Proving Newton's Propositio Kepleriana using Geometry and Nonstandard Analysis in Isabelle

Jacques D. Fleuriot and Lawrence C. Paulson

Computer Laboratory – University of Cambridge
New Museums Site, Pembroke Street
Cambridge CB2 3QG
{jdf21,lcp}@cl.cam.ac.uk

Abstract. The approach previously used to mechanise lemmas and Kepler's Law of Equal Areas from Newton's **Principia** [13] is here used to mechanically reproduce the famous *Propositio Kepleriana* or Kepler Problem. This is one of the key results of the Principia in which Newton demonstrates that the centripetal force acting on a body moving in an ellipse obeys an inverse square law. As with the previous work, the mechanisation is carried out through a combination of techniques from geometry theorem proving (GTP) and Nonstandard Analysis (NSA) using the theorem prover Isabelle. This work demonstrates the challenge of reproducing mechanically Newton's reasoning and how the combination of methods works together to reveal what we believe to be flaw in Newton's reasoning.

1 Introduction

The reasoning of Newton's *Philosophiæ Naturalis Principia Mathematica* (the *Principia* [14]), as it was originally published, is a mixture of geometric and algebraic arguments together with Newton's own proof techniques. These combine to produce a complex mathematical reasoning that is used to explain the physical world. The demonstrations of *Lemmas* and *Propositions* in the *Principia* are, in fact, proof sketches that require a lot of work on the part of the reader for a detailed understanding. There are several reasons that make the *Principia* a very difficult text to master. First of all, the proofs are very involved and one requires an adequate knowledge of geometry to be able to understand many of the steps made by Newton. Secondly, Newton's exposition can be tedious and difficult to grasp in places. Many mathematicians contemporary to Newton, despite their grounding in ancient Greek geometry and familiarity with the style of the exposition, had difficulties understanding Newton's mathematical reasoning. This gives an indication of the demands that a thorough study of the *Principia* has on the modern reader.

As we mentioned in a previous paper, Newton's geometry is also notable for his use of *limit* or *ultimate* arguments in his proofs [13]. These are implicit notions of differential calculus that are at the core of Newton's treatment and give Newton's geometry an infinitesimal nature. Newton further adds motion to his reasoning and enriches the geometry with various kinematics concepts that enable points to move towards points for example. Thus, Newton's geometry consists in studying the relations, such as ratios, between various parts of the constructed diagrams as certain of its elements tend towards limiting positions or become infinitely small.

In this paper, we build on the tools and techniques that we presented before [13]. In section 2, we review the geometric methods and concepts that we have formalised in this work. We also give examples of theorems proved in Isabelle using these techniques. Section 3 is a brief introduction to the concepts from NSA that we use; it also outlines the infinitesimal aspects of our geometry. We then present in Section 4, as a case study, the proof of the **Kepler Proposition**. This is a key proof of the Book 1 of the *Principia* and our work follows, in its steps, the analysis made by Densmore [9]. This extended case study shows our combination of techniques from geometry and NSA at work to provide a formal proof of a major proposition. The challenge inherent to the mechanisation of Newton's reasoning– especially in an interactive environment such as Isabelle where the user guides the proof– will become obvious as we highlight the steps and difficulties encountered. Section 5 offers our comments and conclusions.

2 Geometry

We use methods that are based on *geometric invariants* [5,6] and high level geometry lemmas about these invariants. A particular property is ideal as an invariant if it ensures that the proofs generated are short. This enables some of the proofs to be derived automatically using the powerful tactics of Isabelle's classical reasoner. Also, the methods should be powerful enough to prove many properties without adding auxiliary points or lines. The other important aspect is to achieve diagram independence for the proofs, that is, the same proof can be applied to several diagrams.

2.1 The Signed Area Method

In this method, there are basic rules about geometric properties called signed areas. These can be used to express various geometric concepts such as collinearity (`coll`), parallelism ($\|$) and so on. Moreover, the basic rules can be combined to prove more complex theorems which deal with frequently-used cases and help simplify the search process.

We represent the line from point A to point B by A—B, its length by $\texttt{len}(A-B)$, and the *signed* area $\mathtt{S}_{\mathtt{delta}}ABC$ of a triangle is the usual notion of area with its sign depending on how the vertices are ordered. We follow the usual approach of having $\mathtt{S}_{\mathtt{delta}}ABC$ as positive if A—B—C is in anti-clockwise direction and negative otherwise. Some of the rules and definitions used are:

$$a - b \parallel c - d \;\equiv\; (\mathtt{S}_{\mathtt{delta}}\, a\, b\, c = \mathtt{S}_{\mathtt{delta}}\, a\, b\, d)$$

$$\texttt{coll}\, a\, b\, c \Longrightarrow \texttt{len}(\mathtt{a}-\mathtt{b}) \times \mathtt{S}_{\mathtt{delta}}\, p\, b\, c = \texttt{len}(\mathtt{b}-\mathtt{c}) \times \mathtt{S}_{\mathtt{delta}}\, p\, a\, b$$

We can also introduce new points using the following property and define the *signed area of a quadrilateral* $\mathtt{S_{quad}}\, a\, b\, c\, d$ in terms of signed areas of triangles:

$$\mathtt{S_{delta}}\, a\, b\, c = \mathtt{S_{delta}}\, a\, b\, d + \mathtt{S_{delta}}\, a\, d\, c + \mathtt{S_{delta}}\, d\, b\, c$$
$$\mathtt{S_{quad}}\, a\, b\, c\, d \equiv \mathtt{S_{delta}}\, a\, b\, c + \mathtt{S_{delta}}\, a\, c\, d$$

We have proved a number of theorems about the sign of $\mathtt{S_{quad}}\, a\, b\, c\, d$ that depend on the ordering of the vertices, for example $\mathtt{S_{quad}}\, a\, b\, c\, d = -\mathtt{S_{quad}}\, a\, d\, c\, b$.

When dealing with geometry proofs, we often take for granted conditions that need to be stated explicitly for machine proofs: for example, two points making up a line should not coincide. The machine proofs are valid only if these conditions are met. These are known as *non-degenerate* conditions and are required in many cases to prevent the denominators of fractions from becoming zero in the various algebraic statements.

2.2 The Full-Angle Method

A full-angle $\langle u, v\rangle$ is the angle from line u to line v measured anti-clockwise. We define the equivalence relation of *angular* equality as follows:

$$x =_a y \equiv \exists n \in \mathbb{N}.\ |x - y| = n\pi$$

and can use it to express that two lines are perpendicular

$$a - b \perp c - d \equiv \langle a - b, c - d\rangle =_a \frac{\pi}{2}$$

Other properties of full-angles concern their sign and how they can be split or joined. The same rule therefore either introduces a new line or eliminates a common one from the full angles depending on the direction in which it is used.

$$\langle u, v\rangle =_a -\langle v, u\rangle$$
$$\langle u, v\rangle =_a \langle u, x\rangle + \langle x, v\rangle$$

Full-angles are used instead of traditional angles because their use simplifies many proofs by eliminating case-splits. Moreover, as we have already mentioned, these methods are useful to us since they relate closely to the geometric properties used by Newton [13]. They preserve the intuitive nature of his geometry and can easily be extended with infinitesimal notions, as we will see shortly.

2.3 A Simple Example: Euclid I.29

Euclid's proposition 29 of Book I [10], can be easily proved using the full-angle method. The proposition states that if $A - B \parallel C - D$ and the transversal $P - Q$ intersects $A - B$ and $C - D$ then $\langle A - B, P - Q\rangle = \langle C - D, P - Q\rangle$.

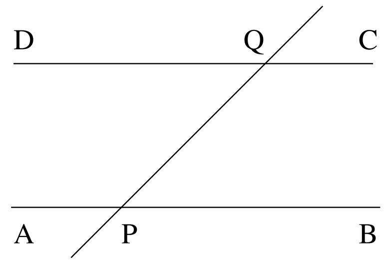

Fig. 1. Euclid Proposition I.29

We prove this theorem easily by using the rules about full-angles given in Section 2.2 and the fact the angle between two parallel lines is zero.
Proof:

$$
\begin{aligned}
A - B \parallel C - D \Longrightarrow & \langle A - B, C - D\rangle =_a 0 \\
& \langle A - B, P - Q\rangle + \langle P - Q, C - D\rangle =_a 0 \\
& \langle A - B, P - Q\rangle =_a -\langle P - Q, C - D\rangle \\
& \langle A - B, P - Q\rangle =_a \langle C - D, P - Q\rangle.
\end{aligned}
$$

2.4 Extending the Geometric Theory

The main aim of the *Principia* is to investigate mathematically the motion of bodies such as planets. Thus, we need to have definitions for geometric figures such as the circle, the ellipse and their tangents. The ellipse is especially important for this work since Kepler's Problem is concerned with elliptical motion. The circle can be viewed as a special case of the ellipse where the foci coincide.

$$
\begin{aligned}
\texttt{ellipse}\, f_1\, f_2\, r &\equiv \{p.\, |\texttt{len}(f_1 - p)| + |\texttt{len}(f_2 - p)| = r\} \\
\texttt{circle}\, x\, r &\equiv \texttt{ellipse}\, x\, x\, (2 \cdot r) \\
\texttt{arc\_len}\, x\, a\, b &\equiv |\texttt{len}(x - a)| \times \langle a - x, x - b\rangle \\
\texttt{e\_tangent}\, (a - b)\, f_1\, f_2\, E &\equiv (\texttt{is\_ellipse}\, f_1\, f_2\, E \;\wedge\; a \in E \;\wedge \\
& \quad\; \langle f_1 - a, a - b\rangle =_a \langle b - a, a - f_2\rangle)
\end{aligned}
$$

We need to prove a number of properties relating to the ellipse such as the one stating that *all parallelograms described around a given ellipse are equal to each other* (Figure 2).

This relationship appears (in slightly different wording) as **Lemma 12** of the *Principia* where it is employed in the solution of Proposition 11, the famous *Propositio Kepleriana* or Kepler problem. Newton refers us to the "writers on the conics sections" for a proof of the lemma. This lemma is demonstrated in

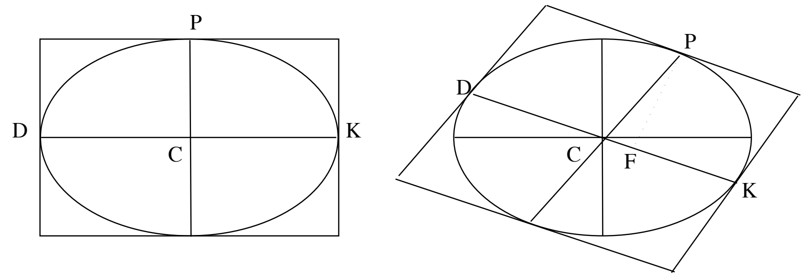

Fig. 2. Circumscribed Parallelograms

Book 7, Proposition 31 in the *Conics* of Apollonius of Perga [1]. Of course, unlike Newton, we have to prove this result explicitly in Isabelle to make it available to any other proof that might use it.

3 Infinitesimal Geometry

In this section, we give a brief overview of our geometry containing infinitesimals. We first give formal definitions for the various types of numbers that exist and which can be used to describe geometric quantities.

3.1 The Nonstandard Universe $\mathbb{R}^*$

Definition 1. *In an ordered field extension* $\mathbb{R}^* \supseteq \mathbb{R}$*, an element* $x \in \mathbb{R}^*$ *is said to be an* ***infinitesimal*** *if* $|x| < r$ *for all positive* $r \in \mathbb{R}$*;* ***finite*** *if* $|x| < r$ *for some* $r \in \mathbb{R}$*;* ***infinite*** *if* $|x| > r$ *for all* $r \in \mathbb{R}$*.*

The extended, richer number system $\mathbb{R}^*$ is known as the *hyperreals*. It has been developed in Isabelle through purely definitional means using an *ultrapower* construction. We will not give more details of this substantial construction in the present paper so as to concentrate on the geometric aspects only.

Definition 2. $x, y \in \mathbb{R}^*$ *are said to be* ***infinitely close,*** $x \approx y$ *if* $|x - y|$ *is infinitesimal.*

This is an important equivalence relation that will enable us to reason about infinitesimal quantities. For example, we can formalise the notion of two points coinciding by saying that the distance between them is infinitely close to zero. Two geometric quantities that become ultimately equal can also be modelled using it. The relation and its properties are used to formalise ultimate situations that might be considered degenerate by ordinary GTP methods [13].
Using the relation, we can also define the concept of two full-angles being infinitely close:

$$a_1 \approx_a a_2 \equiv \exists n \in \mathbb{N}.\ |a_1 - a_2| \approx n\pi$$

Various other *new* properties, such as *ultimately similar* and *ultimately congruent* triangles, can then be defined as we showed previously [13]. These are then used to prove various Lemmas that are needed to demonstrate the important Propositions of the *Principia.*

4 An Overview of the Proof

This is **Proposition 11** of Book 1 of the *Principia.* This Proposition is important for both mathematical and historical reasons as it lays the foundations for Kepler's first law of Gravitation. It provides the mathematical analysis that could explain and confirm Kepler's guess that planets travelled in ellipses round the sun [15].

The proof of this proposition will be studied in detail as it gives a good overview of the mixture of geometry, algebra and limit reasoning that is so characteristic of Newton's *Principia.* It also gives an idea of the depth and amount of mathematical expertise involved in Newton's proof. The proof that Newton describes, though relatively short on paper, becomes a major demonstration when expanded and reproduced using Isabelle. The elegance of many of the constructions, which could be glossed over, are revealed through the detailed analysis.

We give formal justifications of the steps made by Newton in ultimate situations through our formal and logical use of infinitesimals. Infinitesimal reasoning is notorious for leading to contradictions. However, nonstandard analysis is generally believed to be consistent and hence ensure that our mechanisation is rigorous. We will give the enunciation of the Proposition and the proof (sketch) provided by Newton. We will then expand on the sketch and provide detailed proofs of the steps that are made by Newton. This will require the use of the rules from the geometric and NSA theories developed in Isabelle.

4.1 Proposition 11 and Newton's Proof

Proposition 11 is in fact stated as a problem by Newton at the start of Section 3 of the *Principia.* This section deals with "*the motion of bodies in eccentric conic section*". Particular orbits and laws governing forces that are relevant to the universe are investigated. The mathematical tools are developed for later use in Book III of the *Principia* when natural phenomena of our world are investigated. Our task consists in expressing Newton's *result* as a goal which is then proved. Figure 3 shows Newton's original diagram used for this Proposition.

Proposition 11 *If a body revolves in an ellipse; it is required to find the law of the centripetal force tending to the focus of the ellipse*
Newton's Proof:
Let S be the focus of the ellipse. Draw SP cutting the diameter DK of the ellipse in E, and the ordinate Qv in x; and complete the parallelogram $QxPR$. It is evident that EP is equal to the greater semiaxis AC: for drawing HI from the other focus H of the ellipse parallel to EC, because CS, CH are equal, ES,

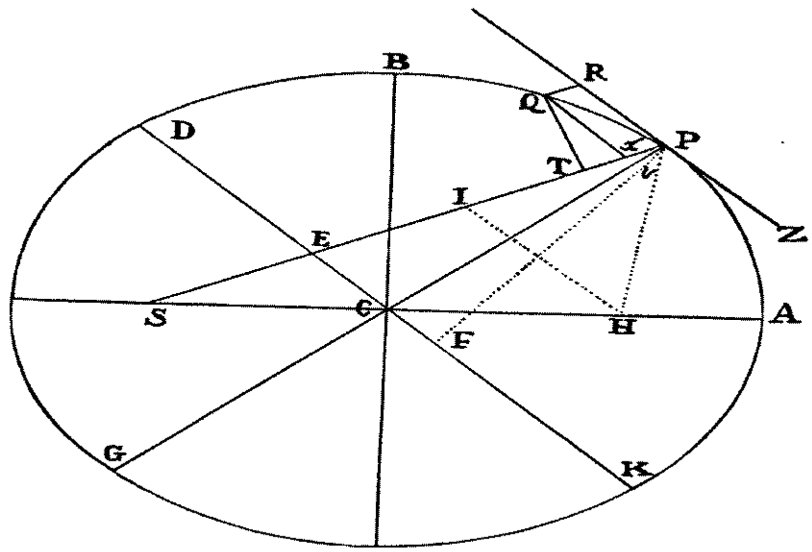

Fig. 3. Newton's Original Diagram for Proposition 11

EI will be also equal; so that EP is the half-sum of PS, PI, that is (because of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis $2AC$. Draw QT perpendicular to SP, and putting L for the principal latus rectum of the ellipse (or for $\frac{2BC^2}{AC}$), we shall have

$$L \cdot QR : L \cdot Pv = QR : Pv = PE : PC = AC : PC,$$
$$\text{also, } L \cdot Pv : Gv \cdot Pv = L : Gv, \text{ and, } Gv \cdot Pv : Qv^2 = PC^2 : CD^2$$

By Corollary 2, Lemma 7, when the points P and Q coincide, $Qv^2 = Qx^2$, and Qx^2 or $Qv^2 : QT^2 = EP^2 : PF^2 = CA^2 : PF^2$, and (by Lemma 12) $= CD^2 : CB^2$. Multiplying together corresponding terms of the four proportions, and by simplifying, we shall have

$$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2 = 2PC : Gv,$$

since $AC \cdot L = 2BC^2$. But the points Q and P coinciding, $2PC$ and Gv are equal. And therefore the quantities $L \cdot QR$ and QT^2, proportional to these, will also be equal. Let those equals be multiplied by $\frac{SP^2}{QR}$, and $L \cdot SP^2$ will become equal to $\frac{SP^2 \cdot QT^2}{QR}$. And therefore (by Corollary 1 and 5, Proposition 6) the centripetal force is inversely as $L \cdot SP^2$, that is, inversely as the square of the distance SP. Q.E.I.

Newton's derivation concludes that the centripetal force, for a body moving in an ellipse, is inversely proportional to the square of the distance.

Our proof proceeds in several steps where we set up various relationships that we will need for the conclusion. This involves proving Newton's intermediate results and storing them as intermediate theorems (we avoid calling them lemmas so as not to confuse them with Newton's own Lemmas).

4.2 A Geometric Representation for the Force

An investigation of the Proposition and Newton's result indicates that our goal is to prove that $\exists\, k \in \mathbb{R}.\, force \approx k \times \frac{1}{SP^2}$ (i.e. $force \propto \frac{1}{SP^2}$ ultimately). We now demonstrate through a combination of geometric and infinitesimal procedures how to prove the theorem.

Our combination of methods was previously used to prove *Kepler's Law of Equal Areas* [13]. This is an important result which states that a body moving under the influence of a centripetal force describes equal areas in equal times. Using this result we can now derive a completely **geometric** representation for the force acting on the orbiting body.

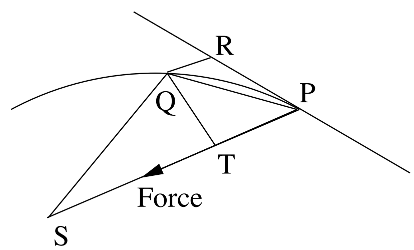

Fig. 4. Diagram for Geometric Representation of Force

Consider Figure 4 in which a point P is moving along an arc of finite curvature under the influence of a centripetal force acting towards S. Let Q be a point infinitely close to P, that is, the length of the arc from P to Q is infinitesimal. QR, parallel to SP, represents the displacement from the rectilinear motion (along the tangent) due to the force acting on P. QT is the perpendicular dropped to SP. From Newton's **Lemma 10, Corollary 3**, we have that displacement "in the very beginning of motion" is proportional to the force and the square of the time, and hence (for some real proportionality constant k_1) that

$$force \approx k_1 \times \frac{\mathtt{len(Q - R)}}{\mathtt{Time}^2} \tag{1}$$

Since the distance between P and Q is infinitesimal, the angle $\langle P - S, S - Q\rangle$ is infinitely small, and hence the area of the sector SPQ ($\mathtt{S_{arc}\,S\,P\,Q}$) is infinitely close to that of the triangle SPQ:

$$\begin{aligned}\langle \mathtt{P - S, S - Q}\rangle \approx_{\mathtt{a}} 0 &\Longrightarrow \mathtt{S_{arc}\,S\,P\,Q} \approx \mathtt{S_{delta}\,S\,P\,Q}\\ &\Longrightarrow \mathtt{S_{arc}\,S\,P\,Q} \approx 1/2 \times \mathtt{len(Q - T)} \times \mathtt{len(S - P)} \qquad (2)\end{aligned}$$

From Kepler's Law of Equal Areas, we can replace $\texttt{Time}$ by $\texttt{S}_{\texttt{arc}}\texttt{SPQ}$ [13] and, hence, using (1) and (2), we have the following geometric representation for the force (for some new proportionality constant k)

$$force \approx k \times \frac{\texttt{len}(\texttt{Q}-\texttt{R})}{\texttt{len}(\texttt{Q}-\texttt{T})^2 \times \texttt{len}(\texttt{S}-\texttt{P})^2} \tag{3}$$

This is a general result (**Proposition 6** of the *Principia*) that applies to any motion along an arc under the influence of a central force. We justify the use of a circular arc for the general arc by the fact that it is possible to construct a circle at the point P that represents the best approximation to the curvature there. This circle, sometimes known as the *osculating circle*[1], has the same first and second derivative as the curve at the given point P. Thus, the osculating circle has the same curvature and tangent at P as the general curve and an infinitesimal arc will also be same. We refer the reader to Brackenridge for more details on the technique [2, 3].

With this result set up, to prove the Kepler Problem, we need to show that the ratio involving the infinitesimal quantities QR and QT is equal or infinitely close to some constant (finite) quantity. Thus, the proof of Proposition 11 involves, in essence, eliminating the infinitesimals from relation (3) above. This relation is transformed using the geometry of the ellipse to one involving only macroscopic (i.e. non-infinitesimals) aspects of the orbit. We show next how the various GTP and NSA techniques are applied to the analysis of an elliptical orbit to determine the nature of the centripetal force.

4.3 Expanding Newton's Proof

A detailed account of our mechanisation of Newton's argument for Proposition 11 would take several pages since the proof sketch given by Newton is complex and we would have to present a large number of derivations. We will highlight the main results that were proved and, in some cases, details of the properties that needed to be set up first. We will also mention the constraints that needed to be satisfied within our framework before the various ratios that were proved could be combined. Our mechanisation was broken down into several steps that roughly followed from Newton's original proof. The main results that are set up are as follows (see Fig. 3):

- $\texttt{len}(\texttt{E}-\texttt{P}) = \texttt{len}(\texttt{A}-\texttt{C})$
- $\texttt{len}(\texttt{A}-\texttt{C})/\texttt{len}(\texttt{P}-\texttt{C}) = \texttt{L} \times \texttt{len}(\texttt{Q}-\texttt{R})/\texttt{L} \times \texttt{len}(\texttt{P}-\texttt{v})$
- $\texttt{L} \times \texttt{len}(\texttt{P}-\texttt{v})/(\texttt{len}(\texttt{G}-\texttt{v}) \times \texttt{len}(\texttt{P}-\texttt{v})) = \texttt{L}/\texttt{len}(\texttt{G}-\texttt{v})$
- $\texttt{len}(\texttt{G}-\texttt{v}) \times \texttt{len}(\texttt{P}-\texttt{v})/\texttt{len}(\texttt{Q}-\texttt{v})^2 = \texttt{len}(\texttt{P}-\texttt{C})^2/\texttt{len}(\texttt{C}-\texttt{D})^2$
- $\texttt{len}(\texttt{Q}-\texttt{v})^2/\texttt{len}(\texttt{Q}-\texttt{T})^2 \approx \texttt{len}(\texttt{C}-\texttt{D})^2/\texttt{len}(\texttt{C}-\texttt{B})^2$

Step 1: Proving $\texttt{len}(\texttt{E}-\texttt{P}) = \texttt{len}(\texttt{A}-\texttt{C})$

[1] from the Latin *osculare* meaning to kiss– the term was first used by Leibniz

This result shows that the length of EP is independent of P and Newton's proof uses several properties of the ellipse. We will give a rather detailed overview of this particular proof as it gives an idea of the amount of work involved in mechanising Newton's geometric reasoning. Moreover, the reader can then compare Newton's proof style and prose with our own proof and see the GTP methods we have formalised in action.

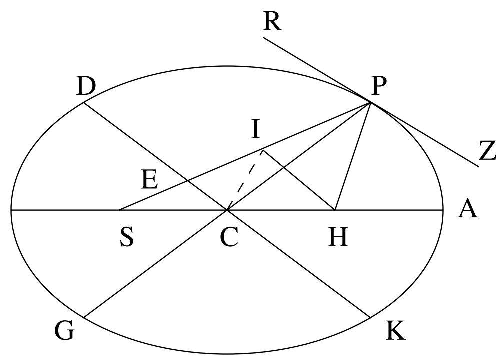

Fig. 5. Construction for Step 1 of Proposition 11

In Figure 5, the following holds

- C is the centre of the ellipse with S and H the foci
- P is a point of the curve
- RZ is the tangent at P
- the conjugate diameter $D - K \parallel P - Z$
- $P - S$ intersects $D - K$ at E
- $H - I \parallel E - C$ and $H - I$ intersects $P - S$ at I

Since $\mathtt{H} - \mathtt{I} \parallel \mathtt{E} - \mathtt{C}$, the following theorem holds,

$$\mathtt{H} - \mathtt{I} \parallel \mathtt{E} - \mathtt{C} \Longrightarrow \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,E\,I} = \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,E\,H} \tag{4}$$

But the foci are collinear with and (by Apollonius III.45 [1]) equidistant from the centre of the ellipse; so the following can be derived using the signed-area method,

$$\begin{aligned} \mathtt{coll\ S\,C\,H} &\Longrightarrow \mathtt{len(S - C)} \times \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,E\,H} = \mathtt{len(C - H)} \times \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,S\,E} \\ &\Longrightarrow \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,E\,H} = \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,S\,E} \end{aligned} \tag{5}$$

Also, points S, E and I are collinear and therefore combining with (4) and (5) above, we verify Newton's **"ES,EI will also be equal"**

$$\begin{aligned} \mathtt{coll\ S\,E\,I} &\Longrightarrow \mathtt{len(S - E)} \times \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,E\,I} = \mathtt{len(E - I)} \times \mathtt{S}_{\mathtt{delta}}\,\mathtt{C\,S\,E} \\ &\Longrightarrow \mathtt{len(S - E)} = \mathtt{len(E - I)} \end{aligned} \tag{6}$$

Next, the following derivations can be made, with the help of the last result proving Newton's **"EP is the half-sum of PS, PI"**

$$\begin{aligned}
\texttt{coll E I P} &\Longrightarrow \texttt{len(E} - \texttt{P)} = \texttt{len(E} - \texttt{I)} + \texttt{len(I} - \texttt{P)} \\
&\Longrightarrow \texttt{len(E} - \texttt{P)} = \texttt{len(S} - \texttt{E)} + \texttt{len(I} - \texttt{P)} \\
&\Longrightarrow 2 \times \texttt{len(E} - \texttt{P)} = \texttt{len(E} - \texttt{P)} + \texttt{len(S} - \texttt{E)} + \texttt{len(I} - \texttt{P)} \\
&\Longrightarrow 2 \times \texttt{len(E} - \texttt{P)} = \texttt{len(S} - \texttt{P)} + \texttt{len(I} - \texttt{P)} \\
&\Longrightarrow \texttt{len(E} - \texttt{P)} = \frac{\texttt{len(S} - \texttt{P)} + \texttt{len(I} - \texttt{P)}}{2} \qquad (7)
\end{aligned}$$

Note the use of the following theorem in the derivation above

$$\texttt{coll S E P} \Longrightarrow \texttt{len(S} - \texttt{E)} + \texttt{len(E} - \texttt{P)} = \texttt{len(S} - \texttt{P)}$$

Next, Newton argues that in fact (7) can be written as

$$\texttt{len(E} - \texttt{P)} = \frac{\texttt{len(S} - \texttt{P)} + \texttt{len(H} - \texttt{P)}}{2} \qquad (8)$$

So, a proof of $\texttt{len(I} - \texttt{P)} = \texttt{len(H} - \texttt{P)}$ is needed to progress further. This will follow if it can be shown that $\triangle\texttt{PHI}$ is an isosceles, that is

$$\langle \texttt{P} - \texttt{H}, \texttt{H} - \texttt{I} \rangle = \langle \texttt{H} - \texttt{I}, \texttt{I} - \texttt{P} \rangle \qquad (9)$$

To prove (9), both $\texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{Z}$ and $\texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{R}$ are derived first using

$$\texttt{H} - \texttt{I} \parallel \texttt{E} - \texttt{C} \wedge \texttt{E} - \texttt{C} \parallel \texttt{P} - \texttt{Z} \Longrightarrow \texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{Z} \qquad (10)$$

$$\texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{Z} \wedge \texttt{coll P Z R} \Longrightarrow \texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{R} \qquad (11)$$

From (10), (11), and the proof of Euclid I.29 given in Section 2.3

$$\texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{Z} \Longrightarrow \langle \texttt{P} - \texttt{H}, \texttt{H} - \texttt{I} \rangle = \langle \texttt{H} - \texttt{P}, \texttt{P} - \texttt{Z} \rangle \qquad (12)$$

$$\begin{aligned}
\texttt{H} - \texttt{I} \parallel \texttt{P} - \texttt{R} &\Longrightarrow \langle \texttt{H} - \texttt{I}, \texttt{I} - \texttt{P} \rangle = \langle \texttt{R} - \texttt{P}, \texttt{P} - \texttt{I} \rangle \\
&\Longrightarrow \langle \texttt{H} - \texttt{I}, \texttt{I} - \texttt{P} \rangle = \langle \texttt{R} - \texttt{P}, \texttt{P} - \texttt{S} \rangle \qquad (13)
\end{aligned}$$

From the definition of the tangent to an ellipse and the collinearity of P, I, and S (also recall that full-angles are angles between *lines* rather than rays and are measured anti-clockwise),

$$\begin{aligned}
\texttt{e_tangent (P} - \texttt{Z) S H Ellipse} &\Longrightarrow \langle \texttt{H} - \texttt{P}, \texttt{P} - \texttt{Z} \rangle = \langle \texttt{R} - \texttt{P}, \texttt{P} - \texttt{I} \rangle \\
&\Longrightarrow \langle \texttt{H} - \texttt{P}, \texttt{P} - \texttt{Z} \rangle = \langle \texttt{R} - \texttt{P}, \texttt{P} - \texttt{S} \rangle \qquad (14)
\end{aligned}$$

From (12), (13) and (14), the following is deduced as required

$$\langle \mathtt{P} - \mathtt{H}, \mathtt{H} - \mathtt{I} \rangle = \langle \mathtt{H} - \mathtt{I}, \mathtt{I} - \mathtt{P} \rangle$$

Thus, we have $\mathtt{len}(\mathtt{I} - \mathtt{P}) = \mathtt{len}(\mathtt{H} - \mathtt{P})$ (Euclid I.6 [10]), and hence (8) is proved, that is, Newton's assertion that **"[EP is the half sum of] PS, PH"**.

Next, it follows from the definition of an ellipse that the sum of $\mathtt{len}(\mathtt{S} - \mathtt{P})$ and $\mathtt{len}(\mathtt{P} - \mathtt{H})$ is equal to the length of the major axis, that is,

$$\mathtt{P} \in \mathtt{Ellipse} \Longrightarrow \mathtt{len}(\mathtt{S} - \mathtt{P}) + \mathtt{len}(\mathtt{P} - \mathtt{H}) = 2 \times \mathtt{len}(\mathtt{A} - \mathtt{C}) \qquad (15)$$

From (15) and (8), we can finally derive the property that Newton states as being evident: **"EP is equal to the greater semiaxis AC"**

$$\mathtt{len}(\mathtt{E} - \mathtt{P}) = \mathtt{len}(\mathtt{A} - \mathtt{C}) \qquad (16)$$

The first step has shown Newton's geometric reasoning in action. For the next steps, as the various ratios are derived, we will not always show the detailed derivations of the geometric theorems. We will concentrate on the setting up of the proportions and how everything is put together to get the final result. We will state Newton's Lemmas when they are used and theorems about infinitesimals that we use.

Step 2: Showing $\frac{L \cdot QR}{L \cdot Pv} = \frac{QR}{Pv} = \frac{PE}{PC} = \frac{AC}{PC}$

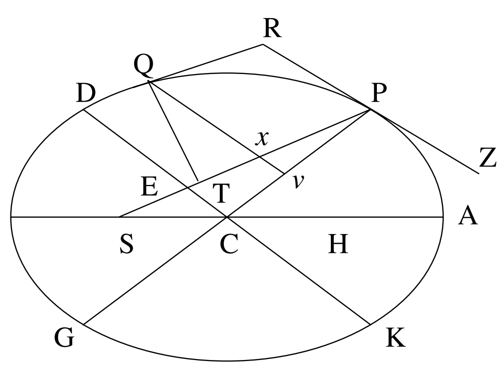

Fig. 6. Construction for Steps 2—4 of Proposition 11

In Figure 6, in addition to properties already mentioned, the following holds

- $QT \perp SP$
- $QxPR$ is a parallelogram
- Q, x, and v are collinear
- Q is infinitely close to P

It is easily proved that $\mathtt{v}-\mathtt{x} \parallel \mathtt{C}-\mathtt{E}$ and so the following theorem follows

$$\mathtt{v}-\mathtt{x} \parallel \mathtt{C}-\mathtt{E} \Longrightarrow \langle \mathtt{P}-\mathtt{v}, \mathtt{v}-\mathtt{x}\rangle = \langle \mathtt{P}-\mathtt{C}, \mathtt{C}-\mathtt{E}\rangle \tag{17}$$

From (17) and the fact that $\triangle\mathtt{Pvx}$ and $\triangle\mathtt{PCE}$ share P as a common vertex, it follows that they are *similar*. Also, since $\mathtt{QxPR}$ is a parallelogram, we have $\mathtt{len}(\mathtt{Q}-\mathtt{R}) = \mathtt{len}(\mathtt{P}-\mathtt{x})$. Thus, the following derivations follow

$$\mathtt{SIM\ P\,V\,x\,P\,C\,E} \Longrightarrow \frac{\mathtt{len}(\mathtt{P}-\mathtt{E})}{\mathtt{len}(\mathtt{P}-\mathtt{C})} = \frac{\mathtt{len}(\mathtt{P}-\mathtt{x})}{\mathtt{len}(\mathtt{P}-\mathtt{v})} = \frac{\mathtt{len}(\mathtt{Q}-\mathtt{R})}{\mathtt{len}(\mathtt{P}-\mathtt{v})} = \frac{\mathtt{len}(\mathtt{A}-\mathtt{C})}{\mathtt{len}(\mathtt{P}-\mathtt{C})} \tag{18}$$

One of the substitution used in (18) follows from (16) proved in the previous step. The equations above verify Newton's ratios.

Step 3: Showing $\frac{L\cdot Pv}{Gv\cdot Pv} = \frac{L}{Gv}$

The proof of the ratio

$$\frac{\mathtt{L}\times\mathtt{len}(\mathtt{P}-\mathtt{v})}{\mathtt{len}(\mathtt{G}-\mathtt{v})\times\mathtt{len}(\mathtt{P}-\mathtt{v})} = \frac{\mathtt{L}}{\mathtt{len}(\mathtt{G}-\mathtt{v})} \tag{19}$$

is trivial and we will not expand on it. We only note that the constant L is known as the *latus rectum*[2] of the ellipse at A.

Step 4: Showing $\frac{Gv\cdot Pv}{Qv^2} = \frac{PC^2}{CD^2}$

By Apollonius I.21 [1], *if the lines DC and Qv are dropped ordinatewise to the diameter PG, the squares on them DC^2 and Qv^2 will be to each other as the areas contained by the straight lines cut off GC, CP, and Gv, vP on diameter PG.* Algebraically, we proved the following property of the ellipse,

$$\begin{aligned}\frac{\mathtt{len}(\mathtt{D}-\mathtt{C})^2}{\mathtt{len}(\mathtt{Q}-\mathtt{v})^2} &= \frac{\mathtt{len}(\mathtt{G}-\mathtt{C})\times\mathtt{len}(\mathtt{C}-\mathtt{P})}{\mathtt{len}(\mathtt{G}-\mathtt{v})\times\mathtt{len}(\mathtt{P}-\mathtt{v})}\\ &= \frac{\mathtt{len}(\mathtt{P}-\mathtt{C})^2}{\mathtt{len}(\mathtt{G}-\mathtt{v})\times\mathtt{len}(\mathtt{P}-\mathtt{v})}\end{aligned}$$

Rearranging the terms, we get the required ratio,

$$\frac{\mathtt{len}(\mathtt{G}-\mathtt{v})\times\mathtt{len}(\mathtt{P}-\mathtt{v})}{\mathtt{len}(\mathtt{Q}-\mathtt{v})^2} = \frac{\mathtt{len}(\mathtt{P}-\mathtt{C})^2}{\mathtt{len}(\mathtt{D}-\mathtt{C})^2} \tag{20}$$

Step 5: Showing $\frac{Qv^2}{QT^2} \approx \frac{CD^2}{CB^2}$ and intermediate ratios

In Figure 7, we have the additional property,

[2] The latus rectum is defined as $L = 2\times\mathtt{len}(\mathtt{B}-\mathtt{C})^2/\mathtt{len}(\mathtt{A}-\mathtt{C})$

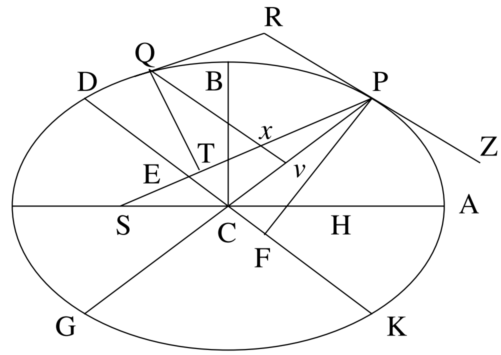

Fig. 7. Construction for Step 5 of Proposition 11

- $PF \perp DK$

Again, it can be easily proved that $\mathtt{Qx} \parallel \mathtt{EF}$. The following theorem then follows from Euclid I.29 as given in Section 2.3

$$\begin{aligned} \mathtt{Q-x} \parallel \mathtt{E-F} &\Longrightarrow \langle \mathtt{Q-x}, \mathtt{x-E} \rangle = \langle \mathtt{F-E}, \mathtt{E-x} \rangle \\ &\Longrightarrow \langle \mathtt{Q-x}, \mathtt{x-T} \rangle = \langle \mathtt{F-E}, \mathtt{E-P} \rangle \end{aligned} \tag{21}$$

Since $\langle \mathtt{P-F}, \mathtt{F-E}, \rangle = \langle \mathtt{x-T}, \mathtt{T-Q} \rangle = \pi/2$ and (21), it follows that $\triangle\mathtt{PEF}$ and $\triangle\mathtt{QxT}$ are similar. The next theorems (using (16) where needed) then hold and verify Newton's intermediate results for the current Step.

$$\mathtt{SIM\ P\,E\,F\,Q\,x\,T} \Longrightarrow \frac{\mathtt{len(Q-x)}^2}{\mathtt{len(Q-T)}^2} = \frac{\mathtt{len(P-E)}^2}{\mathtt{len(P-F)}^2} = \frac{\mathtt{len(C-A)}^2}{\mathtt{len(P-F)}^2} \tag{22}$$

Newton's **Lemma 12** (See Figure 2) is now needed for the next result. According to the Lemma, the parallelogram circumscribed about DK and PG is equal to the parallelogram circumscribed about the major and minor axes of the ellipse. Thence, we have the following theorem

$$\mathtt{len(C-A)} \times \mathtt{len(C-B)} = \mathtt{len(C-D)} \times \mathtt{len(P-F)} \tag{23}$$

Rearranging (23), we have $\mathtt{len(C-A)}/\mathtt{len(P-F)} = \mathtt{len(C-D)}/\mathtt{len(C-B)}$ and substituting in (22), leads to

$$\frac{\mathtt{len(Q-x)}^2}{\mathtt{len(Q-T)}^2} = \frac{\mathtt{len(C-D)}^2}{\mathtt{len(C-B)}^2} \tag{24}$$

By Newton's **Lemma 7, Corollary 2**, when the distance between Q and P becomes infinitesimal as they coincide, we have the following result [13]:

$$\frac{\texttt{len}(\texttt{Q} - \texttt{v})}{\texttt{len}(\texttt{Q} - \texttt{x})} \approx 1 \tag{25}$$

Now, to reach the final result for this step, we need to substitute $\texttt{len}(\texttt{Q} - \texttt{v})$ for $\texttt{len}(\texttt{Q} - \texttt{x})$ in (22). However, we cannot simply carry out the substitution even though the quantities are infinitely close. Indeed, one has to be careful when multiplying the quantities on both sides of the $\approx$ relation because they might no longer be infinitely close after the multiplication. Consider, the non-zero infinitesimal ϵ,

$$\epsilon \approx \epsilon^2 \text{ but } \epsilon \times 1/\epsilon \not\approx \epsilon^2 \times 1/\epsilon$$

It is possible, however, to multiply two infinitely close quantities by any finite quantity; the results are still infinitely close. This follows from the theorem:

$$x \approx y \wedge u \in \texttt{Finite} \Longrightarrow x \times u \approx y \times u \tag{26}$$

Now, assuming that $\texttt{len}(\texttt{C} - \texttt{D})$ and $\texttt{len}(\texttt{C} - \texttt{B})$ are both finite but not infinitesimal (for example, $\texttt{len}(\texttt{C} - \texttt{D}), \texttt{len}(\texttt{C} - \texttt{B}) \in \mathbb{R}$), then $\texttt{len}(\texttt{C} - \texttt{D})/\texttt{len}(\texttt{C} - \texttt{B})$ is $\texttt{Finite}$. Hence, the ratio of *infinitesimals* $\texttt{len}(\texttt{Q} - \texttt{x})/\texttt{len}(\texttt{Q} - \texttt{T})$ is $\texttt{Finite}$. Therefore, from (25), (26), and using (24) the following theorem is derived:

$$\frac{\texttt{len}(\texttt{Q} - \texttt{v})^2}{\texttt{len}(\texttt{Q} - \texttt{T})^2} \approx \frac{\texttt{len}(\texttt{C} - \texttt{D})^2}{\texttt{len}(\texttt{C} - \texttt{B})^2} \tag{27}$$

This gives the result that we wanted for the fifth step of the proof of Proposition 11. We are now ready for putting all the various results together in the next and final step. This will then conclude the formal proof of the Proposition.

Step 6: Putting the ratios together

Combining (20) and (27), with the help of theorem (26) and some algebra yields,

$$\begin{aligned} &\frac{\texttt{len}(\texttt{Q} - \texttt{v})^2}{\texttt{len}(\texttt{Q} - \texttt{T})^2} \approx \frac{\texttt{len}(\texttt{C} - \texttt{D})^2}{\texttt{len}(\texttt{C} - \texttt{B})^2} \;\wedge\; \frac{\texttt{len}(\texttt{G} - \texttt{v}) \times \texttt{len}(\texttt{P} - \texttt{v})}{\texttt{len}(\texttt{Q} - \texttt{v})^2} \in \texttt{Finite} \\ &\qquad \Longrightarrow \frac{\texttt{len}(\texttt{G} - \texttt{v}) \times \texttt{len}(\texttt{P} - \texttt{v})}{\texttt{len}(\texttt{Q} - \texttt{T})^2} \approx \frac{\texttt{len}(\texttt{P} - \texttt{C})^2}{\texttt{len}(\texttt{C} - \texttt{B})^2} \end{aligned} \tag{28}$$

which is combined with (19) to derive the next relation between ratios. The reader can check that both sides of the $\approx$ relation are multiplied by finite quantities ensuring the results are infinitely close:

$$\frac{\mathtt{L} \times \mathtt{len(P-v)}}{\mathtt{len(Q-T)}^2} \approx \frac{\mathtt{len(P-C)}^2 \times \mathtt{L}}{\mathtt{len(C-B)}^2 \times \mathtt{len(G-v)}} \tag{29}$$

The next task is to combine the last result (29) with (18) to yield the following ratio which is equivalent to Newton's "$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2$"

$$\frac{\mathtt{L} \times \mathtt{len(Q-R)}}{\mathtt{len(Q-T)}^2} \approx \frac{\mathtt{len(P-C)} \times \mathtt{L} \times \mathtt{len(A-C)}}{\mathtt{len(C-B)}^2 \times \mathtt{len(G-v)}} \tag{30}$$

But, we know that $\mathtt{L} = 2 \times \mathtt{len(B-C)}^2/\mathtt{len(A-C)}$, so (30) can be further simplified to give Newton's other ratio "$L \cdot QR : QT^2 = 2PC : Gv$"

$$\frac{\mathtt{L} \times \mathtt{len(Q-R)}}{\mathtt{len(Q-T)}^2} \approx \frac{2 \times \mathtt{len(P-C)}}{\mathtt{len(G-v)}} \tag{31}$$

Once these ratios have been derived, Newton says **"But the points Q and P coinciding, $2PC$ and Gv are equal. And therefore the quantities $L \cdot QR$ and QT^2, proportional to these are also equal."**

We formalise this by showing that $\mathtt{len(P-v)} \approx \mathtt{0}$ as the distance between Q and P becomes infinitesimal; thus, it follows that $2 \times \mathtt{len(P-C)}/\mathtt{len(G-v)} \approx \mathtt{1}$ and so, using (31) and the transitivity of $\approx$, we have the result

$$\frac{\mathtt{L} \times \mathtt{len(Q-R)}}{\mathtt{len(Q-T)}^2} \approx 1 \tag{32}$$

The final step in Newton's derivation is **"Let those equals be multiplied by $\frac{SP^2}{QR}$ and $L \cdot SP^2$ will become equal to $\frac{SP^2 \cdot QT^2}{QR}$"**. This final ratio gives the geometric representation for the force, as we showed in Section 4.2, and hence enables Newton to deduce immediately that the centripetal force obeys an inverse square law.

We would like to derive Newton's result in the *same way*, but remark that

$$\begin{aligned} &\mathtt{len(S-P)} \in \mathtt{Finite-Infinitesimal} \ \wedge \ \mathtt{len(Q-R)} \in \mathtt{Infinitesimal} \\ &\qquad \Longrightarrow \frac{\mathtt{len(S-P)}^2}{\mathtt{len(Q-R)}} \in \mathtt{Infinite} \end{aligned} \tag{33}$$

as Q and P become coincident. So, there seems to be a problem with simply multiplying (32) by Newton's ratio SP^2/QR since we cannot ensure that the results are infinitely close. Our formal framework *forbids* the multiplication that Newton does as the result is not necessarily a theorem!

Therefore, we need to find an **alternative** way of arriving at the same result as Newton. Recall from Section 4.2, that we have proved the following geometric representation for the centripetal force:

$$\mathit{force} \approx k \times \frac{\mathtt{len(Q-R)}}{\mathtt{len(Q-T)}^2} \times \frac{\mathtt{1}}{\mathtt{len(S-P)}^2} \tag{34}$$

Now from (32), we can deduce that since $\mathtt{L} \in \mathtt{Finite} - \mathtt{Infinitesimal}$, the following theorems hold

$$\frac{\mathtt{len(Q-R)}}{\mathtt{len(Q-T)}^2} \in \mathtt{Finite} - \mathtt{Infinitesimal} \tag{35}$$

$$\frac{\mathtt{len(Q-T)}^2}{\mathtt{len(Q-R)}} \approx \mathtt{L} \tag{36}$$

$$\frac{\mathtt{len(Q-T)}^2}{\mathtt{len(Q-R)}} \in \mathtt{Finite} \tag{37}$$

Since (35) holds and $\mathtt{1/len(S-P)}^2 \in \mathtt{Finite}$, it follows that $\mathit{force} \in \mathtt{Finite}$ and so we can now use the following theorem about the product of *finite*, infinitely close quantities

$$a \approx b \ \wedge \ c \approx d \ \wedge \ a \in \mathtt{Finite} \ \wedge \ c \in \mathtt{Finite} \Longrightarrow a \times c \approx b \times d$$

with (34), (36), and (37) to yield

$$\begin{aligned} \mathit{force} \times \mathtt{L} &\approx k \times \frac{\mathtt{len(Q-R)}}{\mathtt{len(Q-T)}^2} \times \frac{\mathtt{1}}{\mathtt{len(S-P)}^2} \times \frac{\mathtt{len(Q-T)}^2}{\mathtt{len(Q-R)}} \\ &\approx k \times \frac{\mathtt{1}}{\mathtt{len(S-P)}^2} \end{aligned} \tag{38}$$

Note that we also used the fact that $\approx$ is symmetric in the derivation above. Finally from (38), we get to the celebrated result since L is finite (real) and constant for a given ellipse,

$$\begin{aligned} \mathit{force} &\approx \frac{k}{\mathtt{L}} \times \frac{\mathtt{1}}{\mathtt{len(S-P)}^2} \\ \mathit{force} &\propto_{ultimate} \frac{\mathtt{1}}{\mathtt{len(S-P)}^2} \end{aligned} \tag{39}$$

5 Final Comments

We would like to conclude by mentioning some important aspects of this mechanisation and possible changes to the geometry theory that could improve automation. We also briefly review what we have achieved.

5.1 On Finite Geometric Witnesses

We have made an important remark about steps involving infinitesimals, ratios of infinitesimals and the infinitely close relation. Whenever we are dealing with such ratios, care needs to be exercised as we cannot be sure what the result of dividing two infinitesimals is: it can be infinitesimal, finite or infinite. We notice, when carrying out our formalisation, that whenever Newton is manipulating the ratio of vanishing quantities, he usually makes sure that this can be expressed in terms of some finite quantity as in the proof for Step 5 of Section 4.3. Thus, the ratio of infinitesimals is shown to be infinitely close or even equal to some finite quantity. This ensures that such a finite ratio can be used safely and soundly within our framework. The importance of setting up such *finite geometric witnesses* cannot be under-stated since the rigour of NSA might prevent steps involving ratios of infinitesimals from being carried out otherwise. Indeed, we have seen that the lack of a finite witness in the last step of Newton's original argument prevents us from deriving the final result in the same way as he does. The alternative way we went about deriving the result, however, is sound and follows from rules that have been proved within our framework.

5.2 Further Work

In our previous work [13], we mentioned the existence of other methods, such as the Clifford algebra, that provide short and readable proofs [4, 11]. Although these algebraic techniques are more difficult to relate with the geometric concepts that are actually used in Newton's reasoning, interesting work done by Wang et al. has come to our attention in which powerful sets of rewrite rules have been derived to carry out proofs in Euclidean geometry [12, 16]. It would be interesting to see how these could be integrated with Isabelle's powerful simplifier to provide a greater degree of automation in some of our proofs. In a sense, such an approach would match in the level of details some of the results that Newton states (as obvious) and does not prove in depth.

As an interesting observation, it is worth noting that the Kepler Problem can be proved, or even discovered, using algebraic computations. This has been demonstrated through the work on mechanics done by Wu [17], and also by Chou and Gao [7, 8], in the early nineties.

5.3 Conclusions

We have described in detail the machine proof of Proposition 11 of the *Principia* and shown how the theories developed in Isabelle can be used to derive Newton's geometric representations for physical concepts. We have used the same combination of geometry and NSA rules introduced in our previous work to confirm, through a study of one of the most important Propositions of the Principia, that Newton's geometric and ultimate procedures can be cast within the rigour of our formal framework. The discovery of a step in Newton's reasoning that could not be justified formally– in contrast with other ones where Newton explicitly sets up finite witnesses– is an important one. The alternative derivation presented in this work shows how to use our rules to deduce the same result soundly.

Once again, the mechanisation of results from the *Principia* has been an interesting and challenging exercise. Newton's original reasoning, though complex and often hard to follow, displays the impressive deductive power of geometry. The addition of infinitesimal notions results in a richer, more powerful geometry in which new properties can emerge in ultimate situation. Moreover, we now have new, powerful tools to study the model built on Newton's exposition of the physical world.

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