Straight Sequences in a Spiralling Grid of Numbers

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1 Introduction

I have had the good fortune to teach a class of able ten-year-olds in a Primary School, weekly for the last seven years. My objective has been to find topics which have not been on their regular syllabus, and which are not expected to be on their syllabus in the future. Thus my policy is enrichment, not anticipation. This is the same policy as in the Masterclasses with which I have previously been involved, for thirteen-year-olds (see Sewell 1997) and eight-year-olds. This article describes extensions of a topic, which I have thus far taught to the ten-year-olds via a particular example (s = 17 in the SW-NE diagonal below), and which I have not seen before.

2 A Spiralling Grid

Begin with a square grid of rectangular boxes. Enter any chosen integer s (standing for "starter"), which may be positive, negative or zero, in a box which we deem to be the centre of the grid. Enter s+1 to the right of s, then s+2 above s+1, then s+3 and s+4 to the left of s+2, then s+5 and s+6 below s+4, then s+7 and s+8 to the right of s+6, thus completing a spiral in the 8 boxes round this centre box.

Now enter s+9 to the right of s+8, and construct the next circuit with s+10, s+11, s+12 upwards from s+9, then s+13, s+14, s+15, s+16 to the left of s+12, then s+17, s+18, s+19, s+20 downwards from s+16, then s+21, s+22, s+23, s+24 to the right of s+20. This circuit occupies 16 boxes, as illustrated in the diagram.

.	.	.	.	.	.	.
.	s+16	s+15	s+14	s+13	s+12	.
.	s+17	s+4	s+3	s+2	s+11	.
.	s+18	s+5	s	s+1	s+10	.
.	s+19	s+6	s+7	s+8	s+9	.
.	s+20	s+21	s+22	s+23	s+24	s+25
.	.	.	.	.	.	.

Now enter s+25 to the right of s+24, then s+26 upwards, and so on round another circuit, which occupies 24 boxes.

Continue to construct circuits in this way. Evidently, if we deem the first circuit to be the one which immediately surrounds the starting box, the m^{th} circuit will contain 8m boxes.

Having constructed several circuits in this way (say eight, which is enough to reveal significant patterns), with any s, we wish to deduce formulae which describe the numbers which occupy the four half-diagonals from the centre, and also those which occupy the four horizontal and vertical half-lines from the centre.

Let w denote the w^{th} box in the south-west (SW) half-diagonal, for w = 0 (the centre box), 1,2,3,... outwards from the centre. Let x, y, z be similarly used as counting numbers to denote the distance from the centre of boxes in the north-west (NW), north-east (NE) and south-east (SE) half-diagonals, respectively.

The topic of immediate interest from the point of view of developing this material in a way which can be explained to ten-year-olds, and which has a clear conclusion, is the method by which we can derive the four algebraic formulae which will reveal the occupants of the four sets of boxes on the half-diagonals as functions of these distances w,x,y and z out from the centre. In one case, for the SE half-diagonal, we shall need a first order difference equation.

Then we shall use a,b,c,d to denote the coordinates of the boxes along the half-lines in the north (N), east (E), south (S) and west (W) directions from the centre (with a = b = c = d = 0 there). We shall find that a single second order difference equation is needed to describe the occupants of these four half-lines.

3 SW Half-Diagonal

Moving SW along the half-diagonal from the centre, the exhibited numbers are

s, s + 6, s + 20, s + 42, s + 72, s + 110, s = 156, s + 210, s + 272,...

We can observe first that alternative forms of these are

$s + 1^2 - 1 = s$, $s + 3^2 - 3 = s + 6$, $s + 5^2 - 5 = s + 20$, $s + 7^2 - 7 = s + 42$,

and so on. Therefore these are typically $s + (2w + 1)^2 - (2w + 1) = W[s,w]$ (say), for successive integers w = 0,1,2,3... acting as counting numbers of the boxes on the half-diagonal, so that a simpler description is

s, s + 2x3, s + 4x5, s + 6x7, s + 8x9, s + 10x11,...., or

$W[s,w] = s + 2w(2w + 1)$ in the w^{th} box SW from the centre.

Thus the entry in the w^{th} box from the centre is obtained by adding to s the product of the adjacent pair of even (2w) and the next odd (2w + 1) numbers.

4 NW Half-Diagonal

Moving NW along this half-diagonal from the centre, the exhibited numbers are

s, s + 4, s + 16, s + 36, s + 64, s + 100, s + 144, s + 196, s + 256,....

Denoting the counting numbers of the boxes along this NW half-diagonal by x = 0,1,2,3,... it is easy to see that the entry in the x^{th} box from the centre can be written

$X[s,x] = s + (2x)^2$.

Having worked through the alternative observations which lead to W[s,w], the pupils were eager to infer this easier formula for X[s,x].

5 NE Half-Diagonal

Moving NE along this half-diagonal from the centre, the exhibited numbers are

s, s + 2, s + 12, s + 30, s + 56, s + 90, s + 132, s + 182, s + 240,....

We can observe first that alternative forms of these are

$s + 1^2 - 1 - 0 = s$, $s + 3^2 - 3 - 4 = s + 2$, $s + 5^2 - 5 - 8 = s + 12$, $s + 7^2 - 7 - 12 = s + 30$,...

and so on. Therefore these are typically $s + (2y + 1)^2 - (2y + 1) - 4y = Y[s,y]$ (say), for successive integers y = 0,1,2,3,... acting as counting numbers of the boxes on this half-diagonal, so that a simpler description is

$Y[s,y] = s + (2y - 1)2y$ in the y^{th} box from the centre.

Thus the entry in the y^{th} box from the centre is obtained by adding to s the product of the adjacent pair of odd (2y - 1) and the next even (2y) numbers.

6 SE Half-Diagonal

Moving SE along this half-diagonal from the centre, the exhibited numbers are

s, s + 8, s + 24, s + 48, s + 80, s + 120, s + 168, s + 224, s + 288,...

Alternative forms of these are

s, s + 8x1, s + 8x3, s + 8x6, s + 8x10, s + 8x15, s + 8x21, s + 8x28, s + 8x36,...

and so on. We see that the *difference* between successive multipliers of 8 in these expressions is 2,3,4,5..., thus increasing by 1 with each step. Therefore this observation leads explicitly to the idea of a first order difference equation Z[z] = 8z + Z[z - 1] for each s. The solution of this can be carried out iteratively, and we find that for successive integers z = 0,1,2,3,... acting as counting numbers of the boxes in this half-diagonal, the entry in the z^{th} box from the centre is

Z[s,z] = s + 4z(z + 1).

One might take the view that, given the spiral method of construction of the grid, it is surprising that these four half-diagonals can be described by such simple, and different, formulae.

7 N, E, S, W Paths

Next we consider the four paths out from the centre along the four principal compass directions.

Moving North from the centre the exhibited numbers A (say) are

s, s + 3, s + 14, s + 33, s + 60, s + 95, s + 138, s + 189, ...

For any given starter s, we denote the typical one of these numbers by A(a), using a as the Northward box coordinate or counting number from the centre. The "first" differences between successive pairs of terms are seen to be

3, 11, 19, 27, 35, 43, 51 = (s + 189) - (s + 138), ...

The "second" differences, between successive pairs of these first differences, are evidently 8 for every pair so that, for example,

A(4) - A(3) - [A(3) - A(2)] = 8, and in general

A(a + 2) - 2A(a + 1) + A(a) = 8 for every a = 0, 1, 2, 3, ...

This is a second order difference equation. Textbooks such as Ferrar (1943, p.121) indicate that it has the general solution

$A(a) = s + 4a^2 + \alpha a + \beta$

where α and β are arbitrary constants. Fitting this to our particular data that A(1) = s + 3 and A(2) = s + 14 leads to the solution

$A(a) = s + 4a^2 - a$

that we require.

Moving East, South and West the set of exhibited numbers B(b), C(c), D(d) (say) in the illustrated grid can be seen to be s + each of

0, 1, 10, 27, 52, 85, 126, 175, 232, ... for B,

0, 7, 22, 45, 76, 115, 162, 217, 280, ... for C,

0, 5, 18, 39, 68, 105, 150, 203, 264, ...for D.

It can be readily verified that B(b), C(c) and D(d) satisfy the same second order difference equation as that for A(a) above. Whether or not this is surprising, it is helpful. It leads, after using the associated three sets of initial data, to the solutions

$B(b) = s + 4b^2 - 3b$,

$C(c) = s + 4c^2 + 3c$,

$D(d) = s + 4d^2 + d$.

8 Persistence of Primes

A question of interest for some of these paths is the following. If the starter s is a prime number, for how long will the sequence of values of W, X, Y and Z continue to be prime numbers as we progress, with increasing positive integer values w, x, y, z respectively, along each of those four paths?

From the structure of the above formulae for W[s,w], X[s,x], Y[s,y] and Z[s,z], we see that we need consider only odd starter integers s, which may be either negative or positive. Zero or even s will induce even and therefore non-prime W, X, Y, Z.

For each s, the values of W[w], X[x], Y[y], Z[z] lie on parabolas defined by regarding w,x,y,z in their formulae as continuous variables. They have minimum values $W = s - 1/4$, $X = s$, $Y = s + 1/4$, $Z = s - 1$ where their gradients are zero, at $w = -1/4$, $x = 0$, $y = 1/4$, $z = -1/2$ respectively. Therefore the values of W, X, Y, Z will increase monotonically for each successive positive integer argument w,x,y,z respectively.

9 Path Termination via a Specified Function

A second question suggested by the above formulae for W, X, Y, Z is the following. Under what circumstances can one of their values become those of a specified "terminator" function t(s) of s, for example t(s) = s^2? In particular, if the sequence of values of W, X, Y, or Z begins with a set of prime values, and if t(s) has a non-prime value, this could be one way of terminating a sequence of primes. In those circumstances, if t(s) = s^2, such a sequence would be terminated by a square number.

10 Example

For any odd starter integer s, we wish to satisfy

W[s,w] = s^2, i.e. s + 2w(2w + 1) = s^2 and therefore

(2w + s)(2w - s + 1) = 0.

No negative odd s could satisfy this equation, so it is enough to consider only positive odd s, for which case W = s^2 will be satisfied in the w^{th} box when

w = (s - 1)/2.

We explore this case to the extent of selecting s to have the sequence of odd positive integer values 1, 3, 5, 7,......, 45, 47, 49, 51, stopping arbitrarily there. The corresponding values of w = (s - 1)/2 are 0, 1, 2, 3,......, 22, 23, 24, 25, and these locate the first 25 boxes on the SW diagonal outwards from the centre. In those boxes W has the square values 1, 9, 25, 49,......,1980, 2162, 2352, 2550. Explicit testing shows that there are only four values of s up to 51, namely 5, 11, 17 and 41, for which these sequences consist entirely of prime numbers before they reach the square value of their starter, namely

W = 5, 11, 25 = 5^2,

W = 11,17,31,53,83,121 = 11^2,

W = 17, 23, 37, 59, 89, 127, 173, 227, 289 = 17^2, and

W = 41, 47, 61, 83, 113, 151,197, 251, 313, 383, 461, 547, 641, 743, 853, 971, 1097, 1231, 1373, 1523, 1681 = 41^2.

11 SW and NE Diagonals Combined

Here let s denote any given integer, and let n be any given positive integer. Use them to construct the integer

N = s + n(n + 1).

We will call this definition of the function N(n) Euler's Formula, for any particular value of s.

If we choose n = 2y - 1 with y = 1,2,3,..., then Y(y) = s + (2y - 1)2y = N(n) in value for each s.

If we choose n = 2w with w = 1,2,3,..., then W(w) = s + 2w(2w + 1) = N(n) in value for each s.

From these definitions we see that for any given starter s, the sequence of Euler numbers N(n) generates alternating values of Y (from the NE diagonal) and W (from the SW diagonal), as n increases through even and odd values respectively.

This result establishes a connection between the spiral construction and Euler's Formula.

The proof is an immediate consequence of the definitions given above. From the discussion in the previous Section we can see that, even if the Euler numbers start as a sequence of primes, eventually that sequence will be terminated.

The 41^2 result in the previous Section is a version of what is called Euler's Formula in the literature, and I have also seen the 17^2 result announced as a fait accompli. These two are in the context of expressions $n^2 + n + 17$ and $n^2 + n + 41$, which deliver a prime number for both odd and even positive integers n until n reaches 17 and 41 respectively, when they deliver 17^2 and 41^2. The background to this is explained here for $(2w)^2 + 2w + s$ in a way which is viable for my class of ten-year-olds.

We can illustrate the result with the first four terms of the sequence, which are

N(1) = s + 2 = Y(1), N(2) = s + 6 = W(1), N(3) = s + 12 = Y(2), N(4) = s + 20 = W(2).

12 References

W. L. Ferrar. Higher Algebra. Oxford University Press. 1943.

Michael Sewell (editor). Mathematics Masterclasses - Stretching the Imagination. Oxford University Press. 1997.