentage change in f will be

2x1 + $\frac{1}{2}$ x 2 + 3 x $\frac{1}{2}$ = 4.5 per cent from the given data above.

73 De Moivre's Theorem

Returning to discuss complex numbers, we recall that de Moivre's Theorem states that

(cosθ + i sinθ)^n = cos nθ + i sinθ

for any real θ and n, with $i = \sqrt{(} - 1)$.

The *proof* is based on our previous axiom that

e$^{i\theta}$ = cos θ + i sin θ for any θ, so that

e$^{in\theta}$ = cos $n\theta$ + i sin $n\theta$

by replacing θ by $n\theta$.

Hence (cosθ + i sinθ)^n = e$^{in\theta}$.

74 Applications of De Moivre's Theorem

1. Find *all* the solutions z of z^3 = 8.

If we were just dealing with *real* numbers then obviously the *only* solution is $z = 2$.

but if we allow for *complex* numbers z = r(cos θ + i sin θ) then we have to find r and θ which satisfy

z^3 = $8(cos2k\pi + isin2k\pi)$ for k = 0, ±1, ±2,...

so z = $2(cos2k\pi + isin2k\pi)^{\frac{1}{3}}$ = $2(cos\frac{2k\pi}{3} + isin\frac{2k\pi}{3})$ by DMT.

The three solutions are therefore

$z = 2(cos0 + isin0)$ = 2 from k = 0;

z = 2(cos $\frac{2\pi}{3}$ + isin $\frac{2\pi}{3}$) = 2(- $\frac{1}{2}$ + i$\frac{\sqrt{3}}{2}$) = - 1 + i$\sqrt{3}$ from k = 1;

z = 2(cos $\frac{4\pi}{3}$ + isin $\frac{4\pi}{3}$) = 2(- $\frac{1}{2}$ - i$\frac{\sqrt{3}}{2}$) = - 1 - i$\sqrt{3}$ from k = 2.

In summary, the three cube roots of 1 are 1 and -1 $\pm$ i$\sqrt{3}$.

[Diagram]

These can be checked by working out the three cubes explicitly.

2. Prove that cos $3\theta = 4cos^3\theta$ - 3cosθ for any real θ.

We use the particular version of De Moivre's Theorem which says that

cos 3θ + i sinθ = (cos θ + i sin θ)^3 = cos$^3\theta$ - 3sin$^2\theta$cosθ - i(sin$^3\theta$ - 3cos$^2\theta$sinθ)

so that by equating real and imaginary parts

cos $3\theta = 4cos^3\theta$ - 3cosθ and also sin 3θ = - $4sin^3\theta$ + 3sinθ.

Thus we get two results for the price of one.

75 Links between hyperbolic and trigonometric functions

We previously defined cosh $x = \frac{e^x + e^{-x}}{2}$ for real x.

By analogy we can replace the real x with imaginary $i\theta$ for real θ, and so define

cosh $i\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

Using $e^{\pm i\theta}$ = cosθ $\pm$ isinθ leads to

cosh iθ = cos θ.

Similarly defining sinh $i\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$ leads to

sinh iθ = i sin θ.

If we replace real θ by imaginary $i\phi$ with real ϕ in these equations we get

coshϕ = cos iϕ and sinhϕ = - i sin iϕ.

Example: find the real and imaginary parts of sin(3 + i).

sin(3 + i) = sin 3 cos i + cos 3 sin i = sin 3 cosh 1 + i cos 3 sinh 1

so Re[sin (3 + i)] = sin 3 cosh 1 and Im[sin (3 + i)] = cos 3 sinh 1.

Example: Find the complex z which satisfies cos z = 3.

Use $\frac{e^{iz}+e^{-iz}}{2} = 3$ so that e^{2iz} - 6e^{iz} + 1 = 0

is a quadratic equation for e^{iz} = 3 $\pm\sqrt{8}$

(cos$x$ + $i$sin$x$)e^{-y} = 3 $\pm\sqrt{8}$

.

Equating imaginary parts gives

e^{-y}sin x = 0 so that $x = k\pi$ for k = 0, $\pm$1, $\pm$2,...

Equating real parts gives

e^{-y}cos x = 3 $\pm$ 2$\sqrt{2}$

so that $\pm$e^{-y} = 3$\pm$2$\sqrt{2}$ and $z = x + iy = k\pi \mp \ln(3 \pm 2\sqrt{2})$.

76 Integration

Integration is the reverse of differentiation.

The basic problem is: if $g(x)$ and $f(x)$ satisfy $g = \frac{\mathrm{d}f}{\mathrm{d}x}$, what is $f(x)$?

Example: if $\frac{\mathrm{d}f}{\mathrm{d}x} = x$, then $f(x) = \frac{1}{2}x^2$ + *any* constant (the *constant of integration*).

The integral sign is the symbol which calls for integration to be performed. Examples are as follows.

$\int x^3 \mathrm{d}x = \frac{1}{4}x^4$ + constant c.

$\int \sqrt{x}\mathrm{d}x = \int x^{\frac{1}{2}}\mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}}$dx + c.

$\int \frac{3}{x^2}\mathrm{d}x = 3\int x^{-2}\mathrm{d}x = \frac{3}{(-1)}x^{-1}$ + c = - $\frac{3}{x}$ + c.

$\int(x+2)\mathrm{d}x = \frac{1}{2}x^2 + 2x$ + c.

$\int \frac{\mathrm{d}x}{x}$ = ln x + c because $\frac{\mathrm{d}}{\mathrm{d}x}(lnx) = \frac{1}{x}$.

$\int b^{ax}\mathrm{d}x = \frac{1}{a}b^{ax}$ + c for constants a,b,c.

$\int$sinaxdx = - $\frac{1}{a}$cosax + c.

$\int$cosaxdx = $\frac{1}{a}$sinax + c.

$\int$sinhaxdx = $\frac{1}{a}$coshax + c.

$\int \cosh ax \mathrm{d}x = \frac{1}{a}\sinh ax + \mathrm{c}.$

77 Integration by parts

This uses the product rule for differentiation, namely

$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}(u)}{\mathrm{d}x}\mathrm{v} + \frac{\mathrm{d}(v)}{\mathrm{d}x}\mathrm{u}$

integrated to give

$uv = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$

or

$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x.$

Examples.

1. $\int xe^x\mathrm{d}x = xe^x - \int e^x\mathrm{d}x = (x-1)e^x + \mathrm{c}$

by choosing $u = x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$ so that $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, with $v = e^x$.

2. $\int x\ln x\mathrm{d}x = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\frac{\mathrm{d}x}{x} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \mathrm{c}$

by choosing $u= \ln x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = x$ so that $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$, with $v = \frac{1}{2}x^2$.

78 Integration by change of variable

Examples.

1. $\int x\mathrm{e}^{x^2}\mathrm{d}x = \int e^u\frac{\mathrm{d}u}{2} = \frac{e^u}{2} + c$

by choosing $u = x^2$ so that $2x = \frac{\mathrm{d}u}{\mathrm{d}x}$ and $x\mathrm{d}x = \frac{\mathrm{d}u}{2}$.

2. $\int(x+1)^3\ \mathrm{d}x = \int(x+1)^3\ \mathrm{d}(x+1) = \frac{(x+1)^4}{4} + \mathrm{c}.$

3. $\int \sin x\cos x\mathrm{d}x = \int \sin x\mathrm{d}(sinx) = \frac{1}{2}(sinx)^2 + \mathrm{c}.$

4. $\int \frac{\mathrm{d}x}{x+3} = \int \frac{\mathrm{d}(x+3)}{x+3} = \ln(x + 3) + \mathrm{c}.$

79 Integration by partial fractions

To find the integral of $\frac{4x+3}{x^2-1}$ seek a version of the integrand in the form

$\frac{A}{x-1} + \frac{B}{x+1}$, which requires

$A(x+1) + B(x-1) = 4x + 3$ and therefore $A + B = 4$ and $A - B = 3$, so that $2A = 7$ and $2B = 1$. Thus

$\int \frac{4x+3}{x^2-1}\mathrm{d}x = \frac{7}{2}\int \frac{\mathrm{d}x}{x-1} + \frac{1}{2}\int \frac{\mathrm{d}x}{x+1} = \frac{7}{2}\ln(x-1) + \frac{1}{2}\ln(x+1)$ + c.

80 Further examples of integration

1. $\int \frac{\mathrm{d}x}{(2x+1)^2} = \frac{1}{2}\int \frac{\mathrm{d}(2x+1)}{(2x+1)^2} = \frac{1}{2}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{2u}$ + c = $-\frac{1}{2(2x+1)}$ + c using $u = 2x + 1$.

2. $\int tanx sec^2 x\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2$ + c = $\frac{1}{2}\tan^2$x + c using $u = tanx$ with $\mathrm{d}u = \sec^2 x\mathrm{d}x$.

3. $\int e^x \mathrm{sin}x\mathrm{d}x = \int u\mathrm{d}v = \int \mathrm{d}(uv)$ - $\int v\mathrm{d}u = uv$ - $\int v\mathrm{d}u$ = - $\mathrm{e}^x\cos x + \int e^x \mathrm{cos}x\mathrm{d}x$ using $u = \mathrm{e}^x$, $\mathrm{d}v = \sin x\mathrm{d}x$, v = - $\cos x$.

4. $\int e^x \mathrm{cos}x\mathrm{d}x = \int u\mathrm{d}w = \int \mathrm{d}(uw)$ - $\int w\mathrm{d}u = uw$ - $\int w\mathrm{d}u = \mathrm{e}^x\sin x$ - $\int e^x \mathrm{sin}x\mathrm{d}x$ using $u = \mathrm{e}^x$, $\mathrm{d}w = \cos x\mathrm{d}x$, $w = \sin x$.

5. Combining the above two results gives

$2\int e^x \sin x\mathrm{d}x = \mathrm{e}^x(\sin x$ - $\cos x)$ + constant.

81 Integration and area

The area under the graph of a function $g(x)$ from $x = a$ to $x = b$ *is* the integral

$\int g(x)\mathrm{d}x$ because the integral *means* the sum of all the vertical strips of width $\mathrm{d}x$ and height g (at that location).

[Diagram]

If we treat the starting point $x = a$ as *fixed* and imagine the end point $x = b$ as variable, then we can think of the integral

$\int g(x)\mathrm{d}x = f(b)$

as *another* function $f(b)$ of the end point value b which, when we imagine the end point to be variable, will have the property

$\frac{\mathrm{d}f}{\mathrm{d}b} = g(b)$.

This fact has the rather grand name of "The Fundamental Theorem of the Calculus", but it just means that integration is the opposite of differentiation.

82 Examples of integration

1. Find the area under the graph $g(x) = 1$ between $x = 1$ and $x = 3$. Draw the picture whenever convenient.

[Diagram]

Area = $\int 1\mathrm{d}x = [x + c] = (3 + c)$ - $(1 + c) = 2$.

Here c is an arbitrary "constant of integration" which cancels out.

2. Find the integral from $x = 1$ to $x = 2$

$\int x^2\mathrm{d}x = [\frac{1}{3}x^3 + \mathrm{c}] = (\frac{8}{3} + c)$ - $(\frac{1}{3} + c) = \frac{7}{3}$.

[Diagram - we have evaluated the shaded area under the parabola]

3. Find the integral from $x = 1$ to $x = 3$

$\int(-1)\mathrm{d}x$ = [- x + c] = (- 3 + c) - (- 1 + c) = - 2.

[Diagram]

This illustrates that any part of a graph which is *below* the axis will contribute a *negative* amount to the integral.

4. Find the integral from $x = -\pi$ to $x = \pi$

$\int$ sin $x\mathrm{d}x$ = [- cos$x$] = (- cos π) - (- cos (-π)) = -(-1) - (- (-1)) = 1 - 1 = 0.

[Diagram]

So this is *not* the area between the curve and the x-axis. That would be four times the integral of sin x between $x = 0$ and $x = \frac{\pi}{2}$.

83 Mean values

Mean value = "average" value, in plain language = $\frac{integral}{length of interval}$.

For example, the mean value of $\cos^2 x$ over one period of cos x is $\frac{1}{2\pi}\int cos^2x\mathrm{d}x$ integrated from 0 to 2π.

[Diagram]

To integrate this we need the formula cos $2x = \cos^2 x$ - sin $2x = 2\cos^2 x$ - 1 so that

$\frac{1}{2\pi}\int cos^2 x\mathrm{d}x = \frac{1}{4\pi}\int(1 + cos2\mathrm{x})\mathrm{d}x = \frac{1}{4\pi}[x + \frac{1}{2}\sin 2x$

evaluated between 0 and 2π which is

$\frac{1}{4\pi}(2\pi$ - 0) $= \frac{1}{2}$.

[Diagram]

Thus the "mean value" is $\frac{1}{2}$, and the "root mean value" is $\frac{1}{\sqrt{2}}$.

84 Integration by substitution

Find $\int(x + 1)^2\mathrm{d}x$ integrated between $x = 1$ and 2.

Introduce $u = x + 1$ which implies $\mathrm{d}u = \mathrm{d}x$ so that

$\int(x + 1)^2\mathrm{d}x = \int u^2\mathrm{d}u = \frac{u^3}{3}$

evaluated between $u = 3$ and 2, so the integral is $\frac{27}{3}$ - $\frac{8}{3} = \frac{19}{3}$.

85 Integration via simplifying fractions

To find $\int f(x)\mathrm{d}x$

where $f(x) = \frac{2+3x}{(1+x)}^2(4 + 3x)$

needs $f(x) = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{4+3x}$

which leads to

$A = 6$, $B = -1$ and $C = -18$ so that

$\int f(x)\mathrm{d}x = \int \frac{6}{1+x}\mathrm{d}x$ - $\int \frac{1}{1+x^2}\mathrm{d}x$ - $\int \frac{18}{4+3x}\mathrm{d}x$

$= 6\ln(1 + x) + \frac{1}{1+x}$ - $6\ln(4 + 3x)$.

We can now choose any limits for the integration.

86 Binomial theorem

This is the generalisation of expansions like

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

whose coefficients form part of a pyramid, with a general form.

87 Application:compound interest

If I invest P pounds for n years at r percent per annum, and if I leave each year's interest invested for subsequent years (so that *compound* interest is earned), instead of removing the interest annually (which would earn simple interest only), then after n years I shall have $\mathrm{P}(1 + \frac{r}{100})^n$ pounds.

This illustrates the Binomial Theorem.

88 Arithmetic progression

Example: 5 + 8 + 11 + 14 + 17 + 17 + 20 + ... has common difference 3.

Generally, $a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + ... + (a + (n - 1)d) = \mathrm{S}_n =$ sum of n terms, has first term a, common difference d and number of terms n. We can prove that the sum of n terms is

$\mathrm{S}_n = \frac{n}{2}(2a + (n - 1)d)$.

This is done by writing out the series in the initial version, and then the same series but in reversed format, and adding the n pairs of terms. Each pair sums to $2a + (n - 1)d$ so we reach the quotes result.

Example: 5 + 8 + 11 + 14 + 17 + 20 = 75 because $n = 6$, $a = 5$, $d = 3$ which delivers the result from the formula.

89 Geometrical progression

The example 2 + 4 + 8 + 16 + 32 illustrates a geometrical progression with common ratio 2.

The general case is

$\mathrm{a} + \mathrm{ar} + \mathrm{ar}^2 + ... + \mathrm{ar}^{(n-1)} = \mathrm{S}_n$

= sum of n terms for a series with first term a and common ratio r.

To find a compact formula for S_n, multiply the series by r to give $r\mathrm{S}_n$. Then by subtracting the two series we find

$\mathrm{S}_n = a\frac{r^n-1}{r-1}$.

Example: 2 + 6 = 18 + 54 + 162 has $a = 2$, $r = 3$, $n = 5$ so that $\mathrm{S}_5 = \frac{2(3^5-1)}{3-1} = 242$.

We can apply these ideas to find a sum to an infinite number of terms if $r < 1$, because then $r^n \to 0$ as $n \to \infty$.

Then $S_n = a\frac{r^n-1}{r-1} \to \frac{a}{1-r}$ as $n \to \infty$.

90 Catch-up problem

Car A with uniform speed u is being chased by car B with uniform speed $v > u$ and initial separation s, so there will be catch-up after time t when B has travelled a distance

$c = vt = ut + s$

and the time taken will be

$t = \frac{c}{v} = \frac{c-s}{u} = \frac{s}{v-u}$

and the catch-up distance travelled by B will be $c = \frac{vs}{v-u}$.

Numerical example: if $s = 100m$, $v = 10ms^{-1}$, $u = 1ms^{-1}$

then $c = \frac{10x100}{10-1} = \frac{1000}{9} = 111.111...$

91 More applications of De Moivre's Theorem

1. Given that $z = 64(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6})$, find all the values of $z^{\frac{1}{6}}$.

[Diagram]

Using $z = 64[\cos(\frac{\pi}{6} + 2n\pi) + \text{i}\sin(\frac{\pi}{6} + 2n\pi)]$ for *any* integer n, De Moivre's Theorem gives

$z^{\frac{1}{6}} = 64^{\frac{1}{6}}[\cos(\frac{\pi}{6} + 2\text{n}\pi) + \text{isin}(\frac{\pi}{6} + 2\text{n}\pi)]^{\frac{1}{6}}$

$= 2[\cos\frac{1}{6}(\frac{\pi}{6} + 2n\pi) + \text{isin}\frac{1}{6}(\frac{\pi}{6} + 2n\pi)]$

$= 2[\cos(\frac{\pi}{36} + \frac{n\pi}{3}) + \text{isin}(\frac{\pi}{36} + \frac{n\pi}{3})]$

for $n =$ 0,1,2,3,4,5.

[Diagram]

The location of the roots is on a circle of radius 2 (= modulus of all the roots) at angles (= amplitudes = arguments) of

$\frac{\pi}{36}$ radians (= $\frac{180}{36}$ = 5 degrees) for $n = 0$,

$\frac{\pi}{36}$ + $\frac{\pi}{3}$ = $\frac{13\pi}{36}$ radians (= 5 + 60 = 65 degrees) for n = 1,

$\frac{\pi}{36}$ + $\frac{2\pi}{3}$ = $\frac{25\pi}{36}$ radians (= 5 + 120 = 125 degrees) for n = 2,

$\frac{\pi}{36}$ + $\frac{3\pi}{3}$ = $\frac{37\pi}{36}$ radians (= 5 + 180 = 185 degrees) for n = 3,

$\frac{\pi}{36}$ + $\frac{4\pi}{3}$ = $\frac{49\pi}{36}$ radians (= 5 + 240 = 245 degrees) for n = 4,

$\frac{\pi}{36}$ + $\frac{5\pi}{3}$ = $\frac{61\pi}{36}$ radians (= 5 + 300 = 305 degrees) for n = 5.

2. Find cos 4θ in terms of cos θ.

Use De Moivre's Theorem in the form

cos 4θ + i sin4θ = (cos θ + i sin $\theta)^4$

= $\cos^4\theta$ - $6\cos^2\theta\sin^2\theta$ + $\sin^4\theta$ + i[$4\cos^3\theta\sin\theta$ - $4\cos\theta\sin^3\theta$].

Equating *real* parts and using $\cos^2\theta + \sin^2\theta = 1$ gives

cos 4θ = $8\cos^4\theta$ - $8\cos^2\theta$ + 1.

We now get a bonus without more work by equating the imaginary parts, which gives

sin 4θ = $4\cos^3\theta\sin\theta$ - $4\cos\theta\sin^3\theta$ = $4\sin\theta\cos\theta(\cos^2\theta$ - $\sin^2\theta)$.

3. Find all the roots z of $z^3 = \frac{i\sqrt{2}}{1+i}$.

The right hand side is $\frac{i\sqrt{2}}{1+i}$ $\frac{1-i}{1-i}$ = $\frac{1+i}{\sqrt{2}}$ = cos $\frac{\pi}{4}$ + i sin $\frac{\pi}{4}$.

So z^3 = cos $\frac{\pi}{4}$ + i sin $\frac{\pi}{4}$ = cos ($\frac{\pi}{4}$ + $2n\pi$) + i sin ($\frac{\pi}{4}$ + $2n\pi$) for n = 0,1,2

and z = (cos $\frac{\pi}{4}$ + i sin $\frac{\pi}{4})^{\frac{1}{3}}$ = cos ($\frac{\pi}{12}$ + $\frac{2n\pi}{3}$) + i sin($\frac{\pi}{12}$ + $\frac{2n\pi}{3}$) for n = 0,1,2

= cos $\frac{\pi}{12}$ + i sin $\frac{\pi}{12}$, cos $\frac{9\pi}{12}$ + i sin $\frac{9\pi}{12}$, cos $\frac{17\pi}{12}$ + i sin $\frac{17\pi}{12}$

with modulus 1 and amplitudes $\frac{\pi}{12}$ = 15 degrees, $\frac{9\pi}{12}$ = 135 degrees, $\frac{17\pi}{12}$ = 255 degrees.

[Diagram]