Mathematics in the Garden

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Abstract

The method of construction of a novel garden ornament is described. The essential mathematics of it is indicated. The use of this as a practical project for a group of 10-year-olds in a school lesson is explained.

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1 The Hardware

The outdoor photographs show a garden ornament made by Richard Sewell in 2001. It consists of straight steel rods, of 1 centimetre diameter, which connect two different cubic curves. The ends of the rods lie on two imaginary parallel planes 1 metre apart. The structure has been used to help sweet peas grow upwards, albeit not vertically. Painted white, and movable, it also provides intriguing contrast, at different times, in a patch of daffodils, or of bluebells, or against an astilbe bush.

The two cubics, each in an x, t plane, belong to the same family $x = (y-2)t - t^3$, for the particular values $y = 0$ and $y = 8$.

The model can be seen to have 15 straight rods, corresponding to the $x(y)$ straight lines provided by successive values of t at intervals of 0.25 in the range $-1.75 \leq t \leq +1.75$. The central rod $t = 0$ is perpendicular, at each end, to the imaginary bounding plane there.

To make the model the two cubics, for $y = 0$ and $y = 8$, were plotted on two hardboard templates, and holes were drilled in the hardboard at those $x(y)$ points of the cubic specified by the above values of t. Each pair of corresponding holes, having the same value of t, was labelled. The templates were clamped in position at the correct spacing of 8 y units, i.e. 1 metre for the size of model required. For each pair of holes, a 1 centimetre diameter steel rod was cut to the correct length and passed through the holes, allowing a short protrusion at each end.

Once the long straight rods were in place, protruding through the hardboard planes at each end, the overall shape of the structure could be seen. Next, adjacent rods had to be joined at their ends to make a rigid structure.

This was done by cutting short sections of rod to the required length, laying them on the template between the holes in the hardboard, and welding them to the protruding ends with a MIG (metal inert gas) welder. Welded together, these short rods then combine to make a good piecewise linear approximation to the cubic curve on each bounding plane $y = 0$ and $y = 8$. Both cubics have an inflexion at $x = t = 0$. The cubic with $y = 8$ also has a local maximum and a local minimum, but that with $y = 0$ does not.

The hardboard templates were then cut away to reveal the finished structure. The welds were ground clean, and the finished structure was brush-painted with zinc primer and then gloss paint.

Two inverted U-shaped pegs hammered into the ground astride the lower cubic are sufficient to secure the structure in an upright vertical position.

Figure 1 shows a view looking, obliquely from above, towards the vertical x, y planes on each of which t is constant and so we see straight line sections of the surface. The upper limiting cubic, without local maximum and minimum, is seen on the plane $y = 0$. The Sun projects a cusped envelope, formed by the rods, onto the ground behind the structure. This evident cusp is the origin of the name “cusp catastrophe” sometimes associated with the surface (see Section 3 below). Different views of the cusped envelope emerge as one walks around the structure.

Figure 2 shows a view looking parallel to the x, y planes on each of which

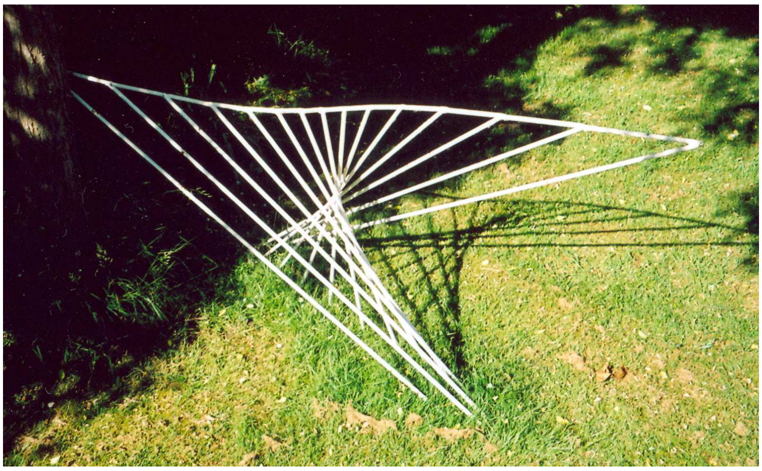

Figure 1: Front view showing cusped envelope shadow on ground

t is a constant. From this aspect the rich curvature contained in the model is concealed, but the simplicity of the construction is shown. The model is next to a white astilbe bush.

Figure 3 shows the steel structure in a different garden situation, next to a silver coloured globe artichoke. Daniel Sewell is sitting next to his Uncle Richard's construction. The structure can readily be moved around the garden so that it has more colourful backgrounds, such as azaleas or foxgloves, but these are not illustrated here.

2 The Software

I have had the good fortune, since retiring from undergraduate teaching, to teach a selected group of about nine 10-year-olds once a week in a primary school. It has proved possible, over a period of five years, to choose topics without repetition during that time, and which are not on the current syllabus, and also which do not deliberately anticipate a future syllabus which the pupils may meet. Some accidental anticipation is, of course, inevitable if certain topics are to be treated coherently. But the purpose is to offer breadth rather than acceleration.

Figure 2: Side view showing spacing of rods

Figure 3: View against a globe artichoke background

Chocolate offers an opportunity. Until very recently, every 100g bar of Lindt Excellence, a fine dark chocolate with 70 per cent cocoa, was sold in a packet stiffened with a cardboard rectangle measuring 19 centimetres by 8.5 centimetres, and 1 millimetre thick. Having accumulated a large collection of these cards, with an eye to possible future usefulness, an opportunity arose.

I taught the children how to plot the two cubics described in the previous Section, on centimetre graph paper. That is, they were required to draw the x, t axes twice, one to use for $y = 0$ and the other to use for $y = 8$. Then they were required to make tables of values of x for each of the 15 values of t specified above. Next the points were plotted on the two pieces of graph paper, and labelled B,C,D,E,F,G,H for $t > 0$, and b,c,d,e,f,g,h for $t < 0$, with the origin $t = 0$ at A and a. This lettering helped in the next stage.

The graphs were glued to a pair of cards (I did this myself at home, between lessons, to save time), and 0.5 millimetre holes were then pierced, by the pupils, at the labelled points using a drawing pin.

Next the pupils had to fix the two cards 8 centimetres apart, and so that the x and t axes on each card lay directly one above the other. We used some more card. Two rectangles were cut, each 10 centimetres by 8 centimetres. On each, the midline was marked which was parallel to the longer edge, and the card was bent to a right angle along that midline. Four notches were cut, 1 millimetre in width, and 1 centimetre from the ends of the longer edges. The pupils could then wedge each end of the cards containing the graphs into these notches, as seen in Figure 4. This fixed the graphs at the required distance apart, and parallel to each other.

Now we were ready for the nice bit. The children used a needle to thread cotton between correspondingly labelled points on the two graphs to create the string model. This is shown in Figure 4. Of course it was possible for there to be "many a slip..." in the last phase, but there was enough success to make the endeavour rewarding.

I carried the 1 metre high steel garden model into the classroom before we began the topic, to help provide motivation.

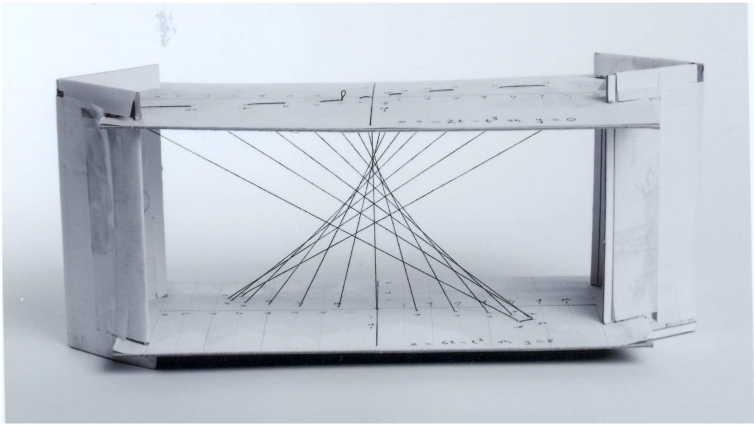

Figure 4: String model constructed in primary class

3 The Applications

Some of my early research papers were concerned with the calculation of eigenvalues which gave the buckling load of straight struts and flat plates compressed into the plastic range of stress. A developing wider viewpoint in the literature introduced the idea of an imperfection as another pertinent geometrical parameter in elastic and plastic structures. I wrote a paper (1966) which introduced the "equilibrium surface" as a geometrical and analytical representation of all the equilibrium configurations which would be possible under the range of possible values of all pertinent physical parameters.

In another part of the forest, as became clear to me when I attended a conference at Warwick University in 1969, the subject which eventually became widely known as catastrophe theory was rapidly germinating. For mechanical structures like struts, flat plates, and curved shells, with imperfections, the surface constructed in the previous two Sections above was found to be representative of the equilibrium surface which described the observed phenomena of the evolution of some of these structures under increasing load. In some cases (e.g. shells) the structure is sensitive to imperfections, which lower the buckling load, and in other cases (e.g. struts and plates) it is not sensitive to them.

A large number of other contexts emerged, some mechanical and some not, which made use of the surface exhibited in the photographs, or associated surfaces. Descriptions of some of these theories are to be found in, for example, the books by Zeeman (1977), Poston and Stewart (1978) and Thompson and Hunt (1984). Further evolution into areas exhibiting multiple bifurcations and so-called chaos have followed. Emotive terminology like "catastrophe" and "chaos" may not have been to everyone's taste, but it cannot be denied that the enormous stimulus which has been provided not only to the statics, but also to the dynamics of natural processes, is continuing.

A slice made by the plane $x = 0$ through the continuous surface represented by the string model will exhibit the parabola $y - 2 = t^2$ and the line $t = 0$ in the y, t plane. Together these display a trident, often called a "pitchfork bifurcation" (although when I was helping to build haystacks, the pitchforks always had two prongs rather than three). In the mechanical context, of a loaded imperfect strut, for example, y represents the axial load, t the sideways deflection, and x an imperfection.

The so-called "swallowtail" is another "elementary catastrophe" for which an attractive string model can be constructed, and photographs of it are shown in Sewell (1977, 1987). Other novel expositions of mathematics for children, in this case for 13-year-olds, are described in Sewell (1997).

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