Mathematics for Ten-Year-Olds

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Introduction

I have had the good fortune to teach a class of capable ten-year-old children in a Primary School, weekly for the past eight years 2001 - 2008. There have been about ten children in the class each year, and sometimes one or two more. My objective has been to find topics which have not been on their current regular syllabus, and which were not expected to be on their syllabus in the immediate future. There will have been accidental overlaps, of course, but my basic policy has been enrichment, not anticipation.

This is the same policy as in the Mathematics Masterclasses with which I was previously involved, for thirteen-year-olds from Berkshire schools at the University of Reading in 1990 - 1999, and for eight-year-olds from Windsor and Maidenhead schools at Bisham School after that. My edited book Mathematics Masterclasses - Stretching the Imagination, published by Oxford University Press (1997), contains the texts of twelve of those Masterclasses for 13-year-olds, selected from about 35 such Masterclasses in all.

After each of my weekly classes for ten-year-olds I have recorded immediately the content in private notes. My purpose here is to make those notes more widely available in their original form. I have edited them slightly, in particular to remove some names of children. Otherwise the notes simply record the course as it evolved. In this format it has not been convenient to reproduce diagrams which have often been used on the white-board. Such a version may come later, but the reader may find it instructive to construct his or her own diagrams. In each year I have taken the view that it was preferable not to prepare material long in advance, but to let it evolve at rather short notice in order to get the benefits of spontaneity. Two examples of this are the discussion of how many houses Bill Gates could afford to buy in Maidenhead with his published fortune; and the reliability or otherwise of supermarket so-called offers as described on their price-labelling.

I am grateful to Mr. Jim Cooke of Bisham School for giving me the opportunity to carry out this enterprise, and for encouraging it from year to year.

Lesson 1.1

Our main item was an investigation about triangles in a circle. Draw a circle. Mark eight points evenly spaced on the circumference. Using any of these as corners, how many different triangles can you draw?

After some trial drawings it was clear that nobody could see how to classify their triangles so we suggested a system of defining the distance (along the circle) between two adjacent points as 1 unit. This enables each triangle to be described as a sequence of numbers, e.g. 1,1,2 (or 1,1,6); 1,2,3; 2,3,3; etc., each number representing a side of the triangle.

This was a hard problem, as we realised beforehand, but most of the pupils did succeed in finding all 5 possibilities. One pupil assumed that the centre of the circle was also a possible vertex, which complicated her answer! Another found it hard to draw a straight line accurately between two points.

We also tried to discover how many squares could be found in any drawing of three overlapping squares. This required, for example, the construction of a workable recording scheme.

The triangles problem enabled us to ask some more questions:

Which are the same? Why?
How can you record your solution so that somebody else understands it?
Who has a different solution?
What would happen if.....(different numbers of points on the circle)?
Have we found them all? (although they were not totally sure this time!).

The foregoing discussion was prompted by acquaintance with NRICH material. All the subsequent material below does not come from any particular identifiable source, but stems from my own acquaintance with mathematics, and in particular from experience of university teaching since 1960. The numbering system used here means that, for example, Lesson 3.7 was the seventh lesson given in the academic year 2003 - 2004.

Lesson 1.2

Subject: Envelopes. What is the job of an envelope? It encloses something, or wraps around it.

Example 1: Ruler rotating, with one side always passing through the same fixed point, and lines drawn along the other side. We saw that, when many lines are drawn, the lines *envelop a circle*. Discussion of *infinity*, via 3, 20, 60, 1000,..., *infinitely* many lines.

Draw one line, then a second, then another, then: *what I am I going to ask you next?*

Example 2: Large circle, with one point marked inside it. Fold so that the circumference touches the point. Make a crease. Repeat, so that a different place on the circumference touches the point. Repeat again, lots of times. *What is the envelope of the creases?* Not a circle. An oval. It is called an *ellipse*. Discussion of the difference between an egg (not symmetrical) and a rugger ball (symmetrical).

Example 3: A tennis ball box is a circular cylinder. It is cut in two by an *oblique* cut. This shows another example of an ellipse. It is the shape of the oblique cross-section cut from a circular cylinder. The *symmetry* of the ellipse is proved by turning the two halves of the box round until they fit next to each other in one other position.

Discussion of the difference between *cylinders* (whose sides can be *curved*, having [non-oblique] cross-sections of any shape – e.g. circles, or ellipses, or anything), and *prisms* (which are cylinders with three or more *flat* sides).

Promise to explore relation between *ellipses* and *eclipses*.

Lesson 1.3

Recap(itulation) [not capitulation]: circle by straight ruler; ellipse by folded circle.

Topic 1: Construction of *ellipse by taut string joined to two fixed pins*. Each pupil fixes the pins at any chosen distance apart, which is less than the length of the string. Holding the string taut with a pencil, draw out a half-ellipse without passing the (extension of the) line joining the pins. Then repeat on the other side to complete the ellipse.

Notice that if the pins are *close* together we get a *fat* ellipse, and if they are *further* apart we get a *thin* ellipse. Notice also that if the pins could *coincide* we would get a *circle* whose diameter is the same as the length of the string.

Discussion of the basic importance of this ellipse construction, which exploits the fact that if one pin is at a point labelled S, and the other pin is at F, and the pencil is at P, then the total distance $SP + PF =$ *a constant* (the length of the string) for every position of P on the ellipse.

Topic 2: *Astronomy*. Discussion of the fact that each planet (name them) moves on an ellipse, under the attracting force of gravity exerted by the Sun (and the planet). The moving planet is at P, and the fixed Sun is at S (a *focus* of the ellipse). So P = planet, or pencil point. The other end of the string is fixed at F, the *second* focus of the ellipse, which the astronomy does not happen to use.

Topic 3: Change one letter, and ellipse becomes *eclipse*. Discussion of the difference between a partial and a total eclipse of the Sun by the Moon. Who saw it two years ago, and from where, and what *precautions* were taken when observing the eclipse?

Topic 4: Why does the Moon *almost exactly* obscure the Sun during a total eclipse, instead of appearing to be too small or too big? It is because of the freak fact or *coincidence* that the Sun is 400 times further away than the Moon, but also 400 times bigger than the Moon. This allows us to use *similar triangles* to calculate the diameter of the Sun if we know the diameter of the Moon (2160 miles). Draw the triangles, and work out 2160 x 400 = 86400 miles. This showed up some weaknesses in multiplication.

Topic 5: We displayed the model of the three dimensional *cubic cusp* surface made of steel rods, i.e. straight lines, in the playground. "We like it".

Lesson 1. 4

Recap: string/pins construction of ellipse. Astronomy – planet on ellipse, attracted to Sun at one focus. Eclipse. Diameter of Sun by similar triangles.

Topic 1. Exhibit *cubic cusp* model made of steel rods, in garden. Commence drawing side view on graph paper. Draw a vertical line down the middle (parallel to short side of A4 paper), and two horizontal lines 8 units apart. Size of unit = 2cm. My published article about this, with a photograph, can be found as Mathematics in the Garden, in Mathematics Today, Volume 42, pages 215 - 216, 2006.

Topic 2: Discussion of *units of length*: children list mm, cm, m, km, (metric); inch, foot, yard, mile (Imperial); cubit = length of forearm from elbow to wrist (ancient) – reading by pupil from I Kings 6, v.2 of specification in cubits of Solomon's Temple; hands (horses); light year (astronomical); nautical mile per hour = 1 knot (nautical – no time for fathom and league); chain = 22 yards = cricket pitch (surveying – no time for furlong).

Topic 3: Returning to graph paper, draw 5 inclined lines by nominating end points B,C,D,E,F on top horizontal at distances of 1.5, 2.9, 4.1, 5, 5.6 units to *right* of centre vertical AA; and B,C,D,E,F on bottom horizontal at distances of 0.5, 1.1, 1.9, 3, 6.4 units to *left* of centre vertical. Join BB, CC, DD, EE, FF. Note that these straight lines *envelop* a curve. Repeat symmetrically about AA, i.e. measure to left at top, and to right at bottom. Note that *the two envelope curves come together at a cusp*. Compare with the side view of the model made with steel rods, and with another but pocket-sized model of the same thing made of cotton threads between two parallel cards. Can we make it?

Lesson 1.5

1. Organisation of previous work, by putting loose-leaf notes in chronological sequence, numbering the pages, and writing a Contents page of work, to be kept up to date.

2. Discussion of fact that a straight line is specified by two things, in one of two ways. Either we are given two points, to be joined with a ruler (as in the example of the model of lines which envelop a curve having a cusp), or we are given one point together with a slope or gradient (as in the rotating ruler next to a fixed point). So there is a recipe or formula, so far implicit, but in the background. We shall want to make such formulae explicit in due course.

3. Construction of cardioid, as the envelope of lines joining 36 equally spaced points on the circumference of a circle. This is done by numbering the points, and joining each point to its double by a straight line, e.g. 11 to 22, 27 to 54 = 36 + 18. The cardioid has a cusp too, like the envelope drawn last week via the lines representing the rods in the steel model. One pupil immediately said "It looks like a bum". (Oh dear, but I suppose this proves they were alert). This was before I had the opportunity to point out that it is heart-shaped and that is why it is called a cardioid. (By then, however, the majority view was otherwise.) Demonstration of cardioid on the bottom of a cylindrical coffee mug, by shining a torch light obliquely into it.

Lesson 1.6

1. In response to an observation from a pupil that what have been doing (e.g. drawing of envelopes) has "not really been maths" (i.e. not "calculating something"), we began with an explanation that maths includes not only arithmetic (calculating); but also geometry = earth measuring (literally, but now often called shape and space in schools), and algebra (working with formulae) too.

2. Chutney Problem. We looked at a vinegar bottle whose label said that it contained 568 ml. An apple chutney recipe requires 2 pints of vinegar. How many bottles should be bought? The children were required to make detailed notes as follows.

   We need to know that "a litre of water is a pint and three quarters" approx., and that 1 ml = 1/1000 litre. We write these facts as equations, using L = 1 litre and P = 1 pint so that (big step) P = (1 + ¾) x L or P = 1.75L approximately (actually P = 1.76L more exactly); and 568 ml = 568/1000 L = 0.568 L.

3. There are two ways to solve the problem, from a graph or from the formula. We only had time for the graph. Draw a horizontal axis (2 cm = 1unit), marking L = 1,2,.. along it. Draw a vertical axis (also 2 cm = 1unit), marking P = 1,2,3,... along it. Plot the point L = 0, P = 0 (origin) and the point L = 1, P = 1.76. Join them by a straight line. Plenty of scope here for lack of precision by some pupils in drawing the axes in the (instructed) place, large blobs for points, and the line not quite through the blobs. On the whole we succeeded, however. Now read off from graph that if L = 0.568 (contents of one bottle in litres), then by drawing a vertical up to the line, and then a horizontal, from the line, to the left, we meet the vertical axis at P = 1 pint. Quite wide variations in this answer, but some were spot on.

4. Conclusion is that, since the recipe requires 2 pints, we need to buy 2 bottles.

Lesson 1.7

1. Draw a table to receive 10 readings of Time in the first row, and the corresponding Temperatures in degrees Fahrenheit in the second row. Filled a mug (having vacuum double-walled sides so that heat loss would only be through the top surface, and therefore slower) with hot water. Taking turns by passing my watch round the room, the children read the temperature every five minutes. We recorded the results on the board, and in their notes. We shall plot the graph next week, and discuss the cooling curve.

2. While this was happening, we recapped a discussion from two weeks ago, about the fact that maths is not just "calculating something", but includes the following.

   Algebra uses letters to represent numbers, known or unknown. E.g. a formula like P = 1.76L, where the juxtaposition of two letters, or of a number and a letter, is understood to mean multiplication P = 1.76 x L, whereas addition, subtraction and division requires explicit use of the corresponding signs, as in 1.76 + L or 1.76 – L.

   Arithmetic, which is what we are doing when we actually put numbers in the formula, like P = 1.76 x 4 = 7.04.

   Geometry, e.g. drawing a graph, for example the straight line P = 1.76L in this case. We examined our previously drawn graph, noted the inaccuracies which could be introduced by drawing blobs instead of points, checked that L = P = 0 and L = 1, P = 1.76 were points which could be joined by a straight line to make the graph, and for the vinegar bottle checked that 1.76 x 0.568 = 0.99... = 1 pint approximately. There was some discussion of the legalities of labelling in metric or Imperial units on bottles of milk, vinegar, etc.

Lesson 1.8

1. Palindromic numbers. We began by noticing that last Sunday's date was 20/1/02. This number is a palindrome, if we ignore the slashes. When is the next palindromic date? Evidently next month, on 20/2/02. But this looks even better if we write it as 20/02/2002.

2. Next we recall that we have some unfinished business from Lesson 1.7, namely to plot the graph of temperature against time from the measurements taken then of the cooling mug of water. First we make sure that we have a record of the measurements in a table

Time	11.15	11.20	11.25	11.30	11.35
Tempdeg F	173	156	149	141	131

Time	11.40	11.45	11.50	11.55	12.00
Temp deg F	130	121	120	117	114

Next we choose the scales and ranges, of 2 cm = 5 mins for time horizontally from 0 to 45 minutes, and of 2 cm = 10 deg F vertically from 100 to 190 degrees, and draw these axes near the left and near the bottom of the graph paper oriented with the long side horizontal.

Next we plot the graph obtained by making crosses (common in science experiments).

Then we join the crosses with a smooth curve.

What do we learn from the curve? First notice that it has bumps in it, so that it is not quite a smooth curve as might have been hoped. This is because the measurements were made in less than ideal conditions. My alternative graph from an experiment done in the more controlled conditions of my kitchen was handed round.

Secondly we noticed that the curve is a falling curve, confirming that the temperature does decrease with time, as we would expect. That is, the water cools.

Thirdly we noticed that the graph flattens out, so that the graph becomes less steep with time. It took some time for the children to articulate this feature. It means that the rate of cooling gets gradually less.

The name of the curve is Newton's Law of Cooling, dating from about 1670. Next week we shall plot a similar curve, which can be described by a simple equation, $y = 1/x$.

Lesson 1.9

1. Recap of Newton's Law of Cooling, by asking one pupil to explain, to a pupil who had been absent, what we did. The absentee was given the graph plotted by the children, and my graph from my more controlled kitchen experiment. The main points were that the curve of temperature falls with time (cooling), and also gets less steep (the rate of cooling also decreases).

2. Next we introduced the equation $y = 1/x$, called a hyperbola. Having discussed what the fraction meant (1 divided by x), we constructed the following table.

x	1/5	1/4	1/3	1/2	1	2	3	4	5
y	5	4	3	2	1	1/2	1/3	1/4	1/5

We constructed this table starting from the middle, then working to the right (the easier part), and then working from the middle to the left so that we had to agree explicitly that, for example, 1 divided by 1/3 was 3 because there are 3 thirds in 3, and so on.

Next we set out to plot the points, as crosses, on A4 graph paper with the long side horizontal. First we drew the x axis horizontally across the middle of the page, and also the y axis vertically down the middle of the page. The scales were nominated as 2 cm = 1 unit on each axis, and the numbers 1,2,3,4,5 were written below, and to the left, of the two axes respectively. It took some time for some people to do all this accurately.

Then we actually plotted the points. Not all of these were in the right place initially, and advice was needed on how to interpolate between adjacent scale values, for example between 4 and 5, and between 0 and 1. The curve was also a smooth falling curve, but with the flattening out feature, rather like the cooling curve. Next week we will deal with negative numbers.

Lesson 1.10

1. We began by addressing a difficulty noticed last week, that some children had incorrectly plotted y = 1/x near the axes. Their plots did not approach the x-axis for small y (nor the y-axis for small x), but instead moved away from those axes again. This meant, for example, that the curve did not flatten out (like the cooling curve) as it should have done.

We dealt with this by examining closely the unit (2cm) square on the graph paper, noticing that it has 10 lines ruled across it (at 2mm intervals) in both directions, so that 1 small square (having 2mm sides) gives us 1/10 of a unit in both directions. This means, for example, that 1/5 of a unit is 2 small intervals, and so on.

2. Now we wish to plot y = 1/x for negative x. For example, we agreed fairly readily that for the formula y = 1/x, x = -1 implied y = -1, and this was explained by the operation

y = 1/(-1) = [1 x (-1)]/[(-1) x (-1)] = (-1)/1 = -1. More discussion was needed to see that

for x = -2, we have y = 1/(-2) = [1 x (-1)]/[(-2) x (-1)] = (-1)/2 = - ½.

We articulated explicitly the two rules that (a) multiplying the top and bottom of a fraction by the same number does not change the fraction, and (b) minus times plus makes a minus, minus times minus makes a plus, plus times plus makes a plus.

Then we had confidence to construct the following table of values, with some guess work based on the pattern of the table and that for positive numbers from last week.

x	-1/5	-1/4	-1/3	-1/2	-1	-2	-3	-4	-5
y	-5	-4	-3	-2	-1	-1/2	-1/3	-1/4	-1/5

We plotted these points, as crosses, on the same graph paper as those for positive x from last week, joined up the crosses with a smooth curve, and finally had both parts of a complete hyperbola with equation y = 1/x.

3. Then we introduced the idea of a complete cone, via two plastic funnels fixed point to point, like two ice cream cones. We exhibited a plane cut through the two halves, and saw that the cuts were shaped like the hyperbola which we had just drawn. Finally we saw that a laser beam, moving along a plane cutting both half-cones in turn, traces out a hyperbola on them.

Lesson 1.11

1. We noticed that at 2 minutes past 8 yesterday evening the time and date could be written

   20.02 20/02/20002 which is a very palindromic number.

2. We discussed the idea of infinity, and in particular that it is a limit, not a number.

We did this via last week's graph of the hyperbola y = 1/x, which has two parts, called branches. We looked at what happened on the positive branch as we passed through the increasingly smaller and larger values of x and y in the following table.

x	1/1000	1/100	1/10	1	10	100	1000
y	1000	100	10	1	1/10	1/100	1/1000

We see that when x gets very small (and stays positive), y becomes very large and positive. Mathematicians express this by saying that, as x tends to zero, y tends to infinity. The arrow notation --> which mathematicians use to mean "tends to" was introduced, for example x --> 0.

We learnt that this is used when a limit is approached, thus introducing the idea of a limit. A limit may or may not be a number. Obviously zero is a number (and also a limit in this example), but infinity is not a number. In this example it is a limit.

The idea that infinity is not a number provoked some discussion of the "is it?, isn't it?"

variety, which served to highlight the idea of a limit. Railway lines are parallel, and appear to meet only "at" infinity. It is not a number which can be attained. So it is a limit.

Also, when y -->0, x --> infinity on the graph of the hyperbola; and also on the negative branch.

3. Next we discussed gradients of roads, which will lead to development of ideas about equations which were coming up in Lesson 1.6. Triangle signs warning of an imminent hill used to have information like "1 in 8" in them, and have recently been modernised to say (12 + half) %. What does this mean? Can you draw the picture? It is the gradient of the tangent to the steepest part of the hill. The definition is

   gradient = [vertical step] / [horizontal step].

   We drew the triangle, and verified that 12.5/100 = 1/8 by the argument that there are 8 12.5s in 100, i.e. 8 x 12.5 = 100.

Lesson 1.12

1. We first noticed a photograph, from last Saturday's newspaper, of the name tag of a baby born on 20 February. It stated that she was born on 20-2-02 at 20.02. As we noticed last week, this could be written more fully 20.02 20/02/2002. So we gave the baby an extra middle name, making her Lily Palindromeda Fry.

2. Continuing our discussion of gradients, we looked at another example of 20% on a road sign, and showed that it is equivalent to 1 in 5 because 20/100 = 1/5. The reason is that 5 x 20 = 100. Put otherwise, we reconcile these by multiplying both sides of the equation by 5.

We noticed that working with an equation is like washing your hands. This means that you must always do the same thing to both sides. Things go wrong otherwise – just as it is very awkward to wash one hand at a time.

We saw, in Monday's Independent cricket report from New Zealand, a reference to "the steepest street in the world", Baldwin Street in Dunedin, which has a gradient of 1 in 1.266. Because 1/1.266 = 79/100, this gradient is 79%, or 38 degrees. Get off your bike and walk.

3. Referring back to the milk carton and chutney bottle problems discussed in Lessons 1.6 and 1.7, we saw that the pints/litres equation P = 1.76L could be rewritten L = P/1.76, by dividing both sides of the equation by the same thing, i.e. by 1.76 in this case. We drew the graph of this second equation, having gradient 1/1.76, and compared with the graph of the first equation having slope 1.76. Then we checked the accuracy of the alternative measures quoted on three milk cartons, as follows.

P	1	2	4
L	0.568	1.136	2.272

4. Each pupil was given a letter from the bank showing the euro/local currency rates for 12 European old local currencies. We shall simplify these to be accurate to the first two digits next week, and identify the appropriate equations.

Lesson 1.13

1. We began with a recap about how to plot the graph of an equation of the form y = 2x. A pupil sketched the graph of this equation on the board. We discussed the fact that the straight line has gradient = (vertical step)/(horizontal step) = 2 in this case, which is the constant that appears in the equation. We saw that the line continues through the origin into the region where the variables have negative values, when there is no intrinsic reason why the variables must be positive (as happens in some cases). Then we expressed x in terms of y, by doing the same thing to both sides of the equation, namely dividing by 2 in this case to give x = y/2. A pupil sketched this on the board as well, and we noticed that the gradient is ½, again the constant that appears in the equation. This recap took some time, because the point about doing the same thing to both sides of an equation had to be reiterated.

2. Then we came to the topic of euro conversion formulae broached last week. We discussed how to approximate the rates which the bank had provided by using the first two digits only, since this was enough for our illustrative purposes (not

pupil	country	rate	L = mE	E = L/m
J	Austria	14	S = 14E	E = 0.071S
O	Belgium	40	F = 40E	E = 0.025F
A	Germany	2.0	M = 2E	E = 0.5M
K	Finland	5.9	M = 5.9E	E = 0.017M
G	France	6.6	F = 6.6E	E = 0.15F
B	Greece	340	D = 340E	E = 0.0029D
S	Ireland	0.79	P = 0.79E	E = 1.3P
M	Italy	1900	L = 1900E	E = 0.00053L
A	Netherlands	2.2	G = 2.2E	E = 0.45G
J	Spain	170	P = 170E	E = 0.0059P

being bank clerks). This rounding up or rounding down did not prove easy, even though it had evidently been seen before, and we shall have to return to it.

We began to compile the above table, wherein L = local currency, E = euros, and m the approximate conversion rate. Each pupil was responsible for one country. We got as far as completing the third column (rate).

Lesson 1.14

We completed the fourth column of the table above, by each child specifying their equation which expresses the local currency (schillings, francs, marks, markka, francs, drachma, punts, lire, guilders, pesetas) in terms of euros. This used the simplified rates, correct to two figures, obtained by rounding (up or down) the rates given in the bank statement. We checked understanding of the equations by using some simple choices such as 2,3,5,10 euros to calculate mentally the equivalent local currency.

Then each child worked out their entry E = L/m for the last column, by dividing both sides of L = mE by the appropriate m.

Next each child sketched their own graph of E = mL on E (for y) and L (for x) axes. These were straight lines through the origin, of gradient m. Some had very large gradients, such as 1900 for Italy and 340 for Greece, one was easy (2 for Germany) and the lowest gradient was 0.79 for Ireland.

Lesson 1.15

Sharing the cake.

This problem was prompted when my wife and I visited the Royal Horticultural Society Garden at Wisley, two days before. After looking at plants, and watching the RHS grass snake (two feet long) cross the path just in front of us, we retired to the cafeteria for a pot of tea and cake. I chose a large slice of coffee sponge cake with chocolate filling, whereat my wife said "I only want a third of that". After some calculations on the disposable paper napkin, in the time honoured fashion of mathematicians, the following easier problem emerged.

Divide the slice of cake into four equal pieces (this is easier to solve than three pieces). We do this as follows, and this problem was presented to the children. First make the problem even easier by removing the curved circular edge with a straight cut joining the two points on the perimeter. We are left with an isosceles triangle to divide into four equal pieces. The children knew what an isosceles triangle is. They were each invited to draw one on the board, and to show how, by making one straight cut, they could remove one quarter of the triangle. It is done by joining the midpoints of any pair of sides. Some could do this, one very precisely by measuring with her ruler on the board; some had some idea but could not articulate it very clearly; and some could not begin.

The solution was then explained, and a proof was given by showing that joining all three midpoints divides the isosceles triangle into four identical smaller isosceles triangles. It was explained that any two triangles which are identical are called *congruent* to each other. The children confirmed the congruence of the four smaller triangles by drawing them, cutting them out with scissors, and laying them on top of each other.

Of course there is another approach, which is just to cut the slice in half along the centre line of the circular sector, and then cut it in half again. This is an exact solution, but it does not give scope for discussing the approximation (of cutting off the curved edge), nor for talking about isosceles triangles.

Losing money to the bank.

It was found, by phoning Lloyds bank, that on

13 March the bank would sell 1.3611 US dollars per UK pound,

and would buy 1.5011 \$ per £.

Yesterday, five weeks later, it was found that on

17 April the bank would sell 1.3884 \$ per £,

and would buy 1.4520 \$ per £.

So the customer gets more \$ per £ now than before, and also needs to produce less \$ now to get a £ than before.

The following problem was posed. If we bought dollars on 13 March for a trip to the United States, and came back having spent all but \$100, how much do we lose on that \$100 by selling them back to the bank on 17 April. The children were asked to think, before next week, how they would solve this problem.

Lesson 1.16

We began by recapping the problem posed at the end last week, namely that if 100 dollars is bought from the bank at the rate of 1.3611 \$ per £ on 13 March, and sold back to the bank at the updated rate of 1.4520 \$ per £ on 17 April, how much does the customer lose?

The children were invited to spend 5 minutes just thinking about how they would address this problem, without actually doing the calculation, and then describe in words what they would do. It became clear quite quickly, and perhaps disappointingly, that the children were not at all comfortable with the idea of just stopping to think like that. They much preferred to jump in with a speculative suggestion, such as multiplying the two rates together, or dividing them, which made no logical sense.

So the steps which are required had to be explained, as follows. First, define some easy notation, such as D for dollars and P for pounds. Secondly, express the problem, using this notation, in terms of equations as follows.

I (the customer) buy at the rate of D = 1.3611P, so

D = 100 cost P = 100/1.3611 = 73.47 (rounding from 3 calculated decimal places).

This involves some conversation about dividing both sides of an equation by the *same* thing, to ensure that the result is true.

Then I sell at the different rate of D = 1.4520P, so for

D = 100 I receive P = 100/1.4520 = 68.87.

Thirdly we can now calculate the customer's *loss* to be 73.47 – 68.87 = 4.60 pounds for every 100 dollars.

We then set out to illustrate some of these equations by straight line graphs, to illustrate how much profit the bank makes, this time on the *same* day (13 March). So using the two rates that the bank buys dollars at the *steeper* rate D = 1.5011P to the customer than it sells, namely D = 1.3611P, we want to see that the *other* version of the equations and graphs tells us *more easily* that the bank makes a profit, namely the version that

the bank sells dollars at the steeper rate P = D/1.3611 = 0.73D

than it buys dollars at the lower rate of P = D/1.5011 = 0.67D.

Lesson 1.17

We recapped the discussion of last week leading to the two equations above, and then we plotted these two graphs. We used axes over the ranges 0 < D < 100 and 0 < 80 < P, and scales of 2 cm = 10P and 2cm = 10D. The graphs are both straight lines, joining the origin D = P = 0 to D = 100, P = 73 and to D = 100, P = 67. From these we see that for every 100 dollars which the bank sells and then buys back, it gains 6 pounds.

Anticipating a visit by me to Italy, we repeated the similar calculations for euros. On 1st May the bank was selling euros at 1.5517 per £, and buying them at 1.7008 per £, so their selling and buying equations were E = 1.5517P and E = 1.7008P respectively. The children calculated that these can be rewritten as P = 0.64E and P = 0.59E, and that the customer therefore would lose £5 on every 100 euros bought and then sold back to the bank. These latter two graphs were also plotted.

Spacing of spokes

We finished with a bicycle wheel problem. On some bicycle wheels the spokes are arranged in pairs coming from (each side of) the hub. The two spokes in each pair cross before they reach the rim. It is rather easy to count the pairs by counting their crossover points. There are 9 on each side of the hub, so 18 pairs altogether, so 36 spokes in all. Adjacent connection points on the rim therefore subtend an angle of 360/36 = 10 degrees.

Lesson 1.18

Construction of the cubic cusp surface

In Lesson 1.3 and 1.4 we displayed a model of this, made of steel rods about 3 feet high. We drew a side view of it, and showed that the lines enveloped a cusp in that view. The full size model now has sweet peas growing up it in the garden. It was explained that the cubic curves at the ends of the model did not always keep turning in the same direction, but always had a point where they began to turn in the opposite direction.

The object now is for each child to make a small scale model of the surface, 8 cm high.

We first list the steps required.

1. Calculate the positions of the points on the two end planes.
2. Plot the points on two graphs, from given formulae, and label them.
3. Stick the graphs to stiff rectangular 85 x 190 mm cards from Lindt chocolate packets.
4. Pierce the cards at the plotted points, with a drawing pin.
5. Fix the cards 8 cm apart.
6. Thread cotton though the holes with a needle, joining points having the same label on each card, and pull the cotton taut.

The formula is $x = (y - 2)t - ttt$. We needed to explain what ttt, or t cubed, means. This was done by speaking of volume. Some children began by thinking that it was 3t, but this was corrected.

Now we have first to put $y = 0$, and plot the consequent cubic curve $x = -2t - ttt$, as follows. The children computed the values of ttt on their calculators, which required some discussion of rounding up or down, since we only required two decimal places, and then they worked out each x.

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75
ttt	0	0.02	0.12	0.42	1	1.95	3.37	5.36
x	0	- 0.52	- 1.12	- 1.92	- 3	- 4.45	- 6.37	- 8.86
label	A	B	C	D	E	F	G	H

Lesson 1.19

We reviewed the fact that last week we had worked out the values of x = -2t – ttt for a range of positive values of t, and found that the corresponding values of x were negative. Then we predicted, by examining the formula, that if we choose the same numerical values of t but change their sign to negative, the new values of x would be the same as before except that their sign would change to positive. Thus we obtained the table below.

t	0	-0.25	-0.5	-0.75	-1	-1.25	-1.5	-1.75
x	0	0.52	1.12	1.92	3	4.45	6.37	8.86
label	a	b	c	d	e	f	g	h

We observed via a sketch that these two tables contain enough information to allow us to plot our first graph next week.

Next we went back to the controlling formula x = (y – 2)t – ttt, and this time inserted y = 8 (instead of the previous y = 0) to obtain x = 6t – ttt. A reminder of the cubic (e.g. volume) interpretation of ttt was necessary (it is not the same as 3t). With this new formula we calculated the following table.

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75
6t	0	1.5	3	4.5	6	7.5	9	10.5
x	0	1.48	2.88	4.08	5	5.55	5.63	5.14
label	A	B	C	D	E	F	G	H

Lesson 1.20

We began by recalling that last week we calculated a table of values of x = 6t – ttt, and noting that the value for t = 2 would be easily found to be x = 12 – 8 = 4. Then we saw that if we change the sign of t (i.e. use t = -2) we get x = 6.(-2) – (-2).(-2).(-2) = - 4. That is, we just change the sign of x. With this clue we can easily calculate our final table, from the previous one, to be as follows, just by changing signs.

t	0	-0.25	-0.5	-0.75	-1	-1.25	-1.5	-1.75
x	0	-1.48	-2.88	-4.08	-5	-5.55	-5.63	-5.14
label	a	b	c	d	e	f	g	h

Now we have all the values needed to plot the graphs of x = -2t –ttt and x = 6t – ttt, which we shall subsequently fix 8 units apart to make our string model. The children began to plot these graphs, first drawing the axes, and the using help provided by photocopies of the locations of the points. This help was desirable. It certainly speeded up the process, because they were able to just copy the results if they felt a little lazy or daunted, which some of them did (guess which). But some understanding was gained. For example, some children observed without prompting that the second cubic turned over and began to decrease, which the first one had not. I took their graphs home to stick onto card. Then they will be in a position to make the holes ready for threading the cotton through.

Lesson 1.21

This was the final session of the year. Before the session I had checked the points plotted by the children on their two graphs, and marked in red any errors or omissions, so that everyone could start from the same position. The first task for the children was to use (notice board) pins to make holes through the cards at the plotted points. This was done quite quickly, using soft board to press on so that the pins could go right through without damaging the table. Next the children were given spacers previously prepared from card by me, so that the two graphs could be held at 8 cm apart precisely. It was explained that this 8 = 6 – (- 2) arises from the coefficients of t in the equation x = (y – 2)t – ttt plotted for y = 8 and y = 0 on the two cards. The children wrote the name **cubic cusp model** on one of the pair of cards (the one with y = 0).

We then used the spacers to fix the pair of graphs on the cards at the 8 cm separation. The final task was to thread cotton to join the correspondingly marked holes. My wife had bought the correct needles for this, and a spool of thread. We first asked the children to guess the total length of cotton that would be needed to make the model. The guesses varied from 72 cm to 175 cm. Then a more careful measurement was made, which showed that 200 cm would be needed. The children then threaded the cotton to complete their models. Some needed help from my wife and I, but they addressed the task very well, and finished with evident pleasure.

I revealed that the cards used had been saved from 100 g packets of Lindt 70% dark Extra Fine chocolate, where the cards were used as stiffeners at that time. This information was received with interest. Some photographs of the group were taken, holding their models.

Lesson 2.1

We removed the picture cards from two packs of playing cards, to leave eight sets of the numbers 1 to 10. The group of pupils was divided into four pairs, and each pair had two sets of cards numbered 1 to 10. Colours and suits were irrelevant for our purpose. One member of each pair shuffled the 20 cards. The other member took 2 at random, multiplied the numbers so selected, wrote down that product, and noted whether it was even (E) or odd (O). This was repeated 20 or 30 times, and the results entered in a table:

X	9	9	12	12	36	27	70	16	2	20
E / O	O	O	E	E	E	O	E	E	E	E

We then had a relaxed discussion of what each group had found, and whether the results were what might have been expected. Was there a pattern to be observed, or expected? The results of one group began with five 20s, for example, which had to be discounted as a fluke. Eventually it was realized that there were more evens than odds, and some began to suspect that this was to be expected. They seem not to have thought about the question previously.

The question was now posed: can we *prove* what will happen? The point was made to the group that mathematics develops like building a wall with bricks. We must verify that each brick is reliable before it is used, or the wall might fall down. This was understood, and we then proved as follows that if the experiment above were continued long enough, we should expect 3 times as many even products as odd ones. The following multiplication table was readily constructed and agreed.

X	even	odd
even	E	E
odd	E	O

It was accepted that this is a genuine proof. An explanation was given that we could therefore write, at the bottom of this second table and therefore at the end of the proof, the letters Q.E.D., which stands in Latin for *Quod erat demonstrandum*, meaning *Which was to be proved.* This used to be very common in older textbooks.

Thus we emphasised the importance of constructing a proof in mathematics whenever we can.

Lesson 2.2

The children were asked who would like to summarise what was done last week, and we listened to comments from two or three children.

Picking up the point about the importance of a proof, and the use of Q.E.D. to indicate when the end of a proof was reached, a book was shown round which contained numerous examples of this. This was Modern Geometry (The Straight Line and Circle) by C.V.Durell, a 1955 reprint (used by the presenter when he was a student) of a 1920 book, but still valuable for all that, even if some fashions have changed. We focussed on a particular result, that the three altitudes of a triangle all pass through the same point. (An altitude is the line from a corner which meets the opposite side at right angles). We illustrated on the white-board what this meant, and why there is something to prove (the result is not "obviously" true). We will judge later whether it is practicable for us to go through the proof.

Then we returned to carry forward the main activity of last week, that the product of two numbers selected at random from two sets of cards labelled 1 to 10 was three times more likely to be even than odd. This time the object was to see how this result emerges when we draw a suitable graph.

First we carried out the same experiment as before, but much more quickly, and each pair of pupils grouped their results in eight sets of five, in the following pattern.

X	6	6	8	10	90		16	15	21	72	54		.....	.....	.....
E O	E	E	E	E	E		E	O	O	E	E				

Next we listed the cumulative totals of E and O in the first 5, then the first 10, then 15, ..., and listed the ratio E/O of these cumulative totals.

Tries	5	10	15	20	25	30	35	40
Tot E	5	8	11	14	17	22	26	31
Tot O	0	2	4	6	8	8	9	9
E/O	Infin.	4	2.75	2.33	2.125	2.75	2.88	3.44

Two particular points of interest came up in the discussion of this. The first was "What is 5 divided by 0?" Answer 5. This provoked a discussion during which we worked out 1 divided by ½, etc. It was not long before we agreed that there are 2 halves in 1, then ten tenths in 1, then 100 hundredths in 1, and so on. Rather quickly the fact that 1 divided by 0 is *infinity* emerged. We had introduced it in Lesson 1.11, and I completed the picture with the symbol. "Wow. How do you draw it?". "Eight on its side."

The excited comment "Now I know the symbol for infinity." was heard as they left the room.

Next week we shall draw the graph of "Number of tries" (vertical axis) against "Cumulative ratio E/O" (horizontal axis), and hopefully show that it oscillates towards 3.

The second point of discussion was why we did not use more than the first two decimal places offered by the calculator. The children soon saw that that was no point in including things like, for example, the sixth place, because this was only something like 3 millionths, which was plainly negligible compared with, say 7 tenths and 5 hundredths.

Lesson 2.3

We began by noticing that today's date was 20/11/02. What is interesting about that? It is "palindromic". The name was introduced in Lesson 1.8. What does it mean? Two suggestions emerged. One was that this date is "reflective", which means that you could put a mirror in the middle of the numbers, and see the other half of them in the mirror. The other was that the sequence of numbers is the same whether you read it forwards or backwards. Words like "bob" are palindromic; but there was a discussion about where we could put the mirror in that case. It would have to go through the middle of the "o". Eventually it was agreed that the idea of getting the same letters or numbers by reading forwards and then backwards might be a preferable definition of "palindromic". We will return to this topic soon.

Before that we need to plot graphs of the table of data which we compiled last week, to show how the *cumulative totals* of 5, 10, 15,....tests of the product of pairs of numbers picked at random, from two sets of cards 1 to 10, were related to the *ratio of the totals* of evens to odds in those results. We know that the ratio should eventually be 3, from what we proved before. So the children set out, on graph paper divided into inches and tenths, a vertical axis labelled 1,2,3,4,...on each inch for the ratios; and a horizontal axis labelled 5,10,15,20,... on each half-inch for the cumulative totals. They then plotted their data with crosses (not blobs, after a discussion). Some difficulties were experienced with setting out the axes, and with plotting, and this topic will need to be revisited another time. But enough was done to see that the graphs oscillated (in some cases starting at infinity on the vertical axis); not quite enough data had been collected to see that the graphs would converge towards 3, which we know would be the expected result, but the exercise provoked helpful discussion. (We need to have another discussion about infinity, because some children still were not confident with the idea that 5/0 was not 5, for example).

Imperial measures like inches are still an instructive variant to metric measures like centimetres.

We need to do a similar experiment and graph plotting for tossing a coin, which would be simpler, and a little less time consuming to collect more data. For this there are only two outcomes, heads or tails, which are equally likely, so the oscillation of the ratio H/T should converge to 1.

Finally we returned again to palindromic dates. We noted that last January we had 20/1/02, and in February 20/2/02, which looks even better if we write it as 20/02/2002. We can get more elaborate palindromes by adding a time, namely 2 minutes past 8 in the evening. A photograph appeared in the press of a baby called Lucy Fry who was born at that time, so that the identification tag on her wrist had 20.02 20/02/2002 on it. We debated an appropriate middle name for her, and one suggestion from a member of the class was Palindromica. Another possibility is Lucy Palindromeda Fry.

Lesson 2. 4

We began by asking the children to indicate briefly what topics we treated last time, two weeks ago. There were palindromic numbers; and the graph oscillating to 3 which we found for the ratio of the numbers of evens to odds, which we found when plotted against the cumulative number of multipliers tested with the pairs of cards.

Then we moved on to discuss what the outcomes ("what does that mean?" " Results.") would be if, instead of multiplying pairs of numbers, we added them. The following table emerged.

+	E	O
E	E	O
O	O	E

So the ratio E/O of the number of evens to the number of odds in the table is 2/2. Most people agreed before too long that this was 1, or 50% as some put it. Georgie tried to insist, for quite a long time, that 2/2 would be 2. Some convincing was needed, and there was some debate.

Then we did the same kind of thing for subtraction, and the following table was agreed.

-	E	O
E	E	O
O	O	E

The new feature this time was that the difference of two numbers could produce a negative number, and we needed to agree that negative numbers could also be even or odd. However, this emerged relatively slowly, because the first examples of differences which the children wrote down, when requested to supply examples, all had the larger number first, so that the differences came out to be all positive at first. For this table too it was agreed, after discussion, that the ratio E/O of the outcomes was 2/2 = 1. So a graph would oscillate to 1 as for addition, but distinct from the 3 which we had previously found for multiplication.

Next we posed a similar question for division, i.e. if even is divided by even, or odd by even, or even by odd, or odd, by odd, do we get an even answer or an odd answer, so how do we fill in the following table this time?

/, i.e. divide	E	O
E	?	?
O	?	?

The first few examples which the children gave all had a larger number divided by a smaller one, but even so it soon became clear that the ratios were not always going to be whole numbers, so the answers were often *neither* even or odd. It was necessary to point out that we could also envisage small numbers divided by large numbers in these tests. The example 2/54 exercised us for a little while, because we first of all had to kill the idea that the answer was 27. This at least allowed us to introduce the "not equal to" sign (= with a sloping line through it, which Freddy knew). Eventually it was agreed that 2/54 = 1/27, which is neither odd nor even. It was clear that more discussion about fractions which are different from the most familiar ones of ½ , ¼, 1/8, etc. would be desirable before long. [It looks as if "Word" is limited in what it is expecting in this respect too.]

The opportunity was taken to introduce the fact that a whole number is also called an *integer* in mathematics.

With two minutes remaining the children asked to play "fuzz, buzz", which they had to explain to me. Numbers with a 3 in them, like 31, are "fuzz" numbers. Those which have 3 as a factor, like 15, are "buzz" numbers. We go round the room counting 1,2,3,4..., and you have to say quickly whether the number that reaches you is fuzz, buzz, or neither. A wrong answer counts you out. Grown-ups learn things too.

Lesson 2.5

This was the last session before Christmas, so we chose a fresh topic designed to be entertaining, interesting and instructive (after answering "Yes" to the question "Shall we be meeting like this after Christmas?" from several pupils).

The children were asked to draw out the following table, which was revealed in several stages, three rows at a time. They were asked to work out what language is being used in each column, thus filling in the top row which began with question marks. As the idea became apparent, they were asked to fill in what words they could, and what numbers should appear in the fourth column. Several of them were fairly ready with the French words, rather little was known about the German, and the Italian was at first asserted by one pupil to be Spanish. It may be similar to that, or to Portuguese, but I did not know to what extent that is the case, so of course that was admitted.

What is this to do with mathematics? The children were each asked to write down a one-line sum, using one number written in each language, and any two of the four symbols + , - , x and / (divide), with the answer in the right hand side of the = sign in ordinary numerals (it might easily require more than the first ten whole numbers).

French	German	Italian	Number
un	ein	uno	1
deux	zwei	due	2
trois	drei	tre	3
quatre	fier	quattro	4
cinq	funf	cinque	5
six	sechs	sei	6
sept	seben	sette	7
huit	acht	otto	8
neuf	neun	nove	9
dix	zehn	dieci	10

We then collected on the white board the sums which the children had written, and we found that there was something mathematical to be learnt. Sums like

dix + neun - dieci = 10 and quatre + ein - tre = 2 which only used the + and - operations were not ambiguous, and sometimes also gave minus numbers in the answer on the right. But other sums, which used x or / , like

cinq x acht + otto = ?   dix - acht x due = ?

were ambiguous. The answer depends on which operation is carried out first. In the first example it could be 48 or 80; in the second it could be 2 or - 6.

This lead to a discussion on the need for and the importance of *brackets* when multiply and divide operations are present, and which enclose the operation required to be done first. For there is usually a difference in the answer, for example between

(5 x 8) + 8 = 48   and   5 x (8 + 8) = 80.

In fact there can still be cases when such a difference does not occur, when x and / are *not* mixed with + and - , as in

(10 x 8)/5 = 16   and   10 x (8/5) = 16.

The last example revealed a point of evident difficulty for some pupils. I was confidently and blithely told that 8/5 = 1.3 (confusion between remainder and decimal). So I wrote 8/5 = 16/10 = 1.6 and this was accepted. But they seemed not secure about division except in very simple cases. Recall the difficulty last week with 2/54 = 1/27.

Lesson 2.6

We began a planned sequence of lessons about **Geometry**. We first had a discussion of what the word means, by splitting it into two parts. The easier part was "metry", which we soon agreed had something to do with measuring. The "Geo" part prompted suggestions that it had something to do with Geography and also with Geology. With some help, and a reference to the fact that the Earth (as a planet, not as soil) was shaped like a sphere (? no, not quite, because it is flattened at the Poles) or perhaps a rugby ball (ellipsoid – what's that?), and so could be said to be the Geoid, we eventually agreed that

Geometry means *Earth measuring.*

The discussion was pursued a little further by agreeing that Geography involves the study of activities on the Earth (town planning is part of Geography these days), and that Geology is the study of rocks. We finished this preamble, which drew interested participation from the pupils, by learning that *Trigonometry* is about *triangle measuring.*

Next we introduced the use of a *pair of compasses,* with the objective of getting practice in this use in several situations. First we checked that the pencils in them were short, so that they would not get in the way when we held the central hinge to draw circular arcs (*arc* had to be explained). It is not a trivial practical skill to keep the compass point in one place on the page as a circular arc is drawn, as we soon rediscovered.

The children were then asked to draw concentric circles with radii 1 cm, 2 cm, 3 cm and 4 cm. During this work it was necessary to explain what was meant by *concentric, radius, radii, diameter* and *circumference* (edge of the circle). Some children were quicker or had greater dexterity than others in this activity, but eventually we moved on.

The next task posed was to draw a *six-point flower* (for want of a better name, but it was well recognised when explained). A circle is drawn, with radius perhaps 2 cm, and then with the same radius the compass point is moved to the circumference, and an arc drawn inside the circle to meet the circle twice. The compass point is then moved to *these* two points on the circumference in turn, and the process repeated, and then three more times, when the flower becomes evident. Again, some children had a much more precise final result than others, but this will only be improved with practice. Perhaps we shall draw the flower on the playground with chalk in a week or two.

Lesson 2. 7

The children recapitulated some main points from last week, including the fact that Geometry means "Earth measuring" literally, and the fact that we could use a pair of compasses to draw concentric circles and the six-point flower.

Each child drew another six-point flower on paper, to improve their technique with compasses. The children were then asked to form pairs, for the purpose of drawing the six-point flower on a larger scale on the asphalt playground. Each pair was given a piece of string about 4 feet or 1 metre long, and requested to tie one end around a piece of chalk and the other end around a pencil. Even this task revealed some ham-fistedness, with one pair, otherwise apparently able, eventually having to admit that they could not tie a double knot successfully around a pencil. I had to do it for them.

Then, with one child holding the pencil upright at a fixed point on the ground, the other pulled and held the string straight while drawing a circle with the chalk. Arcs with the same radius were then repeated six times to draw the flower. There was a wide difference between the four pairs in their ability to do this readily. Two girls were very quick and adept; two boys took a long time to achieve the object; and the other two pairs were in between. Having drawn one flower, the children were asked to extend it and extend it again, by using the first six arcs to make whole circles, and again. This soon revealed how we could cover the playground with integrated patterns of circles and flowers, and photographs were taken with the School camera of some results of this, with their authors. A surprising amount of chalk was consumed. Alas (?) the rain will soon wash these patterns away, but at lunchtime other children were seen to be observing the patterns with curiosity.

We then returned inside to the circles and flowers on paper, and the children were asked to join the six points on the circumference with straight lines, to make hexagons. We noticed how these hexagons also covered the plane, with no gaps. Does this pattern occur in Nature? One child soon suggested a bee's nest. I happened to have a wasp's nest with me, and we examined that closely to affirm the hexagonal honeycomb pattern. This nest was actually a stack of four layers of hexagons, each on top of the next. The children made a note of what we had been doing, e.g. of how the repeating hexagonal pattern covers the plane.

Lesson 2.8

Four photographs of children working in the playground last week, and a paragraph stating what we had been doing, have just appeared on the School web-site. We now move forward from the idea of tessellating the plane with hexagons, to some more constructions designed to develop confidence in the use of a pair of compasses and a straight edge.

The meaning of "equilateral triangle" was explained (Latin equi = equal, lateral = side). The children were asked to draw one side (about 3 cm), and then to use the compasses with the same separation of 3 cm, and centred on the two ends of the line in turn, to draw two intersecting arcs. Joining the intersection point to the two ends of the line completes the equilateral triangle. This was repeated a few times to build up a tessellation of the plane by equilateral triangles. Some of the children had met some of this before, but the consolidation was worthwhile. Next the meaning of "isosceles triangle" was explained (Greek iso = equal, sceles = skeletal). Compasses and straight edge were then used to draw isosceles triangles of two sorts, with the equal legs longer, and then shorter, than the base line. Some tessellation with isosceles triangles was carried out.

Next we wished to construct the perpendicular bisector of a given straight line (chosen to be about 5 cm in length). Some discussion took place of "perpendicular", "right" angle (not left, or wrong), 90, 180 and 360 degrees. Then the construction was explained, of setting the compasses further apart than half the length of the line, and with unchanged separation drawing two pairs of intersecting arcs centred on the two ends of the line, and then joining the intersection points. The children did it, but not too readily in every case. Why is the resulting join of the intersections of the arcs the perpendicular bisector of the first line? Symmetry. This reason was understood when explained, but not readily offered by children.

With five minutes left we posed the problem to be solved next week. Given a circle with unknown centre, find the centre using only the compasses and a straight edge. Guesswork or fudging was quickly offered, but not the solution. So that leaves an instructive debate for next week.

Lesson 2.9

We began by noticing that the day after last week's lesson was 30/1/03, which was a palindromic date. We discussed when would be the next few palindromic dates. Someone suggested 30/2/03, but this was quickly ruled out with the observation that 30th February never happens, even in a leap year. Then 3/02/03, at the beginning of this week, was noticed. March examples of 3/03/03 and 30/3/03 also emerged. The discussion lighted on the fact that we were entitled to write 3 = 03 = 003 etc. in appropriate places to make the palindromic idea work.

We moved on to the problem posed at the end of Lesson 2.8, namely how to find, just by construction with a straight edge and a pair of compasses alone, the unknown centre of a given circle. The given circles were achieved by drawing round the perimeters of the plastic tops of cylindrical boxes with a circular cross-section, which could contain tennis balls but had contained flavoured "crisps" called Pringles, but not before an unscheduled investigation of the flavour of the aforesaid Pringles, by smelling the tops and announcing (wrongly) the supposed flavour.

The intended method was to draw any straight chord of the circle and then find its perpendicular by the construction learned last week. This perpendicular would be one diameter, by symmetry. Then draw another chord, and repeat the process. The two diameters would usually intersect, and that point would be the required centre of the circle.

There were significant differences in the readiness with which the pupils appreciated the validity of the procedure, and also in their ability to carry it through with facility. Sometimes the two chords were drawn nearly parallel, for example, so it was hard to see where the consequent diameters intersected. Time had to be given to repeat explanations to some individuals, and to allow them a second or even third attempt to carry out the construction. Eventually we got there, and the sequence of steps was written on the board for them to write in their notes. It might be profitable to repeat this exercise in the future, but hopefully more quickly.

We promised to find the radius of the Earth next week. This created a buzz, and some remarks that the Earth is an ellipsoid anyway. They also asked when they could make the string model which last year's class had constructed. I said that so far we had done nothing that last year's class had done. Aside from palindromic numbers, this was true.

Lesson 2.10

First we recapped how, last week, we had found the unknown centre of a circle using only a straight edge and a pair of compasses. This was done by drawing two different chords, and finding their perpendicular bisectors, which would both be diameters. Then the centre is the intersection of those two diameters. "Did you really eat all those

Pringles?" I explained that I hoarded the boxes over a long period, because they have the shape of circular cylinders, which sometimes occur in maths lessons – as in this instance, where we just needed to draw round the circular lids to get circles with unknown centres.

In preparation for the next topic, to measure the radius of the Earth, we did some ratio work. A rectangular box (cuboid) with sides 26, 22, 17 units is to be increased in size but end up having the same proportions, with the longest side 93 units. Using calculators, we agreed eventually that the scaling factor would have to be 93/26 = 3.6, so that the other two sides would become 22 x 3.6 = 79 and 17 x 3.6 = 61.

For the Earth radius problem we would need to scale triangles having one circular side, so we next scaled up a quarter circle (angle 90 degrees) with length a ("a" standing for "arc" length) to a half circle (angle 180 degrees) with length therefore a x 180/90 = 2a; and then to a three quarter circle (270 degrees) with length a x 270/90 = 3a; and then to a 10 degree arc with length a x 10/90 = a/9.

Next I showed a globe of the Earth, and explained that we really were now going to see how the radius of the Earth was measured, in 1830 in India, by Colonel George Everest of the British Army. First we discussed what a meridian is, and identified the arc of it which Everest measured, from the southern tip of India to the Himalayas. "Did he have help? Did it take a long time?" These were some questions which came up. His measurement of that arc was about 1871 miles.

Next we drew a triangle with that circular arc as one side, and the verticals to the Earth at the two ends extended inwards to the centre of the Earth. So we have a triangle with two equal straight sides whose length is the radius of the Earth, and one circular side along the meridian.

A heavy lead plumb bob hanging from a wire was shown, as the method for determining the local vertical at different places. The continuation of the wire would go through the centre of gravity of the Earth. ("Why plumb?" was asked. Latin for lead. Why "plumber"? Because they used to work with lead pipes. Lead poisoning was a digression too far to follow at this moment). Knowing the two verticals at the two ends of the meridian, we could measure the angle between them. It was about 27 degrees. Now we have enough information to use scaling or ratio arguments, by a factor 360/27 = 13.33, to find the circumference, and thence radius, of the Earth next week.

Lesson 2.11

First we announced an intention to do a "recap of ratios round a rectangle". This phrase is an example of alliteration in language, as Georgie knew. We tried out two more such examples, "she sells sea shells on the sea shore" volunteered by Danielle; and "round the rugged rocks the ragged rascal ran"; both to be said as quickly as possible.

The recap consisted of looking at two rectangles, one with height H and width W, and the other with height h and width w. They have the same proportions if W/H = w/h, i.e. if these two ratios are the same. For example, if 10/5 = 2/h, then we must have h = 1. In Lesson 2.10 we illustrated such equal ratios for two cuboids, which are the three dimensional version of rectangles.

We need the idea of the same ratio in two similar geometrical figures having different sizes for our objective of working out the radius of the Earth. Colonel George Everest measured, in 1830, the arc a = 1871 miles of a meridian from the Southern tip of India to the Himalayas. The two local verticals at the two ends can be determined by a plumb line, and they will meet, when extended, at the centre of the Earth. The angle between these radii was A = 27 degrees. The ratio a/A = 1871/27 must be the same as the ratio all the way round of c/360, where c = circumference of the Earth, because the smaller triangle based on India is mathematically *similar* to the *all the way round* "triangle".

So we have the equation 1871/27 = c/360, from which we can find c = 360 x 1871/27 miles, by multiplying both sides of the equation by 360. One always must do the same thing to both sides of an equation if it is to remain true (like washing your hands, which only works well if you do the same to each hand at the same time). We chose 360 because we noticed that 360/360 = 1, which isolates c, and the calculator then tells us that c = 24947 miles. Now we have the circumference of the Earth, using Everest's measurements.

To get the radius r, we have to introduce pi and the equation c = 2 x (pi) x r for every circle, where pi = 3.14159 = c/2r is the ratio of the circumference to the diameter. We quoted this formula, and gave a brief justification, but we need to return to pi in the next lesson. We used the formula to work out on calculators the radius of the Earth r = 24947/(2 x 3.24159) = 3970 miles.

This is also 8 x 3970/5 = 6352 km. The children were given a sheet containing the actual results measured and calculated by several people, showing that Everest's 1830 results compared well with more recent measurements, including modern ones by satellite, and also showing the small but discernible difference between polar and equatorial radii. The polar one is smaller because the Earth is not really a sphere, but a slightly flattened ellipsoid.

Lesson 2.12

There were three absentees when we last met three weeks ago. They missed the conclusion of our study of how Everest calculated the circumference of the Earth by measuring the distance along a meridian between two points in India, and the angle between the local verticals at those points. Therefore we began with a rather explicit recap as follows (with accompanying diagrams, e.g., of quarter and whole circles).

A simplified example of such a calculation is if the distance measured were 6000 miles between points on a meridian whose local verticals are at 90 degrees to each other, e.g. at the North Pole and on the Equator. Then if c = circumference, i.e. the distance all the way round and therefore between points whose local verticals are separated by 360 degrees, the ratio equation gives $c = 6000 \times 360/90 = 24000$ miles.

Everest's actual measurement was about 1871 miles, between verticals separated by 27 degrees, so by the same ratio principle $c = 1871 \times 360/27 = 24947$ miles approximately.

The problem which now immediately presents itself is, knowing the circumference of a circle, what is the diameter? (Because we would now like to know the radius of the Earth, and diameter = 2 x radius). Although this may be a year or two down the line in terms of formal syllabus content, it has arisen naturally at this point. And as we shall see, it involves perhaps **the most famous number in the whole of mathematics**, always denoted by the Greek letter pi (= 3.14159 approx.). So we approach the idea of pi slowly, as follows.

We shall look at several polygons, some of them regular. A rectangle is a good start. It provides a tangible definition of the two important quantities of

p = perimeter = distance all the way round the outside, and

d = diameter = distance across, through the centre, from one side to an opposite one.

The children each drew a rectangle, and measured p and (between opposite corners) d. Then they were asked to calculate the ratio p/d. For rather thin rectangles they found values typically of about 2.1. Fine so far.

I then showed them how, if the rectangle is made so thin that the two short sides can hardly be seen, the ratio is approximately 2 itself. That is, if the length of one long side is 1 unit, $p = 2$ + very small number, and $d = 1$ + very small number, so $p/d = 2$ approximately. This is also a starting point to talk about **limits**, but I did not pursue that.

Lesson 2.13

We set out the following table, and the children drew their own polygons as described in the column on the left. They measured the diameter d and perimeter p of each. These were usually different from child to child, because they drew figures of different sizes. They then calculated the ratio p/d, and these *ratios* should be the same for any particular choice of polygon (e.g. the square), although a different same value for another polygon (e.g. the equilateral triangle). This was a useful exercise in drawing, measuring and calculating, using a pair of compasses, a ruler and calculator. Of course, discrepancies and uncertainties were encountered along the way, and these were discussed individually. My measurements of *p* and *d* are in centimetres. The ratio has no units, of course.

polygon	perimeter *p*	diameter *d*	ratio *p*/*d*
thinnest rectangle	42	21	2
equilateral triangle	9	2.6	3.46
thin rectangle	8.4	4	2.1
A5 page rectangle	71.6	25.5	2.81
square (fattest rect.)	84	29.5	2.85
hexagon	15	5	3
dodecagon	15.6	5	3.12
circle	?	?	?

All but the circle were completed. The children were able to refer to and sometimes use figures which they had drawn in previous weeks, such as the equilateral triangle and the hexagon. The dodecagon (12 sides) was constructed by sketching an isosceles triangle onto each side of the hexagon, and its perimeter estimated by multiplying the length of one of these extra sides by twelve. When I posed the question of "What is the fattest rectangle?" there was sustained silence before Megan murmured the suggestion of "a square". To find the diameter of an equilateral triangle the children had to construct, using the compasses, the perpendicular bisector of one side – that length serves as diameter of that triangle. There has been scope for good and varied discussion while compiling this table.

Next week we shall draw a graph of *p*/*d* (on the vertical axis) against the number *n* of sides of the polygon (on the horizontal axis), and point out how the graph shows that *p*/*d* is approaching *pi* = 3.14159 as *n* becomes large (and that infinite *n* defines a circle).

Lesson 2.14

We set out to plot some points on the graph of *p*/*d* against *n* for some *regular* polygons. This needed a discussion, first of all, of what is a regular polygon. It is one with equal sides. So what is a regular rectangle, among all those four rectangles that we discussed last week? It is a square. Regular means that all sides are equal. So we drew the axes on graph paper, and plotted the points corresponding to the following table.

name of regular polygon	n	p/d
equilateral triangle	3	3.46
square	4	2.85
hexagon	6	3.00
octagon	8	3.08
dodecagon	12	3.12

On the vertical axis it was enough to use the range of 0 up to 4 for the p/q values. On the horizontal axis we obviously needed the range of n to be at least up to 12. The children needed some individual attention, but soon clearly understood how to plot the points for $n = 3,4,6,12$.

In fact we had not drawn the regular octagon last week, so this had to be done now. It was another useful exercise in construction with straight edge and pair of compasses. First draw a circle, then a diameter through the known centre. Then use compasses to draw the perpendicular bisector of that diameter (by a method from a previous week), thus giving another diameter at right angles to the first one. Join adjacent pairs of points where the two diameters meet the circle. That makes a square. Now construct, with compasses, the perpendicular bisectors of the sides of the square. Note the points where these bisectors cut the circle. Join those points, each to its nearest neighbour. That completes the regular octagon. Measure one of the sides, and multiply by 8 to get p. Measure the diameter of the circle to get d, and hence p/d for $n = 8$. Again the children needed individual attention during these procedures, and they did make mistakes, but eventually understood what to do. Then we could plot the $p/q = 3.08$ point for $n = 8$ on the graph.

Finally we observed how the values of p/q were levelling out fast as n increases. I told them that as n becomes very large, thus describing a regular polygon of very many sides, we should eventually be dealing with a circle. Then the value which p/q approaches would be about 3.14159...., always denoted by the Greek letter *pi*, and this is perhaps the **most famous number in mathematics.**

I explained that for a circle we alternatively write the perimeter as c for circumference, and $d = 2r$ where r is radius. Therefore we have $c/2r = pi$, and so $c = 2(pi)r$. This is a very useful equation for finding the circumference of a circle when we know the radius. When we were discussing the surveying of a meridian in India, we found the radius of the Earth. We can then use this last equation to find circumference of the Earth..

I gave each child a photocopy of the Greek alphabet, so that that could easily understand that *pi*, although written in that funny way as two sticks with a hat on, was just the Greek letter *p*, and nothing more mysterious than that. Thus ended a term of Geo-metry.

Lesson 2.15

We began by stating, and then solving, a very famous problem. This is called Zeno's Paradox. There was some discussion of what the word "paradox" actually means. We eventually agreed that it is a name for something that looks as though it must be nonsense, but when you look into it you find that it is not. We shall need to look into infinity.

We learnt that Zeno of Eleas was a Greek, from the town of that name, who was living in 450 B.C. So how long ago was that? 2453 years, says Freddy, very promptly.

Zeno's paradox offers us a very important problem: "You cannot get home tonight". What about tomorrow? No. What do you mean? How can it be? Here is the reasoning.

Suppose the distance from Here to Home is 1 unit. Draw a straight line, with two ends, to represent this. Then to get Home we must first go ½ way. Put a blob in the middle of the line. Then we must go ½ of the remainder. Put a blob at that new target on the line. The total so far will be ½ + ¼ = ¾ .Then we must go ½ of what is left. Put another blob at the new target. Total so far is ½ + 1/4 + 1/8 = 7\8. Repeat: 15/16. Repeat: 31/32. By now the children were getting the idea, and guessing the next denominator in the pattern. We can keep repeating the procedure, but *never finish*. There is always half of a small remainder to go. *So we never get there*. Of course, this is hard to accept, and there was rightful resistance, without being able to pick a flaw in the argument. The children were writing down notes from the board as the reasoning was developed.

Now we move to a *resolution* of the difficulty. Introduce a *label* for the distance travelled, so that we can *talk* about it efficiently. Write

D = ½ + ¼ + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + 1/256 +..........

This is the sum of an *infinite* number of *decreasing* steps. Multiply both sides of the equation by 2. This gives

2D = 1 + ½ + ¼ + 1/8 + 1/16 + 1/32 + ...Notice that the right hand side is 1 + D.

This is the *key step* that clarifies the problem, because we now have the simple equation

2D = 1 + D. Subtract D from both sides: 2D – D = 1 + D – D, so that D = 1.

This exactly the distance from Here to Home, so you *do* get home, by taking an *infinite* number of *decreasing* steps in a *finite* time. We have added up the infinite series D = ...

It was possible to see understanding growing of what was going on, and eyes opening with a sense of intrigue at the character of the solution.

Lesson 2.16

A mature student was present; so mature, in fact, that he was able to tell us the age of the Earth (1 000 000 000 years) on request. The children were asked, at the beginning, to introduce themselves to him by giving their names.

The subject this week was the Speed of Dinosaurs. We began by saying that we want to estimate the speed of dinosaurs, so we need first to ask what sort of evidence is actually available to enable us to do this. This provoked a lot of discussion, at the beginning, and later on, about the associated archaeology and geology. Three pupils were especially intrigued, and voluble with their speculations. Not all of these were at all accurate, especially about time scales, and it became clear to me that whole lessons on such things as geology, chemistry, zoology and archaeology would not be out of place for this age group.

The discussion led to the display of a time-line, which emphasised that humans had only evolved in the last one million years (and who knows how long *they* will last? Humans might not always be here!), much to the surprise of the children; that it was as long as 65 million years ago that dinosaurs were wiped out by climate change resulting from an asteroid impact in the Gulf of Mexico; and that different types of dinosaurs did not all overlap, even though some types existed for as long as 100 million years, because there was over 350 million years during which there was room for their evolution to take place.

Eventually it was agreed that the *only* evidence available for the previous presence of dinosaurs is (a) footprints and (b) fossils. It had to be explained that fossils are not the same as bones, because chemical changes have taken place.

What sort of measurements can be made on this evidence? We can measure stride length (call it S). Clearly it is longer for running than walking, so speed (call it v) is related to S. What is that relation? It will be different for different species.

Our strategy will be to use mathematics to make a prediction that can be tested for a wide range of modern animals, and then apply it for dinosaurs too. Reasonable?

The mathematics will also need leg length (call it L), which is the height of the hip when standing. The children were able to measure, with a metre ruler, their own L and S. To save some time I quoted my own S for slow and normal walk and sprinting.

This allows us to introduce the idea of $S/L = r$, the *dimensionless* stride length, or stride per unit leg, so that it does not matter what units (e.g. feet or centimetres) are used in the *scaling*.

Now we quote a discovery by zoologists, that for many species of modern animal (e.g.

ostrich, dogs, sheep, camel, human, elephant, rhinoceros) the *same formula*, having a straight line graph, allows speed v to be calculated from stride length r. The graph was sketched, but there was no time to discuss the formula explicitly.

The next idea is to use the same formula for dinosaurs, which is not too unreasonable.

Some pictures were passed around showing fossilised footprints of large (64 cm, perhaps a theropod like Tyrannosaurus) and small three toed bipeds found in Queensland in 1984; and quadruped prints of fore and hind (76 cm) feet, perhaps Apatosaurus, found in Texas in 1944.

The stride lengths were plainly measurable. In the absence of whole leg bones, foot length F can be measured, and leg length L can be inferred from an empirical formula like $L = 4F$ or $L = 3F$. Then we can find $r = S/L$ and, from the graph or formula, the speed v. On this basis, the Queensland biped might have been travelling at 4.5 m.p.h., and the Texas quadruped at 1.5 m.p.h.

Lesson 2.17

We began a discussion of standard paper sizes. Each child was given a piece of A4 paper, and asked to measure the long side (L) and the short side (S) in millimetres, and then work out the ratio L/S with a calculator. Next they were asked to fold it in half, and repeat the measurements of long side and short side for one of the halves, and then work out the ratio long/short for one of the halves; and then to fold again and repeat; and then again. While doing so, they were asked to compile a table as follows.

L	S	L/S
297	210	1.41
210	149	1.41
149	105	1.42
105	74	1.42

While doing this, the children noticed several things: the L for one piece of paper is the same as the S for the previous piece; the pieces are called, successively, A4, A5, A6, A7; and the quotient L/S, approximated by rounding to only two decimal places from the several offered by the calculator, is almost the same each time, varying from 1.40 to 1.42.

We then tried multiplying this quotient by itself, getting 1.41 x 1.41 = 1.988 in one case, and 1.42 x 1.42 = 2.019 in another. Both answers are approximately 2. We then embarked on a discussion of why this should be so, as follows.

The folding in half property, together with the requirement that the long/short ratio should be the same after each fold , means that

L/S = S/(1/2)L. This can be rewritten L/S = 2S/L, and then L/S x L/S = 2, so that L/S must be the square root of 2, namely 1.4142136, or 1.41 approximately.

This explains, by a *mathematical proof,* the numbers in the L/S column of the table, which we had obtained by measurement. In the process we also learnt the mathematical symbol for "approximately equal to" (a variant of = with a little bump in the upper line), which attracted some interest.

The children were acquainted with the symbol for square root, so we were building on what they knew, with an example.

I then explained a little of the history of the adoption of this root 2 ratio for a paper size. It was first proposed in 1786 by Professor Georg Lichtenberg in Germany, but then forgotten; it was proposed again in 1922 by Dr. Walter Portmann, again in Germany, and adopted as standard in that country; in 1975 it was adopted as an international standard by all countries (including the U.K.), except for the U.S.A and Canada which still use a different ratio. I showed some standard American paper, measuring 274 x 210 millimetres (instead of 297 x 210), and gave each child a piece. This does *not* have the same L/S ratio when repeatedly halved.

Lesson 2.18

As this was the last week of the course, and in view of the interest generated when we treated aspects of the time line back to geological eras in Lesson 2.16, the current lesson was devoted to the topic of "What can be found in the ground". First we drew the time line again, starting from the present, stretching back first for 65 million years, through a mountain building period (Alps, Andes, Rockies), to the extinction of the dinosaurs by climate change caused by the impact of an asteroid in the Gulf of Mexico; then to the Jurassic period between 130 and 160 million years ago, which coincided not only with certain particular dinosaur species but also with various smaller organisms to be found fossilised in limestone rocks; then further back, past coal measures and eventually to the pre-Cambrian period of over 500 million years ago when certain granites were formed.

Then various items "found in the ground" were actually shown to the children, with the remark that it was also still possible for *them* to find things in the ground. First we saw a golden sovereign of 1911, found by Mrs Sewell in her vegetable garden on 17th April 2003. "How much is it worth? Why don't you sell it?" " Because it is more interesting to keep it". So that is about one hundred years old. Then we saw a George III penny of 1806, found by Professor Sewell in his home garden in Grantham in about 1950. That is about two hundred years old. For further interest a halfpenny of the same year was shown (this was cleaner and clearer because it had been in general circulation, not in the ground), and a very heavy copper "cartwheel" twopence of 1797, which was viewed with amazement. Georgie volunteered the fact that her father, a landscape gardener, had found a coin from 1786. Next we examined a flint arrowhead, which was evidently a Stone Age tool, and therefore of the order of 5000 years old or more, which must also have been found in the ground.

The time scale was then shifted from the hundreds and thousands to 130 million years ago, by exhibiting some fossils embedded in honey-coloured limestone from a Lincolnshire quarry, and another fossil of a seashell in grey limestone. We examined two other fossils, perhaps older than the others. One was a seashell in quite hard rock, and another appeared to be the imprint of a fern like bracken. Finally we saw a piece of Lewisian gneiss, picked up from the seashore of the Outer Hebrides three weeks ago, which lacks any fossil but does date from over 500 million years ago. We checked its location in Lewis on the famous geological map of Great Britain.

The children seemed to find this all very interesting, and it was a suitably relaxing way to finish the course, which had been primarily about mathematics. This lesson replaced what had been originally planned, which was as follows.

The original plan for Lesson 2.18 was to continue the theme of the standard paper shape property that *L/S squared = 2* as follows. The result is a quite novel one, as far as I know, and it offers a simple variant of what we did on Zeno's paradox in Lesson 2.15.

Take an A4 page, and halve it by drawing a horizontal line across the middle, parallel to the shorter side. Each half is A5. Halve the bottom half, by drawing a vertical line down the middle. This creates two A6 shapes. [The reader will need to *do* this, to confirm what is happening.] Halve the right hand one by drawing a horizontal line across its middle. This creates two A7 shapes. Half the bottom one by drawing a vertical line down its middle. This creates two A8 shapes. Halve the right hand one by drawing a horizontal line across its middle. This creates two A9 shapes. And so on and so on – that is, continue the process until the rectangles are too small to work with.

Now draw the diagonal of the A4 page from the top left corner to the bottom right corner. This will create an infinite sequence of triangles, each occupying one quarter of the A5, A7, A9, A11, A13,... rectangles.

We can prove, as follows, that the sum of the areas of this infinite number of triangles is 1/6 of the area of the starting A4 page.

The sum of the areas, in terms of fractions of the whole A4 page, is

$\frac{1}{4} \times \frac{1}{2}$ of it + $\frac{1}{4} \times 1/8 + \frac{1}{4} \times 1/32 + \frac{1}{4} \times 1/128 + \ldots\ldots$

$= 1/8 \times (1 + \frac{1}{4} + 1/16 + 1/64 + \ldots)$. As a shorthand write $T = 1 + \frac{1}{4} + 1/16 + 1/64 + \ldots$

Then we see that $\frac{1}{4} \times T = 1/4 + 1/16 + 1/64 + ... = T - 1$.

This proves that $3T/4 = 1$ and $T = 4/3$.

Therefore the sum of the areas is $1/8 \times T = 1/6$ of the area of the whole page.

This is exactly the same type of calculation as resolved Zeno's paradox, in the sense that we are adding an *infinite* number of decreasing terms, but the terms are decreasing fast enough to make the *answer* finite.

Lesson 3.1

We began with a *Modest Test*.

This was a list of questions derived from a recent newspaper (Independent) article, which quoted a DfES survey that 15 million adults would not get a low-grade D to G pass in GCSE Mathematics (so nearly half the UK adult population would fail a GCSE maths exam), and 6.8 million "did not even have the basic skills of the average 11-year-old and would fail their primary school national curriculum tasks". We tried some questions quoted in the newspaper.

From the list 14, 15, 16, 17, 18, 19, 20, (a) write down an odd number; (b) write down two numbers that add to 32, and explain your method of thinking; (c) pick out a square number, and explain its meaning in terms of area; (d) say which numbers have 6 as a factor.

The children had little difficulty with these. I mentioned the newspaper article.

Next I developed a *Surprise*, as follows.

Each child, working alone, was asked to write down answers to the following steps.

(a) Write down a three-figure number. Examples: 124 782 539 691

(b) Check that the difference between the first and last digits is 2 or more. Adjust the choice to achieve this if need be. I explained what "digit" means.

(c) Reverse these numbers: 421 287 935 196

(d) Subtract the smaller from the larger in each pair: 297 495 396 495

(Next week we shall notice that these are all multiples of 99, and explain why).

(e) Now reverse the numbers in (d): 792 594 693 594

(f) Add each pair of numbers in (d) and (e): 1089 1089 1089 1089.

The *Surprise* is therefore that this procedure always delivers 1089 for *any* starting three-digit number whose first and last digits differ by 2 or more. Several of the children obtained this result for their own particular starting number, and were intrigued. Some had made arithmetical slips along the way, which had to be found and corrected.

Lesson 3.2

We began by asking the children to summarise what we had done last week, and noted an alternative format for the calculation, namely the vertical version of

$$762 - 267 = 495, \qquad 495 + 594 = 1089.$$

Then I suggested a game that they could play on their friends and/or parents later. This was, firstly, to tell them that a particular but secret number (1089) had been written on a hidden piece of paper.

The friend or parent is then invited to think of any three-digit number whose first and last digits differ by 2 or more, reverse it, subtract the smaller from the larger of those two numbers; then reverse that answer and add it to that answer. Then the friend/parent is told that the final result is the secret number (1089), no matter what three-digit number they had chosen to start with.

This game was greeted with enthusiastic approval. At the end of the lesson the children began excitedly to make a pact that no-one else must be told about 1089 and the game that they were going to play, because that would spoil the fun.

I then began a discussion along the lines of how remarkable this result is, that I did not know it myself until three weeks ago, and that we ought to try to find a way of understanding why it worked. This is what a mathematician would want to know. Is there a difference between a mathematician (who would explain such a process) and a magician (who might not)?

So how can we approach the task of understanding what is going on? First we notice at the halfway stage something remarkable has happened already, namely that, for example,

$495 = 5 \times 99$. Other examples led to $297 = 3 \times 99$ and $396 = 4 \times 99$.

Every case, at this stage, will have 99 as one factor. We constructed an argument to *prove* this, as follows. Any three-digit number has the structure of

ABC = A x 100 + B x 10 + C x 1.

This caused some puzzlement, and discussion was required of the facts that A is the number of hundreds, B is the number of tens, and C is the number of units, and that we are using the letters A, B, C to represent numbers that we do not yet want to specify. This means that what we say about them can, for the time being, apply to *any* numbers, so that we have *generality*. Later on the children realised that we are actually doing something called *Algebra* here, which they had done before in another context, although we did not at first call it Algebra until the end of the lesson. There was very good participation by the children in this lesson.

If we now reverse ABC we get

CBA = C x 100 + B x 10 + C x 1. Then subtracting we get

ABC - CBA = (A – C) x 100 + (B – B) x 10 + (C – A) x 1

= (A – C) x 100 - (A – C) x 1 = (A – C) x 99.

We are trying to encourage the idea of *proving* something here. This reasoning does *prove* that we will always get a multiple of 99 at this stage of the calculation. So this fact ought to be significant. Another example is

691 – 196 = (6 – 1) x 99 = 495.

We will pursue this to the second stage, i.e. of reversing and adding to get 1089, next week.

Near the end Beth pointed out that the *sum of the digits* in *each* first stage number (like 495) is 18 (= 4 + 9 + 5). And also in 1089 = 1 + 0 + 8 + 9 = 18. This is remarkable.

Lesson 3.3

We started with some revision for the benefit of a new pupil..

We worked on the board, with children performing the steps, the vertical version of

782 – 287 = 495, then 495 + 594 = 1089.

This is an example of starting with *any* 3-digit number whose first and last digits differ by two or more, reversing and subtracting; then reversing the answer and adding. The answer is *always* 1089. This is a *very great surprise*, and our task is to understand *why* it is so. There are several associated points of interest which are well worth an explanation accessible to this age group.

Several children reported that they had tried examples of this calculation out on their parents (often mothers), who had been duly impressed (and baffled). I had suggested that they try it out on some grown-ups.

We notice that at the halfway stage we reach a multiple of 99 (495 = 99 x 5), and we next looked at the first ten multiples of 99, as follows.

1 x 99 = 099, 2 x 99 = 198, 3 x 99 = 297, 4 x 99 = 396, 5 x 99 = 495,

6 x 99 = 594, 7 x 99 = 693, 8 x 99 = 792, 9 x 99 = 891, 10 x 99 = 990.

If we reverse *any* multiple of 99, and add to its starter, we *always* get 1089.

We did this explicitly for all these, and showed it to be the case for every one. Notice that in the first one we need to write 099 rather than just 99, so that we are starting with a 3-digit number.

This was by way of revision, because it had been two months since we met. In Lesson 3.2 we had given a *general proof* of why a multiple of 99 is reached at the halfway stage.

Lesson 3.4

We set out to finish the explanation of some various features which we noticed during our discussion of the surprising emergence of 1089 every time from this procedure, i.e. from every starting number. The children each wrote down their own choice of a multiple of 99, and we examined the structure of those numbers on the board, making the following points.

Questions

1. What is the proof that, if we add any 3-digit multiple of 99 to its reverse, we get 1089?

   Observation of the list of 3-digit multiples above shows that they all have the structure, for *any* number N in the list N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

   N x 99 = [N – 1] 9 [10 – N]. Example: 8 x 99 = 792 = [8 – 1] 9 [10 – 8].

   Here [N – 1] is in the hundreds column, 9 is in the tens column, and [10 – N] is in the units column, so the abbreviation *means*

   [N – 1] x 100 + 9 x 10 + [10 – N] x 1.

Reversing the digits gives

$$[10 - N]\ 9\ [N - 1].$$

Adding the two by the usual rules for vertical addition gives $10 - N + N - 1 = 9$ in the units column, $9 + 9 = 18$ so 8 in the tens column, and carrying 1, we have $1 + N - 1 + 10 - N = 10$ in the hundreds column, so 1089 in all, for *any* N.

This is the **proof** of the main "surprise" with which we started out, because we have already seen that *some* multiple of 99 always appears at the halfway stage, after the reversal and subtraction.

2. What is the proof that the sum of the digits after the first stage, i.e. of any 3-digit multiple of 99, is always 18? [This was Beth's observation in Lesson 3.2].

   We have shown in 1. above that any such 3-digit number has the structure

$$[N - 1]\ 9\ [10 - N] \qquad \text{for any} \quad N = 1,2,3,4,5,6,7,8,9,10.$$

So adding the digits gives

$$N - 1 + 9 + 10 - N = 18.$$

A mathematician might call this result **Beth's Theorem** (named after the discoverer). She was duly pleased.

3. We also notice that 1089 itself has the properties

$$1089 = 1100 - 11 = 100 \times 11 - 1 \times 11 = (100 - 1) \times 11 = 99 \times 11$$

i.e. it is *also* a multiple of 99.

Next we started a new topic.

Think of a number

I wrote a number (2) on a piece of paper, but kept it hidden from the class for the time being. Then I asked them to carry out the following steps.

1. Think of a number (ANY number you like). E.g. 7, just for illustration here.
2. Add 1. $7 + 1 = 8$.
3. Double the result. $2 \times 8 = 16$.
4. Take away 3. $16 - 3 = 13$.
5. Add the number you first thought of. $13 + 7 = 20$.
6. Add 7. $20 + 7 = 27$.
7. Divide by 3. $27/3 = 9$.
8. Take away the number you first thought of. $9 - 7 = 2$.
9. *Everybody* should get this same answer, *whatever* starting number they chose. 2 was the number which I had written down at the beginning, but kept hidden. Now I showed it.

But everybody did not get this same answer. Four people did. Two people did not because they made simple mistakes with their arithmetic along the way. One person tied himself in knots because he insisted on starting with a huge number having about nine digits (something like 597403128), which did not provide convenience when it had to be manipulated in the required ways. I did not seek to ban it. I thought that it would be instructive for him to be seen to have to lie on the bed which he had made for himself. The others noticed what was going on, and there was amusement all round, and lessons learnt from the evident self-inflicted discomfort.

I shall give an explanation, using algebra, of why 2 emerges from any starting number next week. Then the children will do some DIY Algebra to construct their own personal example of the same kind of thing (i.e. a predictable answer from an arbitrary starting point. Why does it work?).

Lesson 3.5

We ran through the same steps used last week for Think of a Number. Then we sought an explanation of why, with any starting number, we always ended up with 2.

We did this by introducing the language of Algebra, which the children were quite happy with, and which in this case involved the use of n as the label for "the number you first thought of". So we listed the previous steps as follows.

1. Think of any number. Write it as n.

2. Add 1. $n + 1$.

3. Double it. $2(n + 1) = 2n + 2$. We agreed that $2n$ meant $2 \times n$, etc.

4. Take away 3. $2n + 2 - 3 = 2n - 1$.

5. Add the number you first thought of. $2n - 1 + n = 3n - 1$.

6. Add 7. $3n - 1 + 7 = 3n + 6$.

7. Divide by 3. $(3n + 6)/3 = n + 2$.

8. Take away the number you first thought of. $n + 2 - n = 2$.

This process shows explicitly why the answer is the same (namely 2) whatever starting number n is used, because the algebra puts n in at first and then later takes it out again. We then revealed that the single formula

$$[2(n + 1) - 3 + n + 7]/3 - n = 2$$

neatly summarizes all the steps.

We then introduced the idea of DIY Algebra. That is, the children were asked to devise their own formula, whatever they liked, as long as it had the feature just illustrated of putting n in and later taking it out again. A simple example would be $n + 1 - n = 1$. We looked at the more complicated example

$$[3(n + 4) - 10 - n]/2 - n = 1$$

for which we wrote out the individual steps on the board. I suggested that the children could try out this game on their friends. A rather complicated formula was devised by one boy, but it worked accurately, showing that he in particular had understood this particular type of DIY Algebra very clearly.

Lesson 3.6

We began a new topic, to find the sum of the first n numbers. This was introduced with

$1 + 2 + 3 = 6$, which everyone did quickly, in their heads, of course. Then

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, which took longer. Then I proposed

$1 + 2 + \ldots\ldots + 98 + 99 = ?$, which it was said would take until tomorrow.

It certainly looks laborious if we go straight at it, so is there an easier way? That is what a *mathematician* would look for. That type of mathematics is sometimes called "the higher laziness, namely some hard work in search of an easy way". Here we have a good example, because we can find an easier way by *doubling the size of the problem*, as follows.

Write $1+2+3+4+5+6+7+8+9 = S$, where S stands for sum. Now
rewrite $9+8+7+6+5+4+3+2+1 = S$. Add the two sums vertically gives
$10+10+10+10+10+10+10+10+10 = 2S$. The adding is now much easier,
giving $9 \times 10 = 2S$ so that $S = 45$.

We indicated the power of this method with other specific examples, such as

$1+2+3+\ldots\ldots+54+55 = S$, giving $55 \times 56 = 2S$, so that $S = 55 \times 28 = 1540$, and

$1+2+\ldots\ldots+98+99 = S$, giving $99 \times 100 = 2S$ and therefore $S = 4950$.

There was evidently some uncertainty working with the arithmetic of larger numbers.

Finally we moved to the general case of allowing the final number in the series to be anything, which we called n, so the problem is now to find

$S = 1 + 2 + \ldots\ldots + (n-1) + n$. This can be written backwards as
$S = n + (n-1) + \ldots\ldots + 2 + 1$. Adding the two lines term by term gives

$2S = (n+1) + (n+1) + \ldots\ldots (n+1) + (n+1)$, and there are n terms in this sum, so that
$2S = n(n+1)$ and therefore

$S = n(n+1)/2$. This is the sum of the first n numbers.

The laborious adding which seemed to be required, for large n, is replaced by the simple multiplication in the formula.

It had to be explained carefully that $(n-1)$ was the number before n.
Some of the children found it less than easy to get their heads round the idea of having *two* variables (letters) in an equation, as with S and n here, and we shall need to do some consolidation exercises next week. We shall also give a geometrical illustration, using areas.

Lesson 3.7

It seemed worth diverting briefly, to treat two topics which turned up a few days before.

The first topic was the description of the names for decreasing units by factors of 1000 at each step. They can be applied to, for example, seconds, or grams, or metres, or other things. We discussed explicitly

1 milligram = 1/1000 gram.

1 microgram = 1/1000 milligram = 1/1000 x 1/1000 gram = 1/1000000 gram.

We showed a packet of medicines which stated that each tablet had 125 micrograms of the active ingredient.

1 nanogram = 1/1000 mgm = 1/1000 x 1/1000000gm = 1/1000000000gm.

1 picogram = 1/1000 nanogm = 1/1000000000000gm.

We also introduced the notation of powers of 10 at this point,
e.g. 10 to the power 3 and 10 to the power 12, to simplify the notation.
The children quickly got used to, or already knew, the idea of adding such indices in multiplication. Then we moved on to

1 femtogm = 1/1000 picogm = 1/1000000000000000 gm.

1 attogm = 1/1000 femtogm = 1/1.....18 zeros...gm.

1 zeptogm = 1/1000 attogm = 1/1......21 zeros...gm.

1 yoctogm = 1/1000 zeptogm = 1/1...24 zeros...gm.

We agreed that the last two would be so small that they would be rarely used. But the discussion showed that the children were comfortable with the idea of adding indices as a description of increasing powers of 10.

The second topic arose from noticing that last Sunday was the fifth Sunday in February. How frequently are there 5 Sundays (or any other particular day of the week) in February? Obviously it has to be a leap year. One boy immediately said that it would happen every 28 years. This was very quick. So we discussed the reasoning arising from the two facts that there are 7 days in a week, and a leap year occurs one year in every 4 (except for a century year), so the result follows from the fact that 7 x 4 = 28.

Lesson 3.8

Our discussion of very small numbers, and of the powers of 10, from last week made it seem timely to introduce some ideas and notations about *index shorthand* in mathematics.

We had used notations like 10^3 x 10^3 = 10^(3 + 3) = 10^6 to describe how "10 to the power 3" multiplies with itself to give "10 to the power 6"; except that here, because Word does not have the facility to write raised numbers, I use ^ to signify that the number following ^ is raised to act as a power. We notice that the numbers which designate the powers are *added.*

More examples are 10 x 10 = 10^2, called "10 squared" because the *area* of a 10 x 10 square is 10^2 = 100 units.

Likewise 10 x 10 x 10 = 10^3 is called "10 cubed" because the *volume* of a 10 x 10 x 10 cube is 10^3 = 1000 units.

Generally, and for higher powers, we say things like "2 *to the power* 5" is 2 x 2 x 2 x 2 x 2 = 2^5 = 32.

The children were happy to accept all this.

Moving on to a combination of powers, we explained that the *definition* 2^1 = 2, 5 = 5^1, 17 = 17^1, etc., is consistent with things like

2 x 2 x 2 = 2^1 x 2^1 x 2^1 = 2^(1 + 1 + 1) = 2^3, etc.

This is another example of the rule that we *add the powers* when *multiplying* a number by itself.

Next we explained that *when dividing we subtract the powers*, as in

100 x 1/10 = 10^2 x 1/10^1 = 10^(2 – 1) = 10 ^1 = 10 and in

9 x 1/3 = 3^2 x 1/3^1 = 3^(2 – 1) = 3^1 = 3.

So fractions work like negative powers, as in 1/10 = 10^(-1) and 1/3 = 1/3^1 = 3^(-1).

There was dialogue, and question and answer, continuing throughout this development.

Next we introduce the special definition that *any* number *to the power zero* is 1. Illustrations are 5^0 = 1, 13^0 = 1, 29^0 = 1. This was readily accepted after examples like

1 = 49 x 1/49 = 7^2 x 1/7^2 = 7^(2 – 2) = 7^0 and

3^2 x 1/3^2 = 9 x 1/9 = 1 = 3^2 x 3^(-2) = 3^(2 – 2) = 3^0.

Finally we summarised with three simple statements.

Positive powers are on the top (numerator), as in 5^2 = 25.

Negative powers are on the bottom (denominator), as in 5^(-2) = 1/5^2 = 1/25.

Zero powers are *all* 1, as in 5^0 = 1, e.g.

1 = 25 x 1/25 = 5^2 x 1/5^2 = 5^2 x 5^(-2) = 5^(2 – 2) = 5^0.

The children seemed very happy with all this, much of which they had not seen before. They responded well during the continuous question and answer style which was used.

Lesson 3.9

Summary of index shorthand.

The rules for combining powers, which we explored last week using various representative numerical examples, can all be summarized in a *single formula* if we are prepared to use *algebra.* It seemed appropriate to take the opportunity to describe this.

First we reminded ourselves of some typical examples, as follows.

100 x 10000 = 10^2 x 10^4 = 10^6 = 1000000 uses 2 + 4 = 6.

9 x 1/3 = 3^2 x 1/(3^1) = 3^(2 – 1) = 3^1 = 3.

7^0 = 1.

Now we introduce some algebra. Suppose

n is *any* number. The children were invited to suggest choices for this, and x,y and others were considered. Any choice is possible, but I eventually said that n is common. Also suppose

a is any power (it could be +, 0, or -) and

b is also any power (we could have either a = b or a *not* = b).

Then the single formula

$n^a \times n^b = n^{(a+b)}$ sums up *everything* about this part of the subject.

Some of the children were plainly a little puzzled by this leap to an abstraction, and some more consolidating examples were debated to offer familiarity with the idea, as follows.

$n = 10$, $a = 2$, $b = 4$ gives $10^2 \times 10^4 = 100 \times 10000 = 1000000 = 10^6$ with $a + b = 6$.

$n = 5$, $a = 4$, $b = -2$ gives $5^4 \times 5^{(-2)} = 625 \times (1/25) = 25 = 5^{(4-2)} = 5^2$.

$n = 7$, $a = 2$, $b = -2$ gives $7^2 \times 7^{(-2)} = 49 \times (1/49) = 1 = 7^{(2-2)} = 7^0$.

Leaving that subject now, we return to a topic discussed in Lesson 3.6, namely the formula $S = (1/2)n(n + 1)$ for the sum of the first n numbers. There is a geometrical way of illustrating the derivation of this which increases the understanding of it.

The children were asked to draw a pile of squares which has 5 on the bottom row, 4 on the next, then 3, then 2, then 1, but with all the left hand edges aligned. The total area is

$A = 1 + 2 + 3 + 4 + 5 = 15$. Then they were asked to draw the same pile, but upside down and so that it fitted onto the first to make a 5×6 rectangle whose area will be $30 = 2A$. From the rectangle we can easily find this area, which then implies $A = 15$ for the sum of the first 5 numbers.

This is a geometrical illustration of the trick used in Lesson 3.6 of writing the series $A = 5 + 4 + 3 + 2 + 1$ backwards, and then adding to the forwards version so that

$2A = 6 + 6 + 6 + 6 + 6 = 5 \times 6$ and therefore $A = 15$. The trick generalises to n terms to prove that

$A = (1/2)n(n + 1)$, but now we have a geometrical view of it too.

Lesson 3.10

We tackled the problem of finding the minimum number of straight fences that are needed to separate several bulls in a field.

We assume that each fence goes to the boundary of the field. Alternatively, and more simply, it is enough to suppose that the field is infinite, and that each fence has infinite length. We discussed these idealisations.

We began by looking at the first few special cases. The children were asked to draw the pictures needed to illustrate the following conclusions which they reached in discussion.

2 bulls need 1 fence. 3 bulls need 2 fences. 4 bulls need 2 fences.

By drawing the pictures for up to 8 bulls, it was seen to be worth tabulating the conclusions so far.

bulls	2	3	4	5	6	7	8
fences	1	2	2	3	3	3	4

By now it was clear that it was not the best idea to draw parallel fences, or to draw fences such that three or more passed through a single point. These cases are too special.

At this stage we also said that it may be better to turn the problem on its head by asking a slightly different question, as follows.

For a given number of *fences*, what is the **maximum** number of *bulls* they can enclose?

We soon saw that the resulting table for this is as follows.

fences	1	2	3	4	5
max. bulls	2	4	7	11	16

The question is *really* about how the *space* on the page can be divided up by *lines*.

After some discussion we noticed that the pattern appearing in the table is that

when the number of fences is increased from n to n + 1

this increases the maximum number of bulls by n + 1, e.g. 11 = 4 + 7.

This observation shows how our systematic approach has informed the problem, because we can now continue the table without having to draw the diagrams of the fences, which would be very elaborate when many fences are involved. So the continuation is

fences	4	5	6	7	8
max. bulls	11	16	22	29	37

There turns out to be a formula that, for the case of n fences, tells us that the maximum number of bulls which can be enclosed is $n(n + 1)/2 + 1$. (We did not have time to actually deduce this fact.)

Checking this with, say, 4 fences, we found $4(4 + 1)/2 + 1 = 11$.

We discussed whether we had seen this sort of formula previously. After a good deal of dredging around in their minds, the children did recognise it.

I repeated the derivation, which we did in Lesson 3.6, of the fact that the sum of the first n numbers is $n(n + 1)/2$, and this was recognised.

Lesson 3.11

We began a new topic, called **Basis and Bases of Arithmetic**. This phrase encapsulates the fact that the basis of arithmetic can appear in various forms, depending on which base one chooses to work in.

We first recalled that our *usual* arithmetic uses *ten* different symbols, 0 1 2 3 4 5 6 7 8 9, no more and no less, to write *all* numbers. It is called the *denary* system (Latin for ten), or *base 10*. The method is to imagine a *row of slots*, and attach a value to each slot as follows.

The examples 10, 324, 59867 mean the following.

10 thousands	thousands	hundreds	tens	units
			1	0
		3	2	4
5	9	8	6	7

We already know about *powers*, so we could rewrite the column heads as

10^4 = 10000, 10^3 = 1000, 10^2 = 100, 10^1 = 10, 10^0 = 1.

So, for example, the meaning of numbers in base 10 is 324 = 3x10^2 + 2x10^1 + 4x10^0.

Addition by columns uses *carrying* from one column to the next.

But the usual arithmetic is not the only possible type.

Computer arithmetic uses only *two* different symbols, 0 and 1, to write *all* numbers. This is called the *binary* system (from the Latin for two), or *base two*.

It is useful in computing because the presence of only two options can represent *on/off* or *yes/no* repeatedly.

It seemed that none of the children had heard of the binary system.

The method again is to imagine a row of slots, and attach a value to each slot, but differently from denary. An example is 1,2,3,4,5,6,7,8 written in binary as

eights 8 = 2^3	fours 4 = 2^2	twos 2 = 2^1	units 1 = 2^0
			1
		1	0
		1	1
	1	0	0
	1	0	1
	1	1	0
	1	1	1

1	0	0	0

The children began to get the idea of this, and we tried a simple vertical addition sum using binary, as follows, and compared it with the same sum in denary.

2^3	2^2	2^1	2^0			10^ 2	10^ 1	10^ 0		
		1	1					3		
	1	0	1	+				5	+	
1	0	0	0					8		

Next week we shall try some more simple examples.

Lesson 3.12

First we recalled the main points of what had been discussed last week, namely the denary and binary base number systems in arithmetic.

The children were then asked, round the room, to exhibit various (small) numbers in binary on their (eight) fingers. This was done by showing the back of the hand to the audience, and holding up a finger to signify 1, and down to signify 0. The four fingers in turn, from right to left on the right hand, signified 2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, and on the left hand 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128.

The children were next asked to do some more simple addition sums in binary on paper.

Next I widened the discussion by pointing out that although the most convenient bases were 10 and 2, any other base is possible. Another example is base 3 (ternary), which would require the use of 0,1,2 only, with place values like 3^0, 3^1, 3^2, 3^3, 3^4 =27, etc. I said that although this system was possible, in fact, it was never used. This provoked one boy to say: "So this might be a rude question, but why are you telling us about it then?" To which the answer was that that it was not rude, but a very fair question, and the answer was that I was trying to widen the scope of their thinking.

As another example, I mentioned that base 16 (hexadecimal system), which might seem useless, was actually used in some types of computer.

Next I asked if anybody had heard of the base 60 system. Only one was able to tell us that it had been used by the Babylonians over 2000 years ago. I pointed out that we use it today in our time measuring system. That is, we have 60^0 = 1 second, then 60^1 = 60 seconds = 1 minute, and 60^2 = 3600 seconds = 1 hour. Therefore 60 minutes = 1 hour.

Then we talked about angles, and I told the class that the base 60 system is also used in that context, as follows. They were quite unfamiliar with most of the following. A full circle has 360 = 6 x 60 degrees. Subdivisions include the fact that 1 degree = 60 minutes of arc (written 60'), and 1 minute = 60 seconds of arc (written 60''). So

a full circle has 360 x 60 = 21600 minutes and 360 x 60 x 60 = 1296000 seconds of arc.

These words do not have the same meaning as the above time measures.

I left the children with the suggestion that they might care to find out something about the Babylonians on the internet. For example, why did they use base 60?

Lesson 3.13

First we recalled the various number bases which we recently discussed The children were able to list, with comments about their uses, bases 10, 2, 3, 16 and 60.

In relation to base 60, last week we had finished with a discussion of angles and the facts that a full circle has 360 degrees, that 1 degree = 60 minutes of arc, and that 1 minute of arc = 60 seconds of arc.

Next we developed several points related to *nautical miles*.

1. The Earth is a sphere (nearly), having polar diameter = 7900 land miles, and
   equatorial diameter = 7926 land miles.

2. So, although flattened at the Poles in that way, it is very close to a sphere, and we next assume it to be so. We sketched a sphere, and a *great circle* on it by imagining a flat slice thorough the centre of the sphere. As a contrast I indicated that a slice at a different latitude would make a circle smaller than a great circle.

3. I then explained that *one minute of arc on the great circle defines one nautical mile.*

4. Also, it is known that 1 nautical mile = 6080 feet. I explained that sailors measure speed in *knots*. The definition is 1 knot = 1 nautical mile per hour.

5. So now we can work out that the *circumference* of the Earth, because there are 360 degrees, each of 60 minutes, *all the way round.* So the circumference is, (using fact 3 above)

   60 x 360 = 21600 minutes = 21600 nautical miles.

   Some of the children were better than others at such multiplications.

6. Next I told them that a *land mile* = 3 x 1760 feet = 5280 feet. There seemed to be rather little knowledge of facts such as 3 feet = 1 yard, and
   1760 yards = 1 land mile.

7. So how many land miles are there round the circumference? This is an exercise in ratios and in estimating with approximations. We are avoiding blind use of calculators here, to emphasise the need sometimes to estimate.

   Each nautical mile = 6080/5280 land miles, and this ratio is approximately
   600/525 = 24/21 = 8/7 land miles.

   So 21600 nautical miles = 21600 x 8/7 = 172800/7 = 24700 land miles approx.

   So this is the length of the Earth's circumference in land miles, calculated by using geometry, ratios and approximations. The children admitted to knowing none of that before, as they departed.

Lesson 3.14

We first recalled that previously we discussed bases 10, 2, 3, 16, 60 and their uses.

Now we began a discussion of *Roman numerals*. The children were able to name most, but not all, of the following table with denary equivalents.

Roman	I	V	X	L	C	D	M
denary	1	5	10	50	100	500	1000

They did not know D, and tried to guess others, like B = billion. We had a discussion of how to make other numbers from these, by *juxtaposition*, with smaller numbers

before, meaning a subtraction, e.g. IV = 4, XL = 40, or
after, meaning an addition, e.g. XV = 15, MDL = 1550, or
mixed, meaning both, e.g. MCM = 1900.

I set the children to try several more examples, such as 19 = XIX, 18 = XVIII, 49 = XXXXIX = IL, 71 = LXXI, 79 = LXXIX, 95 = VC = LXXXXV, 362 = CCCLXII. They displayed some inventiveness. It was not hard to get them to agree that the Roman system was and is

very cumbersome, has no evident base, and *very awkwardly* has no symbol for zero.

Next we moved to a discussion of *money measures*.

I explained to them that in 1971 the British money system was changed to so-called *decimal coinage*, with

100 new pence = £1. This system looks easy because we can just use base 10 and the decimal point, as in £17.54 = 17 pounds and 54 new pence.

It replaced *Imperial money,* which had

12 (old) pence = 1 shilling, 20 shillings = £1.

I showed them some Imperial coins, such as pennies carrying the heads of Edward VII (1907), George V (1918), George VI (1945), Elizabeth II (1963). A child had found one such in a garden. I also showed a halfpenny (1945), sixpence (1957), shilling (1956) and florin (1950). I explained how, with this older system, people would normally have in their pockets some quite old coins, even 50 or 60 years old, unlike today.

From the mathematical viewpoint I explained that the Imperial system had more numerical *sub*divisions, such as 12 = 2 x 6 = 3 x 4 and 20 = 2 x 10 = 4 x 5 than the new decimal money system.

Therefore it was more flexible (and price increasing was less easy?).

To illustrate the mixture of 20 and 12 place values in columns, I posed the addition sum

£	s	d	
7	3	4	
9	18	9	+ which most pupils correctly solved to get
17	2	1	

We compared the process with the addition of metric money, as in £5.73
£4.28 +

which gives £10.01

Lesson 3.15

This was a lesson given to the whole class of about 30 children from Years 5 and 6, instead of the usual selected group of 8 children.

The topic of hexagons was introduced, and the children were asked to draw what they thought to be a hexagon. All drew a six-sided figure with straight sides. Most were convex, as might be expected. We also had a discussion about pentagons and quadrilaterals. A few quadrilaterals were concave at one corner. None of the sides crossed over each other.

It was explained that this was to be an exploratory session, and that it was normal for mathematicians to make mistakes in the process of refining their ideas, and also to be unconventional, so no one should worry about not getting every thing right the first time.

It was pointed out that when the sides of polygons did not cross each other, the polygon could be called "open". So everyone had drawn open hexagons. No corners were missed when drawing the sides in sequence (so it could be called a 000000 hexagon). I then demonstrated on the board that six points around a ring could also be joined by allowing some sides to cross each other. For example, if the third and sixth sides crossed, we should have a hexagon with just one self-intersection, which looked a bit like "scissors", and could be given that name. This was demonstrated on the board (every third side missed 2 corners, so 002002 describes that). Another one with seven self-intersections of the six sides, which we called a "star", was drawn by the sequence 121121 of by-passing corners.

In summary thus far, mathematicians could be adventurous in drawing unconventional hexagons.

Finally we approached a famous result, first discovered in 300 A.D. The children drew two straight lines, at a small angle to each other, but not intersecting. Then they marked three points along each line. Then they drew the star-shaped hexagon which joins the points. Of the seven self-intersections, they were asked to say what they noticed about the middle three, by putting a ruler on them. Very gradually the hands went up to say, eventually, that those three points were all in line. It was explained that this result is called Pappus' Theorem, after the ancient Greek who discovered it.

Lesson 3.16

We began by introducing the famous topic of the Golden Ratio, which was thought by the Greeks to be very satisfying.

A rectangle with long side of unit length and (at first unknown) short side s, say, is divided into a square of sides s x s, and a smaller rectangle of sides s x (1 – s). We require that the ratio short/long is the same for the starting rectangle and the new one, so that

s/1 = (1 – s)/s. This is the definition of the Golden Section.

The 1 on the left can be omitted, and the equation can be multiplied by s to become the simpler form

s x s + s - 1 = 0. The children were told that this has the solution

s = (square root of 5)/2 – ½. With their calculators they found that the

square root of 5 is 2.236 approximately, and thence that

s = 0.618 . They were then asked to check that this made the left side of the equation zero (approximately because of rounding errors), thus

0.618 x 0.618 + 0.618 - 1 = 0.000076.

So the Golden Section rectangle has sides 1 x 0.618.

The diagram can be continued by dividing the successively smaller rectangles, each in the Golden Ratio.

Next we returned to the hexagons of last week, to devise a method of describing how to draw them. We look on the left of each side as it is drawn to see how many vertices are omitted, and write these six numbers down in sequence.

For the open hexagon we get, of course, 000000. But for the scissors (having 1 self-intersection) we find 002002, and for the star (having 7 self-intersections) 121121. The children drew other examples, which had other sequences which they listed, to illustrate several different types of hexagons, also having different numbers of self-intersections.

Lesson 4.1

We removed the picture cards from two packs of playing cards, to leave eight sets of the numbers 1 to 10. The group of pupils was divided into four pairs, and each pair had two sets of cards numbered 1 to 10. Colours and suits were irrelevant for our purpose. One member of each pair shuffled the 20 cards. The other member took 2 at random, multiplied the numbers so selected, wrote down that product, and noted whether it was even (E) or odd (O). This was repeated 20 or 30 times, and the results entered in a table:

X	9	9	12	12	36	27	70	16	2	20
E / O	O	O	E	E	E	O	E	E	E	E

We then had a relaxed discussion of what each group had found, and whether the results were what might have been expected. Was there a pattern to be observed, or expected? Eventually it was realized that there were more evens than odds, and some began to suspect that this was to be expected. Different groups found that there were three, four, or five times as many evens as odds.

The question was now posed: can we *prove* what will happen? We agreed that two times any number would be an even number, that odd numbers were never twice a number, that an even number times an odd number would always be even, and that an odd number times an odd number would be odd. This was understood, and we then proved as follows that if the experiment above were continued long enough, we should expect 3 times as many even products as odd ones. The following multiplication table was readily constructed and agreed.

X	even	odd
even	E	E
odd	E	O

It was accepted that this verbal argument counted as a proof. But next week we need to revisit this question of what is a proof, in various ways.

Lesson 4.2

We began by asking the children to recap by describing briefly what we did last week.

Next we got out the cards again, and carried through the same shuffling and selecting procedures, but this time the children were asked to add, instead of multiply, the two numbers drawn from the pack, and record in a table whether the results were odd or even. An example would be

Sum	+	16	13	18	15	11	5	8	8
E or O		E	O	E	O	O	O	E	E

This test was carried out 20 or 30 times, and the children were asked it they could identify a pattern in the results. The four pairs of children obtained quite different patterns, certainly not just the equal numbers of odd and even results shown in the above example.

Progress was rather slow, and before discussing what the results "should" be on some theoretical basis, the children were asked to carry out another series of tests and record the differences of the two numbers in each pair, instead of the sums. No unambiguous pattern emerged from these tests, and next week we must discuss what should be expected, and why.

One of the differences which emerged was –12, and this produced a long and valuable discussion when I asked whether this is an even or an odd number. One pupil clearly understood that it was an even number, but it required a very long discussion before she was able to say why this was so, namely because it is twice another number, namely –6. This was a valuable dialogue for the other pupils to observe.

Lesson 4.3

We reviewed what was done last week. Then, without using the cards this time, we went round the class asking pupils to pick several pairs of numbers, to add, then subtract, and then divide each pair, and record these results. We then discussed whether the results were even or odd or neither, and whether these conclusions were what could be expected in advance. A lot of detailed individual discussion proved to be desirable, because in some cases certain children were quite uncertain about predicting the outcome. By discussion on the board we eventually agreed on the following tables which summarised these outcomes.

+	E	O		-	E	O		/	E	O
E	E	O		E	E	O		E	?	?
O	O	E		O	O	E		O	?	?

Lesson 4.4

First the children were asked to summarize what was done last week.

Next we began a discussion of what are the names of the more familiar parts of mathematics. The children began by offering their own suggestions. These were

maths, numeracy, mental arithmetic, and mental maths.

I had the objective of using algebra in the next lesson or two to prove some results about odd and even numbers, and I offered the following suggestions. The children were invited to make some notes from the board. The most familiar parts of mathematics have the following names.

Arithmetic. This is the study of numbers and the relations between them, obtained by combinations such as adding, subtracting, multiplying and dividing.

Algebra. This is a method of talking about numbers when we don't say, in advance, which particular numbers we are dealing with. That is either because we don't know, or because we don't want to say so that the statements can apply to a lot of different numbers. Algebra was invented in about 1100 A.D. by an arab called Al Habri. If you say the name quickly, you will see why the subject is called Algebra.

Geometry. This is the study of points, lines, and curves in two dimensions, and surfaces as well in when we are in three dimensions. The name means, taken literally, Earth measuring. We digressed to discuss the related names of Geography (Earth description) and Geology (the study of rocks).

Trigonometry. Literally this means triangle measuring, and we noticed that it would be of special interest to surveyors in particular.

Lesson 4.5.

The children were first asked to recall the definitions of some branches of mathematics which were discussed last week. Then we began a discussion of some simple ideas in Algebra.

Each child was asked to choose a letter to represent an unknown number, or at least one that we did not wish to reveal yet. We went round the room to allow each child to choose a different letter from the others. Then they were each asked to write down an equation of the type

$a + 3 = 0$

and we discussed the principles of how to solve such an algebraic equation for a. Some could guess that $a = -3$, and some could not.

We described how to approach this answer, not by guesswork, but systematically. That is, we subtract 3 from both sides, to find

$a + 3 - 3 = -3$ and therefore $a = -3$ because $3 - 3 = 0$.

It was explained that handling equations was a bit like washing one's hands, in the sense that one must always do the same thing to both sides. Put otherwise, an equation is like a balance. To keep it in balance, one must always do the same thing to both sides of the scales, in order to maintain the balance. In this example we are talking about adding the same number to both sides, or subtracting the same from both sides.

Next we moved on to equations which needed both sides to be multiplied by the same number for them to be solved, such as

$a/3 = 1$ which implies $a/3 \times 3 = 1 \times 3$ and therefore $a = 3$. And $a/3 = 0$ implies $a = 0$.

The children wrote down their own equations of this type, but some made difficulties for themselves by making choices which were too elaborate, such as

$a/6 = 36$, and then finding that they could not do $a/6 \times 6 = 36 \times 6$ with confidence to give $a = 216$.

Next we dealt with equations which required division of both sides by the same number in order to "solve" them, such as

$4 \times a = 8$. Dividing both sides by 4 gives $a = 2$.

Not infrequently children could see what the answers were in these problems, having done the mental arithmetic intuitively, but were not able to explain why, or to explain what their mental process had been.

Next we started a discussion which would need a pair of whole numbers (often called integers) which we wish to denote by n and m. It was immediately necessary to correct a suggestion from one pupil that n = 14 because it is the 14th letter of the alphabet. This is quite a good illustration of the preconceptions which may have to brought out into the open when starting to teach algebra.

Lesson 4.6

First the children were asked to recall and describe the topics which we had discussed in the first half of the term. We spent some minutes talking about the questions of whether the addition, then subtraction, and then multiplication of a pair of even numbers would give a result which would be even or odd; the same problem for a pair of odd numbers; and again for an odd and an even number. The pairs can be selected from a pack of cards, or out of one's head. And what about division?

Then we recalled that some of the important parts of mathematics have the name of arithmetic, algebra, geometry, and trigonometry. The children described what each of these branches was about.

After this revision we moved on to provide an illustration of a use of algebra, namely:

Weighing the baby by algebra

I can stand on my bathroom scales, but my new grandson cannot sit on them without falling off. How can I use these scales, and some algebra, to weigh him?

Introduce the label m for me , or man. I find that $m = 13$ stones.

Next introduce the label b for baby. I hold the baby, and stand on the scales with him. I find that

$$m + b = 14 \text{ stones.}$$

One boy said he could not do this, because the baby would puke on him. I might have said that this would be alright if we counted the puke as part of the baby: but I was not quick enough. We proceeded.

By subtracting m from both sides of the second equation , we find

$m + b - m = 14 - 13$ and therefore $b = 1$ stone.

Evens and odds

What letter could we use for a typical even number? Nick immediately suggested e, which would have been a good idea if it were not for the fact that in mathematics e is reserved for a very special number e = 2.71... approximately. So I had to ban it, and we agreed to proceed as follows.

If m denotes a typical whole number, a typical even number can be written

2 x m, which is often written 2m for short. This notation means "2 times something", so it must be even. We discussed several particular examples, such as 26 = 2 x 13.

This was particularly pertinent because of the danger of falling into the trap that

2m = 26 means that m = 6. This is not so, and we discussed why?

Now we can see that if 2m = 2 x m or 2n = 2 x n represent typical even numbers, then

typical odd numbers can be written 2m –1 and 2m + 1. We illustrated these with m = 13

so that 2m = 26 is even, and the *next door numbers* are 2m + 1 = 27 and 2m – 1 = 25.

Lesson 4.7

We used algebra to prove that

the typical product of even numbers is even, because

2 x n x 2 x m = 2 x (2 x n x m), and that even times odd is even, because

2 x n x (2x m + 1) = 2 x [n x (2 x m + 1)], and odd times odd is odd, because

(2 x n + 1) x (2 x m + 1) is not 2 x any whole number.

Similarly we proved that the typical sum of even numbers is even, because

2 x n + 2 x m = 2 x (n + m), and that even plus odd is odd, because

2 x n + 2 x m +1 = 2 x (n + m) + 1, and that odd plus odd is even, because

2 x n + 1 + 2 x m + 1 = 2 x (n + m + 1).

Similarly we proved that the typical difference of even numbers is even, because

$2 \times n - 2 \times m = 2 \times (n - m)$, and that odd minus even is odd, because

$2 \times n + 1 - 2 \times m = 2 \times (n - m) + 1$, that even minus odd is odd because

$2 \times n - (2 \times m + 1) = 2 \times (n - m) - 1$, and that odd minus odd is even because

$2 \times n + 1 - (2 \times m + 1) = 2 \times (n - m)$.

These are genuine and instructive mathematical proofs which I think the children understood. I had to explain at one point that $(-1) \times (-1) = +1$.

Lesson 4.8

We began by recalling that we have been discussing some examples of Algebra in recent weeks, and before that some Arithmetic. Previously we also identified two more branches of Mathematics, namely Geometry and Trigonometry.

Today we initiated a discussion of some of the geometry and trigonometry of Rugby, prompted by something that had recently appeared on television.

Who plays rugby? "We all do", said the class of 5 girls and 3 boys, somewhat to the surprise of the presenter. But this will help our discussion. Who watches television? Not all, again a little surprise, but one girl said they there is no television in her house. Another said that it was not possible to watch anything else when rugby was on, because that took precedence.

First we collected on the board the geometrical facts about the shape of the rugby field. It is rectangular, made up of two squares , each about 50 metres square on either side of the halfway line, with H-shaped goal posts in the middle of each goal-line, and another smaller rectangle behind each goal-line where tries are scored by grounding the ball. The two main squares are each divided in two by a line across the field called the 22.

Why is it called that? It is the same line which was used when the presenter played rugby at school 50 years ago, but then it was called the 25. Why? Because that meant 25 yards, and the advent of the metric system has changed the 25-yard line into the 22 metre line. So this gave us the opportunity to discuss the conversion of metres into yards, by using the facts that

one yard = 36 inches and one metre = 39 inches approximately (a little more, in fact).

So we explored the fact that $25 \times 36 = 22 \times 39$ approximately.

Next we collected, on the board, the scoring system in rugby, and the class provided the information that

1 try = 5 points, 1 conversion = 2 more points, so that 1 goal = 7 points; and also that

1 penalty goal = 3 points, and 1 dropped goal = 3 points.

Now we had enough background information to come to the trigger for starting the discussion of rugby in a mathematics lesson. In the last three or four weeks the television presentation has include a new illustration, which did not appear last season. When a try is scored, and has to be converted, a triangle has been shown on the TV screen whose corners are at the two posts and the kicking point, together with three pieces of information, illustrated as follows.

Distance 41.9 metres; Angle 23 degrees; Visible (or apparent) goal width 5.1 metres.

What do these facts mean? Distance must mean distance to *some* point on the goal-line. But *which* point? Angle must mean the angle seen by the kicker when he looks at the two posts. Visible or apparent goal width is very ambiguous, and there are certainly *two* possibilities for that.

We will discuss these ambiguities next time, and that will lead us to ideas of bisecting a line and bisecting an angle, ideas of symmetry, and the use of compasses.

Lesson 4.9

Before continuing with the Rugby Problem, we did two new topics which had turned up.

First I set the children to do a Christmas Problem, for homework. This was to construct a Christmas Tree by working out the following multiplication sums, and displaying the answers appropriately (meaning symmetrically) on the right hand side of the equal signs.

1 x 1 =	1
11 x 11 =	121
111 x 111 =	12321
1111 x 1111 =	1234321
11111 x 11111 =	123454321
111111 x 111111 =	12345654321
1111111 x 1111111 =	1234567654321
11111111 x 11111111 =	123456787654321
111111111 x 111111111 =	12345678987654321

They were given the left hand sides, and we worked out the first three right hand sides. The children were asked to find the subsequent six answers. “Can we use calculators?” We agreed that calculators would only be of limited use, because they would soon run out of space. The sums could be attempted in School, before the end of term, or during the holidays.

Next we did the Foggy Day Problem, for a reason that will become clear. On boarding a train at Reading, I noticed that one could see the whole of the Sun's disc, and tolerate looking at it through the fog. On could see that the size of the Sun appeared to be almost the same as the full Moon. But that is curious, because the Sun is 93 000 000 miles away, and the Moon is only 240 000 miles away. This fact, with the observation just described, allows us to deduce that the Sun is actually about 400 times larger than the Moon. The reasoning uses "similar triangles", with the fact that the Sun is about 400 times further away than the Moon.

This example of elementary mathematics was evidently triggered by the conjunction of the two points that I was going to that particular meeting and was thinking about it, plus the fact that the fog allowed me to see the Sun's disc.

So I described all this in the class. We talked through the approximation that 93 million could be replaced by 100 million, and that 240 thousand could be replaced by ¼ million within reasonable approximation, so that the Sun was 100/(1/4) = 400 times further away than the Moon. We then drew the diagram showing the two rays of light from opposite ends of the Sun's diameter to the eye, and showing that these passed through the ends of the Moon's diameter, but 400 times nearer than the Sun. Then we introduced the mathematical idea of "similar triangles", and showed how it could be used to calculate that the Sun must be about 400 times bigger than the Moon. These arguments were well accepted by the children.

Finally we returned to the Rugby Problem (see my article: A Rugby Riddle, in Mathematics Today, Volume 42, pages 126 -128, 2005). We drew a plan view of a particular case showing the goalposts, and the kicking point towards the side. I asked the question "Where would you aim your kick?" Answer: "To the middle". I pointed out that this was ambiguous, because the midpoint between the posts is not the same direction as the bisector of the angle which the kicker sees between the posts. The children need to draw their own diagrams next term, to confirm this.

Lesson 4.10

In an lesson abbreviated, we devoted the time to a discussion of the Christmas Problem posed in Lesson 4.9.

We discussed the result, and how to do the successive multiplications, such as

111 x111 = 1 x 111 + 10 x 111 + 100 x 111 and so on.

We then discussed the verbal names of the results, such as

123454321 = 123,454,321 = 123 million 454 thousand 321 and

1234567654321 = 1,234,567,654,321 = 1,234,567 million 654 thousand 321.

We also discussed the definition of

1 billion = 1 thousand million = 1000000000 = 1,000,000,000

almost universally nowadays, although it was previously ambiguous and 50 years ago, when the presenter was a student, some people might have said that

1 billion = 1million million = 1000000000000 = 1,000,000,000,000.

The children seemed to relish all of this, which they could readily get to grips with.

Lesson 4.11

We began with something learnt from the newspaper about the names of big numbers. An article about an Earth Simulator computer in Yokohama which is so big that it needs a floor area of four tennis courts told us that it works at a speed of 36 teraflops, which means 36 trillion operations per second. This is to predict climate change and global warming. So we learn that

1 trillion = 1 000 000 000 000 = 1 million million = 1 thousand billion.

Next, at last, after several weeks of anticipation, prompted by the Rugby Problem which arose before Christmas, we began the use of compasses.

The children brought the school compasses from their classroom, which all had long pencils in them, and I asked them to draw a circle. There are various methods, e.g. holding the compasses still and rotating the paper round, holding the paper still and rotating the compasses by holding the pencil, or rotating the compasses by holding the hinge handle. The last is the best, but a long pencil can get in the way.

To solve this difficulty, I brought a small fretsaw out of my bag and, amid great excitement, I sawed eight long new school pencils in half. I emphasized that we had not wasted anything because I would keep the unused half-pencils until the ones which we had put in the compasses had been used up.

Then we were in a position to draw circles by holding the centre of the compasses without the pencil getting in the way. So first I asked the children to draw some circles. There was some tendency to exaggerate by drawing very large or very small ones. but they had some practice, and used plenty of paper. Then we discussed arcs of circles, and agreed that an arc of a circle meant part of a circle. The initial tendency was to draw rather large arcs.

Next, we approached the Rugby Problem, and I mentioned the idea of a perpendicular bisector of a line. I showed the children how to find this by drawing arcs of circles using the ends of the line as centres, finding where arcs drawn using opposite ends of the line as centres intersected, and then joining with a ruler such intersection points on opposite sides of the line. The line drawn with the ruler is the perpendicular bisector of the line. The children all did this for themselves.

Lesson 4.12

We began, not for the first time, with something learnt from a newspaper. I had noticed it quoted that Bill Gates, the Chairman of Microsoft, was worth $27 billion. As we had been discussing large numbers recently, I thought it would be worth putting the Rugby Problem on hold while we did something useful with his money. I proposed to the children that we should work out how many houses he could buy in Maidenhead with that money.

First we had to convert it to sterling. I showed the children how one can find out, from the Business News in each daily newspaper, what the current rate is for the conversion of pounds into dollars. On 26th January it was

£1 = $1.8821 so on this basis, how many pounds does Bill Gates have?

Approximate first, to make the calculation easier, by supposing that £1 = $2.

Then it is, fairly obviously, a division sum that we have to do to find that

$27 billion = £27/2 billion = £13 500 000 000.

By now the atmosphere in the room was quite animated, and it was readily accepted that the next approximation would be

$27 = £27/1.8 billion = £(3 x 9)/(0.2 x 9) billion = £3/(0.2) billion = £3/(1/5) billion

= £15 billion = £15 000 000 000.

To get the final approximation we would have to use calculators, for the first time in this problem, and we found immediately that they did not have enough decimal places to provide the answer exactly. Instead we found that, approximately,

$27 = £27/1.8821 = £14.345678 billion = £14 345 678 000.

So this is what Bill Gates has in his pocket, expressed in sterling.

How many houses in Maidenhead could he buy with it? I suggested that we might assume that the value of each house, on average, might be

£250 000.

This speculation revealed genuine interest by the children in that aspect of the question and some expressed their view quite firmly that £400 000 would be much more realistic. However, to make a simple first calculation, the children agreed to go along with my suggestion (with the warning that they would be rather small houses).

Thus we calculated that Bill Gates could buy

14 345 678 000/250 000 = 14 345.678/(1/4) = 4 x 145 345.678 = 57 382.712, i.e.

57382 small houses in Maidenhead. But fewer if they were more expensive.

In the few minutes remaining we returned to the geometry of circles, and posed the following question. You have a goat. You also have a square field. And you have a peg, and a piece of rope whose length is half the side of the field. How many different peg positions will you have to use to allow the goat to graze the whole field?

Some variety of guesses were proposed, and we shall have to return to this question next time.

Lesson 4.13

Before continuing with the Goat Problem, we began with an exercise in percentages which arose from a cricket match two days ago. I asked the class to name all types of result which would be possible, and they suggested Win, Loss, Draw, Cancel. They did not suggest Tie, and there was some prolonged discussion before this possibility was agreed. Even then, one girl said later that she still believed that a Tie was the same as a Draw.

The One Day International between England and South Africa had ended in an exciting Tie, with each side scoring 270 runs. During the discussion I drew a map of Africa on the board, so we learnt some Geography by finding out where South Africa is (and also Zimbabwe in answer to a question).

The newspaper had told us that there have been 2218 One Day Internationals, and this was the 20th Tie. So what percentage have been Tied, both approximately, and then exactly?

We had some discussion of what percentage meant, and then worked out that the

approximate percentage = (20/2000) x 100 = (2/200) x 100 = (1/100) x 100 = 1%.

After this convenient approximation, we used calculators to work out the

exact percentage = (20/ 2218) x 100 = 0.90173 %.

Returning to the Goat Problem, there was some curiosity about where the problem had come from. I said that it had arisen out of a breakfast table discussion. For our purposes it gives us an exercise in the use of compasses.

I asked the children to sketch their solutions freehand first, and only two children were able to offer the correct answer. I spent some time commenting on incorrect conclusions from diagrams which contained marked inaccuracies. We shall return to this question next week.

Lesson 4.14

First, in the cause of accurate use of language, I wanted to deal with a point which had been omitted last week, namely to find an accurate description of the difference between a draw and a tie, because someone had said at the end last week that she still believed there is no difference between the two.

So I gave the children three minutes to think, in silence, of how they would describe the difference. It became very clear that they were not used to addressing a problem with such discipline, because several of them took more than half the time fiddling with their pencil or rubber and looking round to see what other people were doing, thus postponing the moment when they got around to actually thinking. That was interesting in itself, and perhaps it was not surprising that I did not get a really clear answer from anyone.

A girl who was not in class last week when the issue was first discussed was keen to say, with only a slightly embarrassed smile, that a tie was what you put round your neck and a drawer was where you kept things!

After listening to some other attempts to explain the distinction, not very convincingly, I made the point that some games had a natural finishing point built in to their structure (like cricket, which in one version finished when the second side were all out if they had not reached or exceeded the score of the first side), so you can have a tie in such a game; but other games, like football, had no in-built end, and in its simplest form always had to finish after a specified time. Cricket could have either a tie, or a draw. In football, a draw and a tie were always the same thing. I mentioned the famous timeless Test Match in Durban in 1939, which was planned to be played to a finish no matter how long it took, but which had to be stopped as a draw when it was still unfinished after ten days, because

England had to catch the boat home.

Next we moved on to some compass work, and the children were required to follow my step by step instructions of how to draw a square using only a straight edge and compasses (and not by measuring). This involved the construction of two (parallel) perpendiculars to a given line, and completion of the square by joining their ends. We had thus constructed right angles with the compasses. The results were checked by measuring the sides of the square and finding that they were (very nearly) equal.

Finally, and continuing with compass work, we returned to the goat in the field problem, and showed how five positions of the peg (the centre and corners of the square) were enough to allow the goat to eat the whole field. This symmetric solution works. But it was then pointed out by one pupil that there is another solution in which four positions of the pin are enough. We need to discuss this next time, and in particular the fact that such a solution does not have to be symmetrical.

Lesson 4.15

In a shortened lesson, we first returned to the problem of the goat in the square field. The children were given centimetre-squared paper, so that it was easy to draw squares. First they drew the 5-point solution, which is symmetric, and readily suggests itself. They used compasses to draw the associated circle (centred on the centre of the square), and the four quarter-circles centred on the corners, which certainly allow the goat to cover the whole field.

But then we looked at the unsymmetric solution suggested previously, and we found that there are several of these which work. For example, two of the centres can lie on one diagonal, each one quarter of the way in from opposite corners. Then the remaining two centres can be at the other two corners, and the goat can cover the field from these four corners.

Several children had suggestions of three-centre solutions, and I looked at these with them, but we agreed that none of them worked.

I made the point about mathematics being, in some cases, "constant hard work in search of the easy way", in relation to the frequent search for a solution which involves the least work, in some sense, and illustrated here by the fact that four centres is less than five.

Next we discussed another example where there is a symmetric solution and an unsymmetric solution. This occurs in the squashing of a rubber block between two parallel plates. If the block is thick, it will "barrel out" symmetrically. If it is thin, like a strut, it will buckle sideways, and therefore unsymmetrically.

Lesson 4.16

This was to be an exercise in the use of compasses and a straight edge.

I asked the children to draw an angle of 180 degrees. They realized that they had to draw a straight line, but they were unsure how to designate the angle on it. Some sketched a semi-circle on it at a particular point. They were unclear that any point on the line, except the two ends, could be regarded as a point where the angle was 180 degrees.

Next I asked them to construct an angle of 90 degrees, by bisecting the 180 degrees. There was a lot of uncertainty and attempted fudges, but eventually it was done by using the compasses at any point P on the line to draw arcs at equal radii cutting the line at C and D, followed by two more intersecting arcs with C and D as centres. Then the join of the new intersection point to P is a perpendicular to the line, and therefore makes two equal angles of 90 degrees on either side. Next the children were asked to bisect one of these right angles by using P as centre, and marking two intersections E and F of arcs of equal radius on one pair of perpendicular lines, and then using E and F as new centres to make the intersection G of two more arcs. Joining G to P bisects the right angle, so we have two 45 degree angles.

Repeating the process gives two 22.5 degree angles, and repeating again 11.25, and so on. Eventually we could see that by repeating this process many times we could make a protractor.

These ideas were then applied to the rugby problem broached earlier in Lesson 4.8. We drew the triangle formed by the kicker at K and the two posts at A and B. We marked the midpoint M between A and B.

We used compasses to bisect the angle AKB, and used C to label the point where this angle bisector intersected AB.

We could see that, unless K is directly in front of the posts, C is not the same as M.

If there is no wind or swerve, does the kicker aim along the line KM or the line KC? They are different.

The current BBC television presentation is ambiguous about this, when it speaks of apparent goal width which it does not define. Presumably it is the length of a line, joining that goalpost nearer to K (suppose this goalpost is A) to a point on the line (KB) to the other goalpost, which is perpendicular to either KM or KC. But which? They are different.

Lesson 4.17

At the start of a new term, we began with a new problem, called the Big Birthday Problem.

Two people that I know are called

Bunty, an old lady, who was born on 10^{th} November 1907, and

Daniel, my grandson, born on 7^{th} January 2004.

First we worked out how old they are now.

B = 2004y – 1907y = 97y approximately, but exactly

= 97years + 5 months + 12 days.

D = 2005y – 2004y = 1y approximately, but exactly

= 1y + 3m + 15d.

What is the **sum** of their ages **now**? 98y + 8m + 27 d exactly.

How old will they be on 7^{th} January 2006?

B = 2005y – 1907y = 98 approximately = 98y + 1m + 28d exactly.

D + 2006y – 2004y = 2y exactly.

So the **sum** of their ages on 7^{th} January = 100y + 1m + 28d exactly.

Question: on what day will the sum of their ages be exactly 100 years?

First clue: it must be between 10 Nov 2005

when B = 98y and D > 1y, so that B + D > 99y, and 7 Jan 2006

when B > 98y and D = 2y, so that B + D > 100y.

How many days are there between the two?

Nov 20 + Dec 31 + Jan 7 = 58.

I worked all this on the board, requiring the children to contribute to the discussion, which they readily did, and to make their own notes from the board.

The method is to use **algebra**, as one child suggested.

Suppose the Big Birthday is X days after B = 98y birthday, and therefore

58 – X days before D = 2y birthday.

So the required fact is that

$(98y + Xd) + (2y - [58 - X]d) = 100y.$

Can we solve this equation to find X?

Lesson 4.18

In a shortened lesson we completed the calculation begun last week. Explaining that a minus time s a minus makes a plus, we simplified the equation to

$100y + Xd - 58d + Xd = 100y.$

Explaining that an equation expresses a balance, so that doing the same thing to both sides of an equation maintains the balance. We deduce that

$2Xd - 58d = 0$ so that $2Xd = 58d$ and therefore the unknown $Xd = 29d$.

This means that the Big Birthday is 29 days after Bunty's 98th birthday and 29 days before Daniel's 2nd birthday.

So it is on 10th November + 29 days = 9th December.

Summing up, on 9th December 2005

Bunty will be 98 years + 29 days, and Daniel will be 2 years - 29 days

So the **sum** of their ages will be **exactly 100 years** on that day.

Lesson 4.19

We introduced the topic of temperature scales, Fahrenheit and Centigrade (= Celsius).

I explained that C had achieved its current emphasis over F in everyday life only relatively recently, and that Celsius (the name of a man) had relatively recently replaced

Centigrade (which means, literally, 100 degrees) in daily emphasis.

After some discussion we agreed that water boils at 212 F, or equivalently at 100 C; and that it freezes at 32 F which is 0 C.

These facts allow us to plot a straight line graph joining the two points, with F on the vertical axis and C on the horizontal axis, also allowing for negative temperatures, and this graph will be the relation between the two scales.

The children did this, after first marking out the axes from about - 60 for both C and F, to about 220 F and over 100 C. Next we asked the question: is there a temperature at which both C and F have the same value? To answer this they plotted, on the same graph, the line along which F = C. Where this crosses the first line gives us the temperature

F = C = - 40 at which both temperature scales have the same value.

Lesson 4.20

I asked for a volunteer, from the four pupils who were present last week, to explain to the other four who had been absent, what we had done.

The volunteer gave a very good account of the facts that we had plotted, on graph paper, a graph of the relation between the Fahrenheit and Centigrade temperature scales, by joining the freezing 0C = 32F and boiling 100C = 212F points of water by a straight line; and then discovered by drawing another straight line C = F and finding where it intersected the first one, that the Fahrenheit and Centigrade temperatures where the same at – 40.

This time we set ourselves to recover the same result using algebra. By writing f and c, as alternatives to y and x, and by expressing the gradient of the line in two ways as

(vertical step)/(horizontal step) = (f – 32)/(c – 0) = (212 – 32)/(100 – 0), and patiently simplifying this to

(f – 32)/c = 180/100 and then f – 32 = 1.8c and then f = 1.8c + 32 ,

we thus found the general algebraic relation between the two temperature scales.

We now applied this to finding when f = c, by showing that the equation c = 1.8c +32

can be simplified to 0.8c = - 32 , and hence to c = - 32 x 5/4, giving f = c = - 40.

Some debate over the intervening manipulations was necessary, but most people were eventually convinced of this algebraic alternative to the previous graphical proof.

Lesson 5.1

A sunflower head was displayed, cut from my garden four weeks before, and therefore drying out with the seed pattern visible, and the children were asked to draw any pattern which they could see. There was a variety of responses. What I was seeking was an observation that the seeds were arranged in a double spiral pattern, with one spiral clockwise and the other spiral anticlockwise, but this was more evident to some pupils than others. Drawing it proved to be quite a difficult exercise, which was not accomplished at all well by some pupils. The whole lesson was taken up with their attempts at sketching, and we shall return to this. They were quite relaxed and interested, however.

Lesson 5.2

The children were asked to make another attempt to draw the sunflower, which was done better this time. I then gave them each a photocopy of a photograph of one. I told them that the name for patterning in plant development is phyllotaxis, and that the sunflower displayed what is called double spiral phyllotaxis.

I asked them each to give an example of a sequence of numbers constructed according to some rule. One example given was 10, 20, 40, 80 .... thus doubling the interval each time, and one boy gave the example

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,....

He was also able to say that this was called the Fibonacci series. This was exactly what I wanted to discuss. I explained that the name is short for Fi Bonacci or Filius Bonacci in Italian, which means Son of Bonacci.The same device is used in some English names like Thomson and Jackson and Robertson, i.e. son of Robert, etc. Bonacci was living in Pisa in northern Italy in about 1202, much earlier than the children's guesses indicated.

So we were in a good position to discuss the mathematics of this series next time. At the end of this lesson one pupil left the room saying words to the effect that this was all very interesting, and how was he going to survive all the boring teaching in front of him during the next week before we meet again!

Lesson 5.3

We discussed the spin-up (to the vertical) of acorns , starting from the position of lying on their side, although no-one had collected any to try it for themselves. I said the same thing happened with Smarties, lemons and eggs. We discussed why spin-up would be easier for a hard-boiled egg than a raw one (it is easier to transmit the imparted spin through a solid than a liquid), and this was appreciated. I gave them a photograph of a spinning acorn, and an article on my TV appearance in which I demonstrated this.

Next we resumed our discussion of the Fibonacci sequence, and we computed it on the whiteboard for the first 15 terms or so.

Then I explained how we could describe the rule, in mathematical notation, for generating the sequence. That is, we write the nth term as F{n} where {n} is a subscript. Then the formula

F{n+2} = F{n+1} + F{n},

applied repeatedly to successive whole numbers n, generates the sequence. We get

0, 1, 1, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181,...

Next we looked at our sunflower for the third week. Although it was beginning to fall apart, we tried to count the spirals, but found that it is not easy. So I referred pupils to the photograph that I had given to each of them last week. I asked them to draw spiral lines on their photocopy which followed the seeds. Some did this on the seeds, which is not as easy as between them. We shall need to try this again.

Lesson 5.4

We resumed discussion of the Fibonacci series (including some revision of the algebraic notation F with a suffix n, for denoting the nth term of the series). I explained that I wanted the children to actually *discover* something about the sunflower. To this end I gave out new copies of a photograph of a sunflower, and I asked the children to try very carefully to draw lines following the spiral patterns in the two directions; and in addition to count the number of spiral lines by writing the number of the spiral line next to it each time they finished one. A variety of success was soon evident in this exercise, some being quickly proficient, and others finding it quite difficult. Eventually we were able to record on the board the number of spirals counted in the clockwise and anticlockwise directions as follows.

J	R	H	B	A	C	J	A	R	MJS
24	23	22	21	25	34	45	23	?	21
34	34	35	34	35	51	46	34	35	35

We were able to draw the clear conclusion that sunflower seeds are usually arranged in two spirals of about 21 and 34 curves in the two opposite directions, thus demonstrating a clear connection with one particular pair of adjacent numbers in the Fibonacci series.

All this is part of learning to ask the ubiquitous scientific question: **what is the pattern?**

Next I introduced a large fir cone which I had found on Brownsea Island in the previous week, and I encouraged the children to observe the pair of opposing spirals formed on it by the individual segments. We counted them and we will return to this topic next week.

Lesson 5.5

I took in to school a pineapple bought for £1.99 from Marks and Spencers, and two different fir cones, the fat Brownsea one and a thinner one obviously from a different type of conifer.

First we reviewed the Fibonacci series, and in particular the notation for labelling the terms in it, as F with a subscript n for the nth term. This took some time to explain, with examples, but the children eventually grasped it, together with the idea of the nth term of the Fibonacci series, with having to specify any particular n. Then we could understand the Fibonacci formula

$F/(n+2) = F/(n+1) + F/n$

where / signifies that terms following it are subscripts being used as labels, representing typical values 1,2,3,4...etc. which we do not wish to specify.

I explained that this was an example of "algebra", a branch of mathematics said to have been invented by an Arab called Al Habri around 1100 A.D.

We could then say that the sunflower was a Fibonacci example with

$F/9 = 21$ and $F/(10) = 34$.

Next the children were all asked to count the number of spirals in the two opposite directions on the pineapple, fat fir cone and thin fir cone. This was an animated exercise, with some variation of results, but with reasonable unanimity as follows.

Pineapple, and also fat fir cone (from Brownsea Island) $F/7 = 8$, $F/8 = 13$

Thin fir cone (Shinfield Road, Reading) $F/6 = 5$, $F/7 = 8$.

Lesson 5.6

We reviewed the abstract notation used for describing the Fibonacci sequence.

We examined another, smaller, fir cone and established that it was another example of 8,13 in the sequence.

I had bought, from the supermarket, Brussels sprouts on a straight stalk. This caused great interest. I asked the children to discover the Fibonacci sequence. This stalk was a very tangible object. After animated discussion most children agreed with me that it exhibited 2,3 as the description of two opposing spirals. But one pupil also pointed out that there was a longer spiral with the Fibonacci number 5. This was a new discovery for us, that there could be more than just a single pair. More discussion will be merited.

We looked at another, smaller fir cone which had 8,13.

We also looked at a further small but more difficult fir cone which seemed to be 3,5.

We listed all the results we had found, as follows.

Sprouts 2,3 and 5. Difficult fir cone 3,5. Large thin fir cone 5,8. Small fir cone 8,13.

Fat (Brownsea) fir cone 8,13. Pineapple 8,13. Sunflower 21, 34.

From 7 objects we have found 5 different Fibonacci pairs.

The children asked if we would find such patterns on animals. I suggested that they look. Where? A tortoise was suggested, and it was thought also that it might be worth looking at the pattern of fish scales on the supermarket shelves.

Lesson 5.7

We discussed how to unfold, draw and count the Fibonacci spirals on the Brussels sprouts, by wrapping paper round them, so that information on a cylinder is converted to a flat page, where it would be easier to handle.

I gave the children some squared paper, and we carried out the following steps.

1. Draw a 2 x 3 rectangle.
2. Draw diagonal of all of the three 1,3 rectangles and the two 2,1 rectangles.
3. Mark their intersections. Do these points mimic the sprouts pattern?
4. Repeating the 2 x 3 rectangle several times might map all the sprouts patterns?
5. Will the steeper connections be part of 5 (and even 8?) spirals?

There was much discussion. I showed them how I had attempted to mark the positions of each individual sprouts on a piece of A2 paper.

Lesson 5.8

We left Fibonacci aside for the time being, and did some mental arithmetic.

First $1 + 2 + 3 = 6$ in our heads. Easy.

Next I asked for the first ten numbers to be added up, in our heads. This created some frowns, hesitation, and apprehension of what might come next. The answer 55 was produced in a reasonable time, more quickly by some than by others.

Then I asked for the sum of the first 100 numbers. Consternation from some. Mentally? Can you be serious? But some of the more thinking ones addressed themselves to it. Various erroneous answers or guesses were produced by some. But it was not long before the youngest member of the class showed his ability to think in a different way by pairing the numbers starting with

first + last, i.e $1+ 99 = 100$, then $2 + 98 = 100$, then $3 + 97$, ending with $49 + 51 = 100$, so we soon get 4900 and adding the loose 50 which only occurs once, 4950.

Similarly $1 + 2 + 3 + ... + 98 + 99 + 100 = 5050$.

Next we generalized to find $1 + 2 + 3 + ... + n = S$ (say). To make some sense of this we rewrote it $n + (n-1) + (n-2) + ... + 3 + 2 + 1 = S$. Now adding vertically we see that we have $n \times (n+1) = 2S$.

This level of generality was unfamiliar, but then acceptable.

Hence $S = n(n+1)/2$ for any number n.

E.g., for $n = 100$, $S = 100 \times 101/2 = 10100/2 = 5050$ as above.

Finally we recalled the spin-up of acorns from Lesson 5.3, and I demonstrated the same effect with the seed of an avocado which is quite large and therefore dramatic. It worked well for me, the children were also able to make it work, and I gave it to them to ask their teacher to keep it safe for them.

Lesson 5.9

We began with another avocado stone, smaller than that of last week, and therefore easier to spin, to see if it would "spin up" (stand on end) when spinning. Every one had a turn, with enthusiasm, and I took some photographs.

Next we discussed a geometry problem, for a change. Everyone could set down three points on the page, and agree that they "usually" made a triangle (unless they happened to be in a straight line, which we discussed).

Then I asked them to set down four points, and we discussed the fact that they "usually" made the corners of a quadrilateral; and that it might be "convex" (no dents) or "concave" (having one inward pointing corner making a dent).

Finally I asked the children to set down five points. We discussed the fact that four of these five **always** made a convex quadrilateral.

Then we moved on to mention Pythagoras. The children had not heard of him, but they have now. He introduced the idea of a "perfect number", defined to be one which is the sum of its divisors. Thus the first two are

$6 = 1 \times 2 \times 3 = 1 + 2 + 3$, and $28 = 1 \times 2 \times 14 = 1 \times 4 \times 7 = 1 + 2 + 4 + 7 + 14$.

The ancient Greeks knew about these and two more, namely 496 and 8128. Seventeen centuries later a fifth was discovered, namely 33550336 which has 8 digits.

Currently 37 are known, the largest having 1819050 digits. All 37 are even. **Nobody** knows whether there is an odd one.

Finally I mentioned the so-called Goldbach conjecture, which I called Goldbach's Guess. It states that **every** even number greater than 2 is the sum of two prime numbers. Nobody has proved this to be true. Examples are $12 = 7 + 5$ and $16 = 13 + 3$. I told the children that they would become famous if they could prove GG to be true.

Lesson 5.10

After a quick look back to Goldbach's Guess, prompted by the children, I introduced the topic of Family Trees. I asked the children if they could draw their family tree just for the most recent three generations, and without siblings or uncles and aunts. This prompted some interested debate, e.g. what to do about stepchildren, people who were dead, people who had been married twice, but I insisted that all such complications be excluded so that we just concentrated on immediate blood relatives. That in itself was a worthwhile exercise in simplification.

The children drew some wonderfully exotic trees, with curved lines, extra lines, lines going round corners, all of which indicated that they had not seen a conventional simple tree diagram. So when they had had their turn, I showed on the board what was required.

D Q Q D Q D Q Q D Q D Q Q 13

Q D Q Q D Q Q D 8

Q D Q D Q 5

Q D Q 3

Q D 2

Q 1

D 1

Then I turned to the question of what is the family tree for Bees. I explained that, just as there are two sorts of people, men and women, there are also two sorts of bees, Queens and Drones. But unlike people, who each have two parents, one male and one female, bees do not all have two parents. Queens have two, a Queen and a Drone, but Drones have only one, a Queen. Then I asked the children to draw the family tree of a drone, going back as many generations as they could, and at least seven, and using only horizontal and vertical connecting lines. This task was embarked upon with enthusiasm, with a result as above. The children did it very well.

Then we counted the number of bees in each generation, starting from the most recent, and found them to be the numbers shown in the diagram, which the children recognised immediately as the Fibonacci sequence **again**. Amazement. Then they could **predict** the numbers to be expected in earlier generations, 21, 34, 55, 89 ....., without having to draw the diagram.

Lesson 5.11

I asked the children to draw a square of side 3 units, and inside it another square of side 2 units. They were allowed to choose their own units. Some chose inches, some chose centimetres, and some made more exotic choices like two inches. These variations were instructive in themselves. The inner square did not have to be parallel to the outer one, or symmetrically placed. Then I asked the children to find the area in between the squares.

The exercise produced interested debate, but an eventually agreed answer of 5 units, namely

$3 \times 3 - 2 \times 2 = 9 - 4 = 5$. A very few noticed that $3 + 2 = 5$ also. So we then explored

$4 \times 4 - 3 \times 3 = 16 - 9 = 7$ and $4 + 3 = 7$, followed by

$5 \times 5 - 4 \times 4 = 25 - 16 = 9$ and $5 + 4 = 9$.

Was this funny fact always true? The children soon found that

$6 \times 6 - 4 \times 4 = 20$ which is **not** the same as $6 + 4 = 10$, and some similar results.

How do we explain this? "I know" said one, and then "we are going to find some sort of Fibonacci explanation" said another, which we were not. But the remarks showed there was some attentiveness and interest present.

I introduced some algebra, which we had to work quite hard at, with noticeably different levels of understanding among the pupils.

We proved, and discussed, the fact that for **any** two numbers x and y,

$x.x - y.y = (x + y).(x - y)$ because the right hand side is

$x.x + x.y - y.x - y.y$ for **all** x and y.

So when $x - y = 1$, as we have in the first squares which we drew at the start, we see that

$x.x - y.y = x + y$. But this is **not** true if x is **not** $y + 1$.

Some careful explanation of the use of brackets, and of the multiplication symbol itself, was required. I used a dot in this case. Evidently more of this is needed.

Lesson 5.12

We introduced the Binomial Square, which has a square of sides a x a in one corner, another of sides b x b in the opposite corner, and with a corner of one touching a corner of the other, so that the whole (a + b) x (a + b) square is completed with an a x b rectangle and a b x a rectangle in the other two corners.

The children chose their own dimensions, and we discussed the variations thus revealed.

Then we worked out the areas of the squares and rectangles, and eventually agreed that we could write the whole *area* as

$(a + b) \times (a + b)$ and also as $a \times a + a \times b + b \times a + b \times b = a^2 + 2 \times a \times b + b^2$.

We then replaced x by . to simplify the multiplication notation to

$(a + b).(a + b) = a.a + 2.a.b + b.b$ and finally to just $(a + b)(a + b) = a^2 + 2ab + b^2$.

This was a useful exercise in the familiarization with algebraic notation, which took the whole hour, with the intervening debate, but which seemed well worthwhile.

Lesson 5.13

We returned to the Fibonacci distribution of Brussels sprouts on a stalk. I modelled the stalk with a cardboard tube 35 cm long and 2.5 cm diameter. To represent the locations of the sprouting points, which we know to lie on 2 distinct spirals in one direction, and 3 in the other, I used 5 different coloured threads of wool.

In the presence of continuing dialogue with the children, I taped one end of a blue thread to the bottom of the tube, and wound it round helically until I could tape the other end to the top of the tube. Then I taped an orange thread similarly, winding it so as to bisect the separation between the blue coils. This meant, as we discussed, that the ends of the two threads would need to have 180 degrees separation at each of the two ends.

Next I wound three threads, black, grey and yellow, in the opposite direction. The ends therefore needed a separation of 120 degrees, to achieve even spacing of the threads.

With the 5 threads in place, we were able to discuss the interpretation of the points where there were crossovers. These were the places from which the sprouts must sprout, to achieve the Fibonacci pattern. More discussion needed next week.

In the meantime we finished with quite a different topic. What is the sum of the series

$1 + 3 + 9 + 27 + 81 + 243 = S$, say?

One girl had no difficulty in getting $S = 364$ longhand. So next I wrote down underneath

$3 + 9 + 27 + 81 + 243 + 729 = 3S$

and showed that by subtraction we get

$729 - 1 = 3S - S$ so that $728 = 2S$ and $S = 364$ again.

We will build on this next week.

Lesson 5.14

We sought the sum of the series

1 + 5 + 25 + 125 + 625 + 3125 + 15625 + 78125 = S, say.

After some discussion, and longhand attempts, we agreed to look at

5 + 25 + 125 + 625 + 3125 + 15625 + 78125 + 390625 = 5S, from which we could then

see that, by subtraction,

4S = 390625 – 1 which implies S = 390624/4 = 97656.

This example has the "common ratio" of number/previous number = 5 (3 last week).

Next we looked at an example in which the common ratio is less than 1, which makes the "naive" approach of trying to add up all the terms directly noticeably more difficult. Find

1+ ½ + ¼ + 1/8 + 1/16 + 1/32 + 1/64 = S. The common ratio here is ½. Multiply by it.

½ + ¼ + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 = S/2. Subtract the two series. Then

S – S/2 = 1 – 1/128 implies S/2 = 128/128 – 1/128 = 127/128 so that S = 127/64,

i.e. S = 1 + 63/64. With these distinct examples, we can now do the general case.

For any r and n, find 1 + r + rr + rrr + rrrr + rrrrr + ...+ r (n times)r = S.

Multiply by r to give rS, and subtract to give rS – S = r(n+1 times) - 1,

S = (r[n+1 times] - 1)/(r – 1) for any r and n.

Lesson 5.15

We addressed the question of how to represent, on a flat piece of paper, the spirals which we had constructed in Lesson 5.13 by winding different coloured strands of wool in opposite directions round a cylinder.

On 2 mm graph paper, with divisions of 1cm and 2cm emphasized with heavier lines, I asked the children to draw rectangles 2 cm by 8cm. Then they were asked to imagine the cylinder rolled out, with the rectangle representing one revolution. So the short side was the circumference, and the long side the height of the cylinder.

One spiral was then represented by drawing diagonals of successive rectangles measuring 3 x 10 units (1 unit = 2mm) up the long rectangle. Then a spiral in the opposite direction was represented by drawing diagonals of successive rectangles measuring 2 x 10 units in the opposite direction. The children found this to be a useful exercise in accurate drawing, which had to be tried more than once in some cases.

The resulting diagram showed the intersection points of the oppositely sloping diagonals.

The children were invited to explore if these intersections lay on more steeply sloping diagonals. These represented new spirals, revealed by the pattern of intersections of the first two.

This geometry mimicked the distribution of sprouts on a stalk, which lay first of all on Fibonacci (2,3) spirals, and then were found to lie on steeper Fibonacci spirals.

Lesson 5.16

The objective today was to introduce the famous so-called Golden Ratio, and to demonstrate its connection with the Fibonacci Series. The children were well able to quote the latter, and we wrote it out using subscript notation, namely that

F/n (Word shorthand for F with a subscript n), so that we begin with

F/1	F/2	F/3	F/4	F/5	F/6	F/7	F/8	F/9	F/10	F/11	and so on, namely
0	1	1	2	3	5	8	13	21	34	55	and so on.

Then the children were asked to use calculators to work out ratios of successive terms:

R/n = {F/n }/{F/(n + 1)} and also G/n = {F/(n+1)}/{F/n}

and make a table as follows:

n	1	2	3	4	5	6	7	8	9	10	11
R/n	0	1	0.5	0.666	0.6	0.625	0.615	0.619	oscillates		
G/n	inf	1	2	1.5	1.66	1.6	1.625	1.615	also oscillates,		

but both oscillations diminish and tend to limits which are

R = ({square root of 5} – 1)/2 and

G = ({square root of 5} + 1)/2 = 1.6180339887...

because {square root of 5} = 2.2360679774...

This G is called the Golden Ratio. The Greeks knew it.

These exercises provoked a lot of useful discussion along the way.

Next term we shall introduce a geometrical application of the Golden Ratio.

Lesson 5.17

The children were first asked to draw a pentagon, not necessarily a regular one, and freehand. This clarified that a pentagon is a five-sided figure.

Then circles of about 3 inches diameter were handed out, marked on their circumference with 72 equally spaced points, labelled at alternate intervals with the numbers 1 to 36, i.e. at every 10 degrees. The children were asked to say what the angular interval would be if we wanted to draw a regular pentagon, and it was agreed that this would be 72 = 360/5 degrees. The children were asked to mark these points on the circumference of their circle, labelled as A,B,C,D,E. Then they were asked to join, with straight lines, alternate points with straight lines across the inside of the circle, i.e. AC, CE, EB, BD, and DA. This makes a 5-pointed star called a pentagram. Each pair of lines from a vertex, such as AC and AD, make the two equal sides of an isosceles triangle. The children were asked to measure the long side (e.g. AC) and the short side (e.g. DC) and work out the ratio AC/DC = 121/76 = 1.592 measuring in millimetres. Within measuring error this value is very close to the Golden Ratio which is 1.6180339887. Denote F to be the intersection of AD and EB, and G to be the intersection of AC and EB. In the triangle AFG which is similar to the triangle ADC, the ratio AG/FG = 46/28 = 1.642. This is another surprising appearance of the Golden Ratio which we encountered before.

Lesson 5.18

We began by noticing that the time at 2 minutes and 3 seconds past 1 o'clock in the morning of 4th May 2006 could be written 01.02.03. 04/05/06. After some discussion and false starts the children quickly got the idea of this, and had little difficulty with realizing that the time at 6 minutes and 6seconds past 6 o'clock in the morning of 6th June 2006 could be written 06.06.06 0/6/06/06.

Next I introduced a difficulty that I had with a piece of cake on the previous Saturday. My wife and I visited Leonardslee Garden in Sussex, and at their tea-shop we bought a slice of a circular sponge cake. I had the task of cutting it in two equal halves, and I chose to do it by a cut across the centre-line, rather than along it to the centre of the cake. This poses the problem of where to make the cut.

The shape of the slice, in plan view, is called a sector of a circle, and it has a middle line from the centre of the circle. I suggested that we simplify the problem by cutting off, by a cut perpendicular to this mid-line, the curved piece of circumference. So now we have what the children recognised to be an isosceles triangle (they knew what that meant). How do we halve that with a cut which is perpendicular to the line of symmetry? Denote the lengths of the base and height of the triangle by B and H respectively. Then the area of the triangle is BH/2, as was readily accepted, and can be proved by enclosing the triangle in a rectangle with sides B and H. Suppose the cut leaves us with a smaller isosceles triangle of height h and base b, and therefore area bh/2. If this is half the larger triangle we shall have bh/2 = BH/4. But the smaller and larger triangle are clearly similar, with the same angle at the centre of the cake, so that the ratios b/h = B/H are the same. These two facts combine to tell us that Bhh/2H = BH/4 and therefore hh = HH/2. Therefore the cut should be made so that H/h is the square root of 2, i.e. so that H = 1.4h approximately.

"Did you work all that out in the cafe before cutting the cake?" said one incredulous pupil with some emphasis.

Lesson 5.19

We began by asking whether today would have any special mathematical time in it. One pupil remembered that it was the 6th day of the 6th month of 2006, and from that point everyone realized quickly that at 6 minutes and 6 seconds past 6 a.m., we could write the time and date as 06: 06: 06 06/06/06.

Next we returned to the question of halving areas, with the intention of indicating, by a different argument from that used last time, that if *any* area were halved and the shape were kept similar, each linear dimension would be reduced by the reciprocal of the square root of 2. Last time we proved this just for an isosceles triangle, but by reasoning rather specifically tailored to that triangle.

Any area is given by a formula of the type A = kBC, where k is a constant and another area of the same shape would be a = kbc for the same k, and corresponding dimensions are B,b and C,c. For example, similar rectangles have corresponding lengths in the same ratio B/b = C/c = r, say, and therefore A = kbrcr = rra, so the area is halved (a = A/2) if rr = 2, and therefore r is the square root of two. In this example k = 1.

For similar isosceles triangles with bases B and b, and heights C and c, k = 1/2. The same argument applies, as we showed last week

For similar parallelograms with bases B and b, and heights C and c we have k = 1. The same argument proves that the area is halved if B/b and C/c are the square root of two.

The parallelogram can be used to show that the area of any triangle (not only an isosceles one) is half the base times the height, and hence it is halved when the linear dimensions of both are scaled by the square root of 2.

Lesson 5.20

We concluded the year with a topic which I had presented once before, in Lesson 1.5. It provided an attractive way to finish the course.

I handed to each pupil a pre-prepared circle of 13 cm diameter with 36 compass points marked on it, i.e. at every 10 degrees. The children were asked, using a pencil and straight edge, to draw a straight line across the circle in 18 different places as follows. Begin at each one of the numbered points in turn, for example 7, and draw a line to the point having twice that number, which will be 14 in this example. This will give 18 lines. There was a variety of quality in these attempts, some not starting accurately on the circle, or not finishing there; some using a pencil which was too thick; some very slow. But for the most part the idea was grasped.

Having used the first 18 starting points, next pupils were required to use the starting points 19 to 39 and from each draw a straight line, preferably in a different colour from the first 18, across to the point having double the starting number. For example, starting at 26 a line is to be drawn to 52, which is already actually numbered as 16 because 36 + 16 = 52.

The lines so drawn revealed that they enveloped a heart-shaped looking curve which is called a cardioid. The adjective cardiac was known to at least one pupil whose grandfather had experienced a heart attack. The cardioid is a famous mathematical curve. I showed how it could also be reproduced by the reflection of light from a torch off the cylindrical internal sides of a mug onto the base. Finally we took the mug outside, and showed that the rays of the Sun also reflected off the internal sides of the mug to form a cardioid on the internal base of the mug. I refrained from putting a circular disc of paper on the bottom of the mug to see if the focussed cardioid was bright enough to set the paper alight.

Lesson 6.1

As an introduction to some topics about prime numbers, we discussed what different meanings there could be for the word "prime". We identified *adjectives,* for example

"best", as in the "prime of life", and illustrated this by quoting Isaac Newton's remark that, between the ages of 20 and 25 , he said he was "in the prime of my age for invention" of mathematical theories; and also

"chief", meaning "most important", as in Prime Minister; and "ingredients" in the sense that "prime numbers" are the ones used to make up other numbers;

and also *verbs*, as in to

"prepare", e.g. a surface for painting by giving it a coat of "primer"; and "priming" a gun for firing.

Moving on to discuss **numbers**, we first agreed that 1 is such a special number that it would be best to leave it out of the discussion to follow about describing other numbers.

A **prime number** is divisible only by itself (and 1), and it has no other factors. So

2 and 3 are prime, but $4 = 2 \times 2$ is not.

In fact no other even numbers are prime, because they all have 2 as one factor.

So only some odd numbers are prime, but not all odd numbers.

E.g. 7 is prime, but $9 = 3 \times 3$ is not.

The pupils were asked to make a list of the primes between 10 and 50. Some were slower than others, and this was a useful exercise.

11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. They were asked to count them (11) as a check, which it did because the count revealed variations.

Lesson 6.2

An animated discussion developed about the values of studying various academic subjects, such as History, as well as Mathematics, and some provocative views were expressed. I judged it to be instructive to let the discussion run, for almost 45 minutes. One child was particularly emphatic and articulate. Global warming was touched upon. Eventually another child said "Let's do some maths". So we did.

I asked the children to evaluate

$2 \times 3 \times 7 \times 17 = 714$ (which they did by calculator) and then

$5 \times 11 \times 13 = 715$. These are two adjacent numbers, and the sum of their prime factors is

$2 + 3 + 7 + 11 = 29 = 5 + 11 + 13$, i.e. the same for both numbers. This is a very strange fact

Lesson 6.3

We verified that the product

714 x 715 = 2 x 3 x 5 x 7 x 11 x 13 x 17,

i.e. the product of those two adjacent numbers is the product of the first seven prime numbers.

A similar property holds in the simpler case of 5 and 6 = 2 x 3 so that 5 x 6 = 2 x 3 x 5, and also 2 + 3 = 5. Thus the product of 5 and 6 is the product of the first three primes.

There are only 26 pairs of consecutive numbers below 20000 which have the strange property that the sum of their prime factors is the same.

Next, on centimetre squared paper, we wrote 17 in the centre; then 18 to the right of it, then in an anticlockwise spiral 19, 20, 21, 22, 23, 24, 25 to complete the 3 x 3 square surrounding 17; then 26 to the right of 25, and in a 5 x 5 spiral 26,..., 41; then 42 to the right of 41, and in a 7 x 7 spiral 43,...65; then 66 to the right of 65, and in a 9 x 9 spiral 67,..., 97; and continued like this for four more such spirals.

Then we noticed that all the numbers in the south-west to north-east diagonal were the prime numbers

227, 173, 127, 89, 59, 37, 23, 17, 19, 29, 47, 73, 107, 149, 199, 257. Strange.

[I subsequently published a paper which proved formulae describing the numbers in these diagonals. This was called "Straight Sequences in a Spiralling Grid of Numbers". It was published in the journal Mathematics in School, Vol. 38, No. 2, March 2009, pp.15-17.]

Lesson 6.4

We noted that if the spiral process of last week were continued , we would get 359 and 289 before 227 in the SW corner, and 323 in the NE corner.

We discussed how we would actually check whether the numbers on this diagonal were prime. For example, we worked out the division of

73 by 2 [36.5], by 3 [24.3], by 5 [14.6], by 7 [10.4], by 11 [6...], by 13 [5...],

i.e. by a succession of primes. None of these quotients are whole numbers, so 73 is a prime number. We do not actually need to go beyond the point in the sequence where the quotient is smaller than the divider before we can draw this conclusion.

Such methods tell us that the whole diagonal from 227 down to 17 and up to 257 are primes. This raises the conjecture whether the whole diagonal consists of primes, but this is not so because we find that

289 = 17 x 17.

Next we looked at the question of whether the formula

n x n + n + 17 delivers a prime number for every n.

It does so for n = 0,1,2,3,….., 15 (in fact these results of the formula are the numbers in the SW - NE diagonal).

But for n = 16 the formula delivers 289 = 17 x 17.

So, just as with the diagonal of the spiralling numbers, this formula delivers primes for a long time before finally delivering a number which is not prime.

This fact warns against the drawing of too hasty plausible conclusions from quite convincing looking but unproved generalisations. We shall use this to emphasize the need for proof in mathematics.

Lesson 6.5

We began with a reminder that we previously looked at the formula

n x n + n + 17 = N

and found that for n = 0, 1, 2, 3, 4,….,15 the result N is a prime number,

But for n = 16, N = 16 x 16 + 16 + 17 = 289 = 17 x 17 which is not prime,

And for n = 17, N = 17 x 17 + 17 + 17 = 293 = 17 x 19 which is not prime.

This warns against a hasty conclusion, and emphasizes the need for proof of results in mathematics.

We looked at a second example, as follows. The formula

n x n + n + 41 = E is called Euler's formula.

It is found that

E is prime for $n = 0, 1, 2, 3, 4, \ldots\ldots, 39$ but

E is not prime for $n = 40$ because

$40 \times 40 + 40 + 41 = 1681 = 41 \times 41.$

So again a good guess is not a substitute for a proof.

I continued the theme of the desirability of proof with the following example.

Goldbach's Guess (1742)

This is a famous guess or conjecture which has never been proved. It says the following.

Every even number greater than 2 is the sum of two primes.

Examples are $8 = 5 + 3$, $10 = 5 + 3$, $12 = 7 + 5$, $14 = 7 + 7$, and so on.

Anyone who proves it will make their name.

Lesson 6.6

We recalled Goldbach's Guess, and the fact that it had never been proved, and then moved onto a famous result which could be proved, and by us, as follows.

Euler's Theorem.

There is no largest prime number.

Proof

We assume that there is a largest prime number , and then show that this assumption leads to a nonsense. This method of proof is called "reductio ad absurdum".

Suppose that there is a largest prime number, and denote it by P (because we do not know its value). Use it to construct another number, which we label Q, as follows:

$Q = (2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times P) + 1.$

No integer from 2 up to P divides into it, because there is always a remainder 1 when we try to do that.

So if Q is not prime, it must be divisible by a prime larger than P.

But if Q is prime, it is certainly larger than P.

These two conclusions together contradict the assumption that P is the largest prime.

So the assumption was wrong, so there is no largest prime, so there must be an infinite number of primes - they just go on and on.

Next we began to talk about Pythagorean Triples.

These are triples of whole numbers, like 3,4,5, or 5,12,13, which have the property that the sum of the squares of the first two is the square of the third one. That is

3 x 3 + 4 x 4 = 5 x 5 because 9 + 16 = 25 and

5 x 5 + 12 x 12 = 13 x 13 because 25 + 144 = 169.

Before I left the School two teachers told me how much the children had enjoyed the lesson.

Lesson 6.7

We began to compile a table of Pythagorean triples by checking, in turn, that the numbers in the following table all satisfied the equation (in which juxtaposition means multiplication here, to avoid ambiguity) $xx + yy = zz$.

x	y	z
3	4	5
5	12	13
8	15	17
7	24	25
9	40	41

These checks were done by calculator, squaring the first two numbers in the trio, adding, and taking the square root of the result.

Then, for the first two trios, the children were asked to draw triangles with those side-lengths, with ruler and compasses. That is, they drew a line of length 3 units (centimetres were agreed to be a convenient size of unit) and then, with compasses set at 4 and then 5 units, drew intersecting arcs centred on the two ends of the line. That intersection point gave the third corner of the triangle, whose sides they then drew with a ruler. And similarly for 5,12, 13.

The children then checked with a protractor that the angle opposite the longest side was, in both cases, a right angle of 90 degrees.

This proved by construction that each Pythagorean triple contained the lengths of the sides of a right angled triangle.

Finally we used the compasses to construct a right angle. This was done by drawing a straight line with a ruler, marking two points on it, using them in turn as centre for the compasses to draw intersecting arcs above and below the line, so that the two pairs of arcs intersected; and then joining the two intersection points with another straight line drawn with a ruler. The two straight lines intersected at right angles, by symmetry.

Some comments heard at the end of the lesson were as follows.

"That was extremely interesting."

"I think I need to go on a compass training course."

"Do you give courses about this in other schools?"

Lesson 6.8

We began with the sequence

2 8 4 16 8 24 32 16 64 32

and the children were asked to identify the number which does not fit the sequence.

This was rather readily seen to be 24, and the reason was given. That is, the sequence is constructed by multiplying by 4, and dividing by 2, in turn. 24 does not fit this rule.

Next I asked the children to construct two lines at right angles, using only a straight edge and the compasses. Performance was variable, and in general the children were not adept at using these tools.

Next I asked them to draw a right-angled triangle with ruler and compasses. Evidently they need more practise at such an exercise.

I asked them to label the side opposite the right angle z, the other two sides x and y, and to work out the square of z, and the sum of the squares of x and of y. We made a table of these values on the board, which the children copied.

The table showed that no child had found that the sum of the squares of x and y was the square of z (as it should be by Pythagoras' Theorem).

So we shall do this again, with me performing the exercise at the same time. It was obvious, for example, that the children needed to be shown individually how to hold the compasses (at the pivot, and not at the pencil and the point) when drawing a circular arc.

Lesson 6.9

We repeated the exercise of last week, with me doing it first this time, to show the children how to hold the compasses correctly, with one hand at the top near the pivot, and not with two hands at the bottom with one hand at the spike and the other hand at the pivot, which is likely to change the radius during the drawing.

The key point, having drawn a straight line with a ruler, was to draw two arcs from one centre on the line, and then to draw two more arcs from another centre on the line, such that the second pair of arcs intersected the first pair. Then we could join the intersection points by a straight line using the ruler. The two lines should then be at right angles. Even then the children often did not get a pair of orthogonal lines, as was rather obvious just visually. So more than one repetition was needed.

Then each child completed a right angled triangle, measured the sides x, y, and hypotenuse (opposite the right angle) z, and checked whether the sum of the squares of x and y was the square of z (Pythagoras Theorem). Mostly it was not because of inaccurate drawing, so we recorded the following table to describe the error.

Name	x	y	z	x^2 + y^2	z^2	(x^2 + y^2)/ z^2
CH	23	17	28	818	784	1.04
CM	45	58	60	5389	3600	1.49
R	67	68	97	9113	9409	0.97
J	40	30	50	2500	2500	1
S	40	56	68	4736	4824	0.98
H	17	30	33	1189	1089	1.09
I	24	48	53	2880	2809	1.003

Lesson 6.10

We actually proved Pythagoras' Theorem at last. I reminded the children that there are at least three ways of constructing a right angle, by using (a) compasses, (b) protractor, or (c) the corner of a sheet of paper. They were asked to choose a way of drawing two squares of the same size on a single page. They were able to do this, although the squares were a little small in a couple of cases. They were then asked to divide the sides of each square up into lengths x + y in two different ways, as follows.

(a) From one corner, following round the sides of the square, divide the each side up into lengths x and y as follows, where the corners of the square are A,B,C,D, and the successive division points along the sides are E,F,G,H as x,y,x,y,x,y,x,y. That is, AE = BF = CG = DH = x and EB = FC = GD = HA = y.

(b) The second sequence is x,y,x,y,y,x,y,x, so that AE = BF = GD = HA = x and EB = FC = CG = DH = y.

Then we divided up the two equal squares in different ways by drawing internal division lines in two different ways as follows.

(a) EF, FG, GH, HE and

(b) EF, EG, FH, GH.

Each identical big square ABCD then contains four identical right-angled triangles, having shorter sides x and y, and a hypotenuse z (say).When the triangles are imagined removed, we are left with two equal areas, which in the first case is a square of area z x z; and in the second case two squares of areas x x x and y x y.

This proves Pythagoras' Theorem. It is important that children see serious results properly proved. I think these children understood quite well what we were doing.

Lesson 6.11

We introduced the topic of angles in the sector of a circle. A semi-circle is half of the circumference of a circle. A sector of a circle is part of a circumference, in general either more than, or less than, a semi-circle. This is an exercise in the use of compasses to draw a circle, the selection of a semicircle and a sector of the circle, and then the construction of angles in the sector as follows.

In a semicircle a triangle is drawn using the diameter as base, and then any point on the circumference as the third point of the triangle. Every such triangle has a right angle (90 degrees) at that third point, no matter where it is on the circumference. The pupils drew several of these, as an exercise in verifying the fact.

If the sector is less than a semi-circle, then for any chosen pair of points A and B on the circle which define the ends of the sector, and any third point C on the sector between them, the angle ACB is always the same, and greater than 90 degrees because the sector from A round to B is less than a semicircle. The children drew several of these, and verified the fact.

If the sector is greater than a semi-circle, then for any chosen pair of points A and B on the circle which define the ends of the sector, and any third point C on the sector between them, the angle ACB is again always the same, but this time less than 90 degrees because the sector from A round to B is greater than a semicircle. The children also drew several of these, and verified the fact.

There were marked differences in the dexterity exhibited by the children in this work.

Lesson 6.12

This was a brief session, because the children had other obligations preceding it.

The children were asked to draw any triangle, with no special properties such as being right-angled. We then set out to find the orthocentre, which is the point where the three altitudes intersect. This required the children to construct the three altitudes in turn. Each altitude was to be drawn by ruler and compass construction, dropping a perpendicular from a vertex to the opposite side. This was not done well, as the children could not use the ruler and compass with dexterity. This exercise will have to be repeated.

Lesson 6.13

We began with the definition of 180 degrees as the angle on a straight line.

I then demonstrated how to use the compasses and ruler to construct a perpendicular to a straight line, by drawing intersecting arcs from two points on the line, and then joining the two intersecting points with a straight line which must, by symmetry, be perpendicular to the starting line.

Next we agreed that the sum of the interior angles in a triangle is 180 degrees. I introduced a construction using isosceles triangles which I had seen on the previous Sunday in a visit to Lord Carrington's garden at Bledlow, where there are some sculptures. This is a tiling pattern using slates shaped as isosceles triangles with angles of 36, 72 and 72 degrees.

The pupils drew the pattern of five such slates laid in an array with their 36 degree vertices all touching, and the longer sides all touching each other or (for the two end ones) making a straight line, from the centre of which the other triangles radiated outwards. Then on the shorter side of each triangle another triangle was put down with its short side touching the first one. Two more triangles were then put down with longer sides touching the ones just put down. In this way three triangles were used to make a double-sized isosceles shape of four triangles altogether; then five more were adding to treble the size of the first one.

This was repeated for the other four of the first set of five triangles. So we achieved a sun-burst shape of five starting triangles, with 15 in the next ring, the 25, then 35, and so on.

Lesson 6.14

I introduced the famous topic of The Nine-Point Circle.

This result is a property of any triangle. So we had a brief discussion of certain special triangles, in order to exclude them. The children were asked to define an equilateral triangle (all three sides the same length), then an isosceles triangle (only two sides the same length), and then a right-angled triangle (one angle is 90 degrees). Next we agreed to exclude triangles which have an obtuse angle, in order to make the drawing easier.

So with these cases excluded, the children were asked to draw a triangle with vertices A,B,C. We now stated the Theorem, that in any triangle,

(a) the mid-points E, F, G of the sides BC, CA, AB,

(b) the feet H,J,K of the altitudes AH, BJ, CK

and (c) the midpoints R,S,T of the lines joining the corners A,B, C to the orthocentre O

all lie on a circle.

Just to state this required a lot of discussion of the ingredients.

For example, to find the midpoint E of the side BC by using the compasses, intersecting arcs are drawn with the equal radius, centred on B and then C. A line joining the intersection points then cuts BC orthogonally at its midpoint E.

The concept of altitude required discussion, to establish that altitude is always measured straight upwards, i.e. at right angles to the base line, and not at an inclination. So next week we shall do some compass work to construct altitudes.

Lesson 6.15

We continued with the geometry directed towards the nine-point circle. I asked the children to draw an obtuse-angled triangle, on half an A4 page. Even this some were scarcely able to do, because they assumed a freehand drawing would be adequate, without use of a ruler. Then I asked them to use compasses to construct the perpendicular bisectors of two of the sides. The compasses were needed to draw intersecting arcs, with the compass point at each end of a side in turn. The join of these intersecting points would be the perpendicular bisector of the side. The intersection point of two of these bisectors would give the centre of the circumcircle of the triangle.

The dexterity shown with the compasses was still poor on the whole, and we shall return to this topic.

Lesson 6.16

This session was devoted to more exercises with the use of compasses, as last week.

Lesson 7.1

As an introduction to some topics about prime numbers, we discussed what different meanings there could be for the word "prime". As in Lesson 6.1, but now for different children, we identified *adjectives,* for example

"best", as in the "prime of life", and illustrated this by quoting Isaac Newton's remark that, between the ages of 20 and 25 , he said he was "in the prime of my age for invention" of mathematical theories; and also

"chief", meaning "most important", as in Prime Minister; and

"ingredients" in the sense that "prime numbers" are the ones used to make up other numbers;

And also *verbs*, as in to "prepare", e.g. a surface for painting by giving it a coat of "primer"; and "priming" a gun to have a good basis for firing.

Moving on to discuss **numbers**, we first agreed that 1 is such a special number that it would be best to leave it out of the discussion to follow about describing other numbers.

A **prime number** is divisible only by itself (and 1), and it has no other factors. So

2 and 3 are prime, but $4 = 2 \times 2$ is not.

In fact no even numbers except 2 are prime, because they all have 2 as one factor.

So only some odd numbers are prime, but not all odd numbers.

E.g. 7 is prime, but $9 = 3 \times 3$ is not.

The pupils were asked to make a list of the primes between 10 and 50. Some were slower than others, and this was a useful exercise.

11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. They were asked to count them (11) as a check, which it did provide because the count revealed variations.

Lesson 7.2

We recalled some of last weeks' work by way of revision.

I then asked the children to evaluate

2 x 3 x 7 x 17 = 714 (which they did by calculator) and then

5 x 11 x 13 = 715. These are two adjacent numbers. Call this Fact A. We then noticed that the sum of their prime factors is

2 + 3 + 7 + 11 = 29 = 5 + 11 + 13, i.e. the same for both numbers.

This is a very strange fact. Call it fact B.

Lesson 7.3

We verified that the product

714 x 715 = 2 x 3 x 5 x 7 x 11 x 13 x 17,

i.e. the product of those two adjacent numbers is the product of the first seven prime numbers. Call this Fact C.

A similar property of Facts A, B,C holds in the simpler case of 5 and 6 = 2 x 3 so that

5 x 6 = 2 x 3 x 5, and also 2 + 3 = 5.

There are only 26 "rare pairs of consecutive numbers" below 20000 which have the strange property that the sum of their prime factors is the same. (Fact B).

Next, on centimetre squared paper, we wrote 17 in the centre; then 18 to the right of it, then in an anticlockwise spiral 19, 20, 21, 22, 23, 24, 25 to complete the 3 x 3 square surrounding 17; then 26 to the right of 25, and in a 5 x 5 spiral 26,..., 41; then 42 to the right of 41, and in a 7 x 7 spiral 43,...65; then 66 to the right of 65, and in a 9 x 9 spiral 67,..., 97; and continued like this for four more such spirals.

Then we noticed that all the numbers in the south-west to north-east diagonal were the prime numbers

227, 173, 127, 89, 59, 37, 23, 17, 19, 29, 47, 73, 107, 149, 199, 257. Strange.

Lesson 7.4

We digressed, for one week, from the topic of last week

We introduced a stalk of Brussels sprouts bought from a Farmer's Market in Woodley, and noticed that the individual sprouts are aligned in two spirals. One set of three distinct spirals circle the stem from bottom to top in an anticlockwise sense, and the same set of individual sprouts also forms a set of five distinct spirals circling the stem in the opposite clockwise sense from bottom to top. It took time to examine the pattern of sprouts, and for the children to be convinced of the spiral pattern. Beforehand I had wound strands of differently coloured wool between the sprouts, three strands in one direction and five strands in the other, to emphasise this double spiral pattern.

After we had agreed the pattern I told the children the rule of repeated addition of the last two numbers, starting with 1,1,2,3..., to generate the famous Fibonacci series

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233,...

They calculated the series, and I asked them if they had noticed other spiral patterns in nature. Eventually we talked about pineapples and fir cones, and agreed that they had spiral constructions too, although possibly different from the sprouts.

We will return to this topic later, but it was timely to introduce it before I ate the sprouts at home.

Lesson 7.5

After the Fibonacci digression, we returned to the topic being treated in Lesson 7.3. We reviewed the spiral process of constructing numbers of which many were found to be prime, and we noted that if the spiral process were continued , we would get 359 and 289 before 227 in the SW corner, and 323 in the NE corner.

We discussed how we would actually check whether the numbers on this diagonal were prime. For example, we worked out the division of

73 by 2 [36.5], by 3 [24.3], by 5 [14.6], by 7 [10.4], by 11 [6...], by 13 [5...],

i.e. by a succession of primes. None of these quotients are whole numbers, so 73 is a prime number. We do not actually need to go beyond the point in the sequence where the quotient is smaller than the divider before we can draw this conclusion.

Such methods tell us that the whole diagonal from 227 down to 17 and up to 257 are primes. This raises the conjecture whether the whole diagonal consists of primes, but this is not so because we find that

289 = 17 x 17. Not only that, 289 is actually a square number.

This warns us against hasty guessing about what to expect in mathematics, and it emphasizes the need for proofs of everything.

Next we looked at the question of whether the formula

$N = n \times n + n + 17$ delivers a prime number for every n.

It does so for $n = 0,1,2,3,\ldots, 15$ (in fact these results of the formula are the numbers in the SW - NE diagonal).

But for n = 16 the formula delivers 289 = 17 x 17, i.e. the square number noted above.

So, just as with the diagonal of the spiralling numbers, this formula for N delivers primes for a long time before finally delivering a number which is not prime.

This fact again warns against the drawing of too hasty plausible conclusions from quite convincing looking but unproved generalisations.

Lesson 7.6

We began with a discussion of the so-called Euler's formula, named after the very famous mathematician Leonard Euler, 1707 - 1783. It is the definition

$E = n \times n + n + 41$ for any integer n.

The children were asked to use their calculators to evaluate E for successive values of

n = 1,2,3,4,5,6,7,8,9,10. Some were quicker than others. Was the result prime or not? It was eventually agreed that such values of E were all prime. The faster ones continued for

n = 11,12,13,14,15,16,17,18,19,20. All these E were prime as well. After more discussion and calculation I eventually told them that E would be prime certainly for n all the way up to n = 39.

Then we checked for n = 40, which gives E = 40 x 40 + 40 + 41 = 1681.

Then I asked them to work out 41 x 41 = 1681 too. Therefore such E is not prime, because it is the product of two numbers. In fact it happens also to be a square number.

So after extended debate and calculations, all worthwhile, we agreed that although it would have been tempting to assume from the first set of extended sample calculations that E is prime for every n, this is not true because we found a counter-example at n = 41.

The lesson is: don't jump to hasty conclusions in mathematics. Every guess, however intelligent, has to be proved.

Next we moved on to Goldbach's Guess, proposed in 1742. It states that

Every even number greater than 2 is the sum of two primes.

Examples are 8 = 5 + 3, 10 = 7 + 3, 12 = 7 + 5, 14 = 7 + 7, and so on.

Nobody has proved this "conjecture" (fancy word for "guess").

Anybody who does might get a cash prize. That does happen in mathematics. I know somebody who won a very large cash prize for proving something called Fermat's Last Theorem.

Thus ended an animated lesson in which the slower ones were kept in the game by waiting for them, even though the quicker ones were quite quick.

Lesson 7.7

The children were asked to make a list of the primes up to 100, which required some debate about certain numbers. We observed that there are more between 1 and 50 than there are between 50 and 100. We discussed whether the number in each succeeding list of 50 would diminish to zero, or whether there would always be some primes in a group of 50 numbers far down the line. Then we proved the following.

Euler's Theorem.

There is no largest prime number.

Proof

We assume that there is a largest prime number , and then show that this assumption leads to a nonsense. This method of proof is called "reductio ad absurdum".

Suppose that there is a largest prime number, and denote it by P (because we do not know its value).

Use it to construct another number, which we label Q, as follows:

Q = (2 x 3 x 5 x 7 x 11 x ...x P) + 1.

No integer from 2 up to P divides into it, because there is always a remainder 1 when we try to do that.

So if Q is not prime, it must be divisible by a prime larger than P.

But if Q is prime, it is certainly larger than P

These two conclusions together contradict the assumption that P is the largest prime.

So the assumption was wrong, so there is no largest prime, so there must be an infinite number of primes - they just go on and on.

Lesson 7.8.

We changed topic in order to talk about graphs. This was motivated by reference to the televising of cricket and tennis, and the desire to understand how the quoted ball speeds are obtained by side-on television cameras, and what those quoted speeds mean.

We discussed the relation between miles and kilometres, first by looking at an Ordnance Survey map, which, as it turned out, did not give a clear combined scale. So we agreed that

5 miles = 8 kilometres.

I used this fact to get the children to plot, on centimetre graph paper, the straight line relation between miles and kilometres, joining the two points

$k = 0, m = 0$ and $k = 8, m = 5$.

We used this to work out that 80 m.p.h. = 8 x 80/5 k.p.h. = 128 k.p.h.

and some similar equivalences.

Lesson 7.9.

We worked on the graph relating the Fahrenheit and Centigrade temperature scales. This is a straight line joining the freezing point 0 C = 32 F and boiling point 100 C = 212 F of water, extending to ice and steam on either side. We have more to do to establish this clearly.

Lesson 7.10

We continued working on the graph relating miles (m) to kilometres (k), and expressed it as $5k = 8m$ and equivalents such as $k = (5/8)m$ and $m = 1.6k$.

We then returned to the graph relating Centigrade to Fahrenheit measures of temperature, and plotted it after carefully selecting the location where the axes would cross. Then we derived the equation

$(F - 32)/C = (212 - 32)/100$, i.e. $F = 32 + 1.8C$.

We then verified that the temperature where both scales have the same value is $F = C = -40$, both via the equation and via the intersection of the two straight lines

$F = 32 + 1.8C$ and $F = C$ on the graph.

Lesson 7.11

We returned to the spiral grid discussed in Lesson 7.3, but this time with 0 as the starter instead of 17.

We looked closely at the pattern of numbers in the SW diagonal, and perceived it to be

W 0, 6 = 2x3, 20 = 4x5, 42 = 6 x7, 72 = 8x9,.....

in the boxes labelled w out from the centre

w 0, 1, 2, 3, 4,....

Thereby, after some discussion of details, we deduced the general formula

$W = 2w(2w + 1)$ when the starter is 0, and

$W = s + 2w(2w + 1)$ when the starter is s (for example $s = 17$ as previously).

This is the result of a piece of research which was unknown when we last discussed this topic. The children displayed a good understanding of what was going on, and could think ahead to generalise as required.

Lesson 7.12

With only four pupils present we repeated the lesson given last week and added to it a discussion of the NW diagonal.

Using $x = 0, 1, 2, 3, 4, \ldots$ to label the boxes out from the centre, we perceived that their occupants are

$X = 0,\ 4 = (2x1)(2x1),\ 16 = (2x2)(2x2),\ 36 = (2x3)(2x3),\ 64 = (2x4)(2x4),\ldots$

So that in general it is not difficult to perceive that in the xth box, $X = (2x)(2x)$ when $s = 0$. One boy had spotted this last week, and was eager to point it out again.

Lesson 7.13

We summarized what we had previously done for the SW and NW diagonals, and then moved on to discuss the NE diagonal. The boxes were labelled $y = 0, 1, 2, 3,\ldots$
Their contents were labelled Y and noticed to be

$Y \quad 2 = 6 - 4 = 6 - 4x1,\ 12 = 20 - 8 = 20 - 4x2,\ 30 = 42 - 12 = 42 - 4x3, \ldots.$

Therefore in general, recalling the SW diagonal, $Y = 2y(2y + 1) - 4y = (2y)(2y) - 2y$,

$Y = 2y(2y - 1)$ when $s = 0$, and in general, $Y = s + 2y(2y - 1)$.

Much discussion took place during this development.

Next we began to look at the SE diagonal, labelling the boxes $z = 0, 1, 2, 3, \ldots$

In them we find (when $s = 0$) the numbers

Z(z)	0,	8,	24,	48,	80,	120,	168,	... and simplifying
Z(z)/8	0,	1,	3,	6,	10,	15,	21,	.....

Then we noticed that

Z(z)/8 - z	0,	0,	1,	3,	6,	10,	...... so that

$Z(z)/8 - z = Z(z - 1)/8$ because the entries have been moved to the next box.

Lesson 7.14

After a review of the formulae for the SW, NW and NE diagonals, we continued with the SW diagonal, and followed out the consequences of repeating

$Z(z) = 8z + Z(z - 1) = 8z + 8(z - 1) + Z(z - 2) = 8z + 8(z - 1) + 8(z - 2) + Z(z - 3)$

$= 8[z + (z - 1) + (z - 2) + \ldots + (z - (z - 1))] + Z(z - z)$

$= 8[1 + 2 + 3 + \ldots + (z - 1) + z] + Z(0)$

but $Z(0) = 0$ and $[1+2+3+\ldots+(z-1)+z] + [z+(z-1)+\ldots+3+2+1] = z(z+1)$.

So $Z(z) = 4z(z+1)$. Much leading by explicit calculations was required, but worked.

Lesson 7.15

We began to study some geometry, and in particular, the use of geometrical instruments, starting with the compasses. I asked the children to draw a set of concentric circles, with radii 2 cm, 4 cm, and 8cm. They had genuine difficulty doing this accurately, and considerable time was spent trying. I had to explain and demonstrate that one just needs to hold the instrument lightly, and slightly inclined to the vertical as one rotates it, otherwise the set radius may change, and the desired circle will not be achieved.

Next I asked them to use the compasses to draw the flower pattern of six petals. That is, after drawing a circle, keep the compasses at the same radius, put the point on the circumference just drawn, and draw an arc inside the first circle, from one place on the circumference to the another. Then use the ends of the arc as centres for two more interior arcs, each ending on the circumference. Doing this six times in all delivers the petals of the flower, having a six-fold symmetry. All this is with the same radius, and therefore distinct from the first exercise above in which the radius was varied.

Lesson 7.16

On 7th June 2008 there had been a two-page spread feature in The Independent called "Maths: does it matter?", expressing concern about the supply of competent teachers and trainees, and how the subject should be taught. It included a set of 30 diverse questions, which I judged would be well worth putting to my pupils, so I began to do that with the first ten questions in arithmetic, algebra and geometry. The children tried them in sets of five questions, and after each set I broke off to discuss progress and how the questions should be tackled. The questions were just within the scope of the children, and the discussions proved to be informative. Example are "Write the fraction five-twentieths as simply as possible" , "Divide -100 by -10" and "Round 0.348 to two decimal places". Several pupils found these to be testing.

Lesson 7.17

We continued with the second set of ten questions from "Maths: does it matter?" These were again worthwhile exercises in arithmetic, algebra and geometry, but framed in an unconventional way. The children were able to do them on the whole, but not always readily, and there was significant discussion led by me prompting the pupils. For example, what is 20 percent of 700? And multiply 2x by 5y. Write your answer as simply as possible.

Lesson 7.18

We continued with the third set of ten questions from “Maths: does it matter?”, involving arithmetic, algebra, geometry and probability, framed in an unusual way. The discussions were again worthwhile. For example, find an approximate answer to 3009 divided by 599. If a racing pigeon can fly at 90 km/h, how far does it go in one minute? If the probability of taking a blue ball out of a bag is five-sixths, and 20 of the balls are blue, how many are not blue?

Lesson 8.1

We had a prolonged discussion about what would be a suitable symbol for the inequality sign in mathematics. The class to were encouraged to think of traffic signs, and how they are designed when it is required to say that something should *not* be done, in particular by putting a diagonal bar across a sign to indicate a negative. The class then offered their own suggestions, for example by putting a cross over an equals sign. Nine different signs were suggested by the nine pupils, but only one was the conventional correct one of the single slanting line across an equal sign.

Lesson 8.2

We began a discussion of equations like $n - 1 = 0$, and how to solve them for the unknown n. We introduced the idea that an equation is a balance, and that it can be solved if one always does the same thing to both sides. In this example, instead of just guessing that $n = 1$ is the answer, we see that it can be obtained by adding 1 to both sides. Thus $n - 1 + 1 = 0 + 1$, and therefore $n = 1$. Keeping a balance like this is like washing your hands: you have to do the same thing to both hands (sides) for the process to work successfully.

We followed this up with further examples, such as

$n + 1 = 0$, which implies $n + 1 - 1 = 0 - 1$ so that $n = - 1$.

$3n = 12$ implies $3n/3 = 12/3$ so that $n = 4$.

$n/5 = 2$ implies $5 \times n/5 = 5 \times 2$ so that $n = 10$.

It can also happen that the unknown n appears twice in the starting equation, such as $3n - 4 = 2 + n$. Then we may have more steps to do, but still we do the same thing to both sides thus: $3n - 4 - n + 4 = 2 + n - n + 4$, so that $2n = 6$ and $n = 3$.

Lesson 8.3

In previous weeks I had been asked whether 1 is a prime number or not, so I set out to answer this. But first, as an introduction to prime numbers, we identified at least eight different meanings there could be for the word "prime". These included, for example:

"best", as in the "prime of life", and illustrated this by quoting Isaac Newton's remark that, between the ages of 20 and 25 , he said he was "in the prime of my age for invention" of mathematical theories;

"chief", meaning "most important", as in Prime Minister;

"ingredients" in the sense that "prime numbers" are the ones used to make up other numbers;

"prepare", e.g. a surface for painting by giving it a coat of "primer"; and "priming" a gun to have a good basis for firing; and in "priming" an engine;

"prime" cut of beef;

"primary" colours;

"premier" league of footballers.

Moving on to discuss **numbers**, I stated the standard definition that

a **prime number** is divisible only by itself and 1, and it has no other factors. So

2 and 3 are prime, but $4 = 2 \times 2$ is not. On the basis of this definition, taken literally as it is meant to be, 1 is certainly a prime number. It is divisible by itself and 1, without leaving any remainder.

In fact no other even numbers are prime, because they all have 2 as one factor.

So only some odd numbers are prime, but not all odd numbers.

E.g. 7 is prime, but $9 = 3 \times 3$ is not.

I went round the 11 members of the class, asking them each to name a prime number; and then again in the opposite direction, asking them to name a number which is not prime.

Lesson 8.4.

I introduced a cutting from The Independent for 19th November 2008, which said that 10% of the 11-year-old cohort (about 30 000 pupils) had the mathematical ability of 7-year-olds. It offered a "mental arithmetic test for 10-year-olds", which I tried out on my class. Nine questions, 3 to be answered in 5 seconds, 3 in 10 seconds and 3 in 15 seconds.

My pupils performed reasonably well, but not perfectly.

5 secs: how many 100s in 1000; subtract 21 from 40; 8 x 8 = ?

10 secs: double 150 and then double the answer; what is halfway between 50 and 80? Add £1.20 to £2.78.

15 secs: 90 + 110 + 120 = ? If N/2 + 10 = 35, what is N? 75 x 20 =?

I then introduced a Greek cauliflower, which I had never seen before until I bought it in Woodley Farmer's Market two days before for 80p. It had a double spiral structure, with 8 spirals in one direction and 13 in the other. I introduced the Fibonacci series 1,1,2,3,5,8,13,21,34,55,89,....which we discussed. Then we observed that the Greek cauliflower illustrated the Fibonacci pair 8,13 not only on the most obvious main scale, but that the individual parts which made up the spirals were all similar to the whole. Thus the Fibonacci pair was illustrated on a sequence of diminishing scales in the structure. Only the first three scales were big enough to see.

Lesson 8.5.

We began with The Medicine Problem, a real-life problem. Each day I have to take 5.5 mg of medicine, in the form of pills which are obtainable in two strengths: A has 3 mg and B has 1 mg. I deal with this by choosing to take 6 mg one day and 5 mg the next day, thus averaging 5.5 mg over every two days. How many pills of each sort do I need in a 28-day supply?

After some debate the children, with correct suggestions from several of them, were able to see that the following strategy would work. Every two days I need the following breakdown of

11 mg = 2 x 3 + 1 x (3 + 1 + 1) = 2 x A + 1 x (A + B + B) = 2A + A + 2B

= 3A + 2B.

So every 28 = 2 x 14 days I need 14 x (3A + 2B) = 42A + 28B.

So I need 42 of A and 28 of B.

Check: 28 x 5.5 = 140 + 14 = 154mg.

42A + 28B = 42 x 3 + 28 x 1 = 154mg.

Next I returned to the topic of Fibonacci numbers, which had emerged last week in the context of the Greek cauliflower.

We first labelled the terms of the "forward" series as

F1, F2, F3, F4, F5, F6, F7 , F8, F9, F10, ...(but using subscripts), standing for

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... We remembered that the formula is

$F6 = F4 + F5$, for example, and generally $F(n + 2) = F(n + 1) + F(n)$.

Next we asked how to rearrange this formula to calculate the next number **backwards.**

This requires $F(n) = F(n + 2) - F(n+1)$.

That is, we subtract the next two forward numbers. We can use this to extend the Fibonacci series back into negative numbers as follows.

$F0 = F2 - F1 = 1 - 1 = 0$, $F(-1) = F1 - F0 = 1 - 0 = 1$,

$F(-2) = F(0) - F(-1) = 0 - 1 = -1$ and so on, so that the Fibonacci series on the negative sideof 0 becomes

$F(-7) = 13$, $F(-6) = -8$, $F(-5) = 5$, $F(-4) = -3$, $F(-3) = 2$, $F(-2) = -1$, $F(-1) = 1$, $F(0) = 0$.

Notice that every third number in the Fibonacci series is even.

Lesson 8.6.

We revisited the discussion of the Fibonacci sequence, including the extension into negative numbers. We recalled the historical origin of the sequence in the book Liber Abaci published in Italy in the year 1202 A.D. by Fi Bonacci, the son of Bonacci.

We exhibited a stick of Brussels sprouts, bought from Woodley Farmer's Market. The children counted the number of distinct spiral arrangements of the individual sprouts on the stick, and found there to be 3 spirals in one direction and 5 spirals in the other direction. These are two adjacent numbers in the positive Fibonacci sequence, which is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

We then returned to the topic of solving equations, begun in Lesson 8.2, and solved the equation $n \times n + 2n = 0$ by writing it as $n(n + 2) = 0$, so that $n = 0$ or $n = - 2$, or both. This involved considerable discussion of the importance of doing the same thing to both sides of the equation, to keep balance.

Lesson 8.7.

We set out to find the size of the Sun, but first I told the children that if they looked into a clear night sky about an hour or more after sunset, after it had become dark, they would see a small bright object. What is it? Venus, because that is quite near the Sun, and it would show before the other stars.

I asked them to name all the planets, in order of distance from the Sun. Eventually we did get the names of nine planets, but not necessarily in exactly the right order.

Then I posed a mathematical problem, which arose when I was driving to Cambridge two weeks ago on a foggy day. It was possible to see the Sun's disc, and to look at it without danger to my eyesight because of the fog. It looked to be the same size as a full Moon, but we know that it is not really so, because it is much further away.

However, this same size effect suggests that we can use the mathematical idea of similar triangles to find the size of the Sun, if we use some data about the Moon which can be found in an atlas, as follows.

So we draw a long thin isosceles triangle, and mark along the long sides the labels m and s for the distances, of the Moon and Sun respectively, from my eye at the vertex. We also mark the diameters M of the Moon and , at the base of the triangle, S of the Sun.

The fact that we now have two similar triangles with the same angle at the vertex expresses the fact that the ratio diameter/distance is the same for both Sun and Moon:

$S/s = M/m$, which we can also write as $S/M = s/m$, and again as $S = sM/m$.

The atlas tells us that $M = 2172$ miles and $m = 239000$ miles. The value of m is actually slightly variable because the Moon's orbit round the Earth is an ellipse and not a circle, but this variation of m is small and we ignore it in our approximation.

We also know that $s = 93\ 000\ 000$ miles. So from the equation

$S = sM/m = 93\ 000\ 000 \times 2172 / 239000 = 93000 \times 2172 / 239$ miles.

We can approximate this by $S = 93000 \times 2000 / 200 = 930\ 000$ miles without much error. So the diameter S of the Sun is nearly a million miles. And
$S/M = s/m = 93000/239 = 100000/250 = 400$ approximately.

Lesson 8.8

We returned to the topic of equation solving first treated in Lessons 8.2 and 8.6.
We had a long discussion of the examples

n x n + 2n = 0 and n x n + 2n - 3 = 0, which can be solved by factorising as

n(n + 2) = 0 and (n + 3)(n - 1) = 0. These implied

n = 0 or n + 2 = 0 in the first case, and n + 3 = 0 or n - 1 + 0 in the second case.

By doing the same thing to both sides (like washing your hands) we then found the solutions as

n = 0 or n = -2, and n = -3 or n = 1. We noticed that these solutions can all be expressed as particular pairs of numbers occupying the real line.

Lesson 8.9

We revisited the idea of displaying the familiar numbers on the real line, illustrated by the solutions of equations such as n x n = +1, which is satisfied by both n = +1 and by n = -1. Then we checked that the solutions of n x n = +4 (n = +2 and -2), and of n x n = +9 (n = +3 and -3) and so on also provide examples of positive and negative numbers lying on the real line.

Next we raised the question of what solutions there might be to the equation n x n = -1. Evidently there are no real numbers n which satisfy this equation. Mathematicians have therefore invented the idea of a so-called *imaginary* number called the square root of -1 denoted by i.

This allows us to write the solution of n x n = -4 as 2i, and that of n x n = -9 as 3i, and so on.

We showed how to display these imaginary numbers along a second axis on the page, perpendicular to the real line, called the imaginary axis, and intersecting the real line at the origin.

Next we introduced the idea of *complex* numbers of the form a + ib, such as 2 + 3i and 4 + 2i. We showed how to plot them on the plane, using horizontal (real) and vertical (imaginary) axes, and then how to add them according to the rule

(2 + 3i) + (4 + 2i) = 6 + 5i, and how to display this addition on that *complex plane*.

Lesson 8.10.

I showed the pupils a stick from which all the Brussels sprouts had been removed, and on which I had marked, with two differently coloured pens, one clockwise spiral and one anticlockwise spiral. This revealed clearly that there were, in all, two distinct spirals in one direction, and three distinct spirals in the other direction.

This was a tangible example of a pair 2, 3 of adjacent Fibonacci numbers in nature, from the positive sequence 0 1 1 2 3 5 8 ... which we had discussed previously.

Then we returned to the topic of complex numbers, by way of revision. We showed how the real line can be regarded as part of the complex plane, and plotted some complex numbers on it, such as 2 + 3i and 4 + 5i and -2 -2i. We showed how these can be added together to give further points on the complex plane, such as

$$4 + 5i + 2 + 3i = 6 + 8i.$$

Lesson 8.11

I handed out graph paper, and the pupils drew the Real Line horizontally across the middle, and then the Imaginary Line vertically down the middle. This put us into position to plot complex numbers on the Complex Plane.

We plotted 2 + i5 and 3 + i4. Then we added them together to give

$$(2 + i5) + (3 + i4) = 5 + i9.$$

We plotted that point on the complex plane, and observed that it is at the opposite corner of a parallelogram which starts from the origin.

Next we discussed subtraction of complex numbers, via the example

$$(3 + i4) - (2 + i5) = 1 - i.$$

Next we discussed multiplication of complex numbers, via the example

$$(3 + i4) \times (2 + i5) = 3 \times 2 + 3 \times i5 + i4 \times 2 + i4 \times i5$$

$$= 6 + 15i + 8i + 20i \times i = 6 + 23i - 20 = -14 + 23i.$$

This last term required much debate about what i x i means, and eventually we agreed that it is -1. This was a very animated lesson, with the novelty appreciated by many.

Lesson 8.12

We concentrated on the problem of how to divide a pair of complex numbers, such as

$$(3 + i4)/(2 + i\,5) = (26 - i7)/29$$

which took the whole lesson to discuss, because of the various reminders that were required.

We also discussed how an income which had both pounds and dollars could be represented on two axes, with £ horizontally and $ vertically.

Lesson 8.13

I introduced an Ordnance Survey map of Ysgyryd Fawr, a mountain just north of Abergavenny in South Wales, which I had climbed the previous Saturday. It is called the Skirrid in English.

I wanted the class to calculate a reasonable approximation to the actual distance covered. This was viable via the facts that the spot heights at the beginning and the end of the walk were shown on the map to be 194 and 486 metres, so the vertical part of the climb was 292 metres; and the 1 kilometre grid lines on the map gave the horizontal scale, showing that the horizontal distance travelled was 2 kilometres or 2000 metres.

So we were able to conclude that the actual walk covered, approximately, the hypotenuse of a right-angled triangle of length d (say) in metres, such that $2000 < d < 2292$.

To work out d we needed a very famous theorem in mathematics called Pythagoras' Theorem, that in a right-angled triangle, the square of the hypotenuse is the sum of the squares of the other two sides.

We proved this Theorem, by subdividing two equal squares in different ways, removing four equal triangles from each, and then comparing the areas of the residual squares in the two starting squares. We shall use this next week.

Lesson 8.14

We continued with the mountain problem, and approximated it with a right-angled triangle of sides 2000 metres horizontally and 400 metres vertically. So to estimate the actual distance walked we needed to calculate the length T (say) of the hypotenuse by Pythagoras Theorem. We deduced that the square of T would be

$4000000 + 160000 = 4160000$, and the square root of this would be

$T = \text{sqrt}(416) \text{ x } \text{sqrt}(10000) = \text{sqrt}(4) \text{ x } \text{sqrt}(10000) \text{ x } \text{sqrt}(104) = 2 \text{ x } 100 \text{ x } 10.2$

approximately, which is $T = 2040$ metres. So this is a reasonable approximation for the actual length of the walk, allowing for the fact that we travelled vertically as well as horizontally, but not allowing for the fact that the individual steps were not up a smooth incline.

Next I presented the children with a pair of right-angled triangles with side lengths 3, 4, 5 and 5, 12, 13, joined together so that they had the side of length 5 in common. The lengths 5 = a (say) and 12 = b(say) were concealed.

The children were asked to find b, by two uses of Pythagoras's Theorem which we had proved last week. They would have to find a = 5 first, and then b = 12.

This took a long time, a lot of debate, some of it not as much to the point as might have been hoped, before the right answer was achieved.

Lesson 8.15

I introduced a new topic, called "Supermarket Offers: Deal or No Deal?" I had previously noted some price labels in a supermarket, and found that some were decidedly ambiguous in what they seemed to offer. Having copied down the precise and full contents of several such labels, I reproduced that information to illustrate some ambiguities where they existed.

In some cases genuine savings per item could be made if more than one were bought. In other cases I concluded that no saving was actually being offered, although the intention was to invite the shopper to believe that it was. Such labels did no more than state a piece of arithmetic which the shopper already knew before entering the store.

We discussed the following four examples.

Passion fruit was priced at 0.69 each, and the offer was "2 for 1.00". Clearly this did deliver an actual saving of 2 x 0.69 - 1.00 = 0.38 per *pair*. **Deal.**

Lychees were priced at 1.99 each, and the offer was "50% extra", with 150g deleted and 225g written next to it, both in print. This is **no deal**: there is no actual saving being offered, because the price is fixed (at "1.99 each"). We already knew that 225 is 150% of 150 before we entered the shop.

Organic raspberries 125g were priced at 1.99, and the offer was "Save 1.00" with, printed next to it, 2.99 and 2.39 both crossed out in print with a sloping line. This is **no deal** because the price is fixed. We already knew that 2.99 - 1.99 = 1.00 before we entered the shop. If there had been a different price on a previous day, that was irrelevant because that was not this day.

In this case there was extra printed information: 15.92 kg; this merely seems to mean that there is a notional background price of
1.592 per 100g = 1.25 x 1.592 = 1.99

which is the already stated price, so nothing new is being stated.

Blackberries 150g were priced at 2.49, with an offer of "2 for 3.00". This did deliver an actual saving of 2 x 2.49 - 3.00 = 1.98 per *pair* of boxes. **Deal.**

In this case also there was extra printed information: 16.60 kg; this merely seems to mean that that there is a notional background price of
1.660 per 100g = 1.50 x 1.660 = 2.49
which is the already stated price for each packet, so nothing new is being stated.

Strasberries 125g were priced as "Save 1.00" with a printed line through 3.99 and 2.99 printed next to it. This is **no deal** because a price of 3.99 is not stated anywhere.

In this case too there was extra printed information: 23.92 kg; this merely seems to mean that there is a notional background price of
2.392 per 100g which implies 1.25 x 2.392 = 2.99
which is merely the already stated price for each packet of 125g.

Lesson 8.16

I thought it would be appropriate to introduce a topic which I had never previously studied myself, just to indicate that there are always boundaries to one's knowledge (mathematical knowledge in this case).

Magic squares is a topic which I had heard of, but not studied. A magic square is a square grid (which can be of any size) of numbers in which the sum of the numbers in every row is the same, and that is also the same as the sum of the columns, and the same as the sum of the two diagonals too.

How do we construct such a thing? There is one case in which we can illustrate the necessary mathematics at the level of ability in the class, but it has to be explained.

This is the case of a 2 x 2 square, in which we can suppose the four numbers to be as shown:

a b
c d

so that we require $a + b = c + d = a + c = b + d = a + d = b + c = m$.

By subtracting three pairs of these equations we obtain $b - c = b - d = a - c = 0$.

These **proves** that $a = b = c = d$. This could have been guessed, but that would not have been a proof, and we would not have been sure that here was not some other solution which we had not noticed.

So, for example, a = b = c = d = 7 is one solution, in which m = 14. There is a solution of this type for every even number m, in which all the entries in the grid are the same and half that number.

This is a useful illustration of the idea of **proof**.

As it turns out, the same problem becomes very much more difficult when we increase the size of the square. The next one would be a 3x3 square with nine numbers, required to have the same sum in the three horizontal, the three vertical, and the two diagonal directions. I quoted the fact that the solution to this is

2	7	6
9	5	1
4	3	8

Now the numbers are no longer the same, but in fact all different, and their sum in all directions is 15.

At this point I thought it worth telling the class that they could find much more about this famous topic if they typed Magic Squares into Google. Many sources of information about this topic are quoted there, one of which is called Wikipedia.

It would widen the pupils' scope if they became aware that information about mathematical topics can be obtained in that way.

Postscript

I am grateful to my wife Bridgid for commenting on these notes, from her own long experience of teaching mathematics at Primary and Secondary School level, and in Adult Education. It will be obvious that there is a small amount of duplication here, where it has seemed appropriate, but that at least will allow the reader to judge how my objective of novelty has been approached.

The Word format used here was convenient for recording my weekly notes after each Lesson, but its limitations on notation will be evident. I do have a longer term ambition to rewrite these notes in a mathematical vehicle such as Latex. That will allow me to use standard mathematical notation, and also to incorporate diagrams and photographs. At the same time I will remove the duplication, which is recorded here because this is a record of what was actually taught to the eight successive year groups.

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