MetiTarski: An Automatic Prover for Real-Valued Special Functions

Behzad Akbarpour and Lawrence C. Paulson
Computer Laboratory, Cambridge

special functions

- Many application domains concern statements involving the functions sin, cos, ln, exp, etc.
- We prove them by combining a *resolution theorem prover* (Metis) with a decision procedure for *real closed fields* (QEPCAD).
- MetiTarski works automatically and delivers machine-readable proofs.

the basic idea

- Our approach involves replacing functions by *rational function upper or lower bounds*.
- The eventual polynomial inequalities belong to a decidable theory: *real closed fields (RCF).*
- Logical formulae over the reals involving $+ - \times \leq$ and quantifiers are decidable (Tarski).

We call such formulae algebraic.

bounds for exp

- Special functions can be approximated, e.g. by Taylor series or continued fractions.
- Typical bounds are only valid (or close) over a restricted range of arguments.
- We need several formulas to cover a range of intervals. Here are a few of the options.

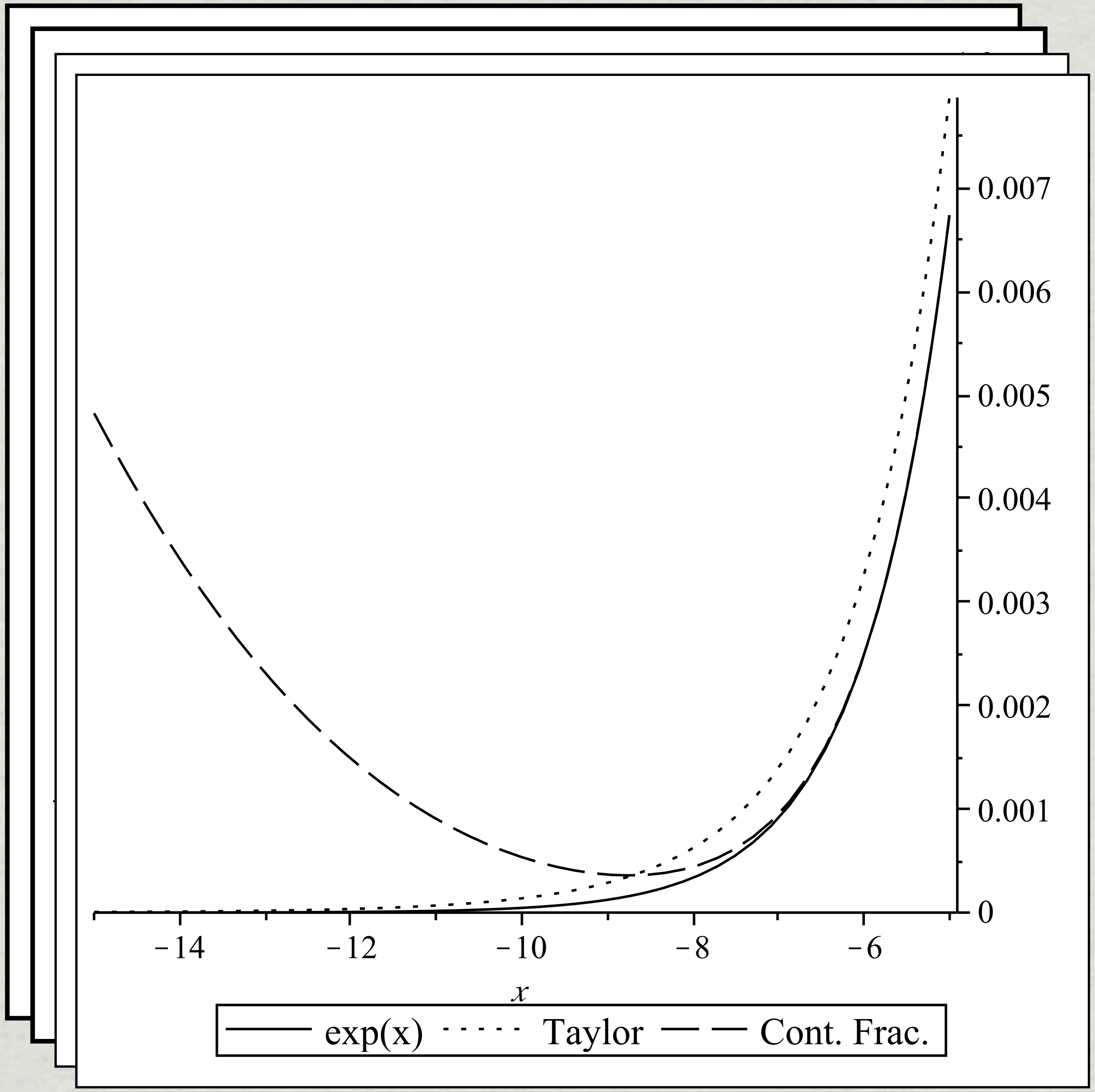
$$\exp(x) \geq 1 + x + \cdots + x^n/n! \qquad (n \text{ odd})$$

$$\exp(x) \leq 1 + x + \cdots + x^n/n! \qquad (n \text{ even}, x \leq 0)$$

$$\exp(x) \leq 1/(1 - x + x^2/2! - x^3/3!) \qquad (x < 1.596)$$

Bounds and their quirks

- Some are extremely accurate at first, but veer away drastically.
- There is no general upper bound for the exponential function.



bounds for ln

- based on the continued fraction for ln(x+1)
- much more accurate than the Taylor expansion

$$\frac{(1+19x+10x^2)(x-1)}{3x(3+6x+x^2)} \leq \ln x \leq \frac{(x^2+19x+10)(x-1)}{3(3x^2+6x+1)}$$

RCF decision procedure

- Quantifier elimination reduces a formula to TRUE or FALSE, provided it has no free variables.
- HOL-Light implements Hörmander’s decision procedure. It is fairly simple, but it hangs if the polynomial’s degree exceeds 6.
- Cylindrical Algebraic Decomposition (due to Collins) is still doubly exponential in the number of variables, but it is polynomial in other parameters. We use QEPCAD B (Hoon Hong, C. W. Brown).

Metis resolution prover

- a full implementation of the superposition calculus
- integrated with interactive theorem provers (HOL4, Isabelle)
- coded in Standard ML
- acceptable performance
- easy to modify
- due to Joe Hurd

resolution primer

- Resolution provers work with *clauses*: disjunctions of *literals* (atoms or their negations).
- They seek to *contradict* the *negation* of the goal.
- Each step combines two clauses and yields new clauses, which are *simplified* and perhaps kept.
- If the *empty clause* is produced, we have the desired contradiction.

a resolution step

$$R(x,1) \vee P(x)$$

$$\neg R(0,y) \vee Q(y)$$

$$R(0,1) \vee P(0)$$

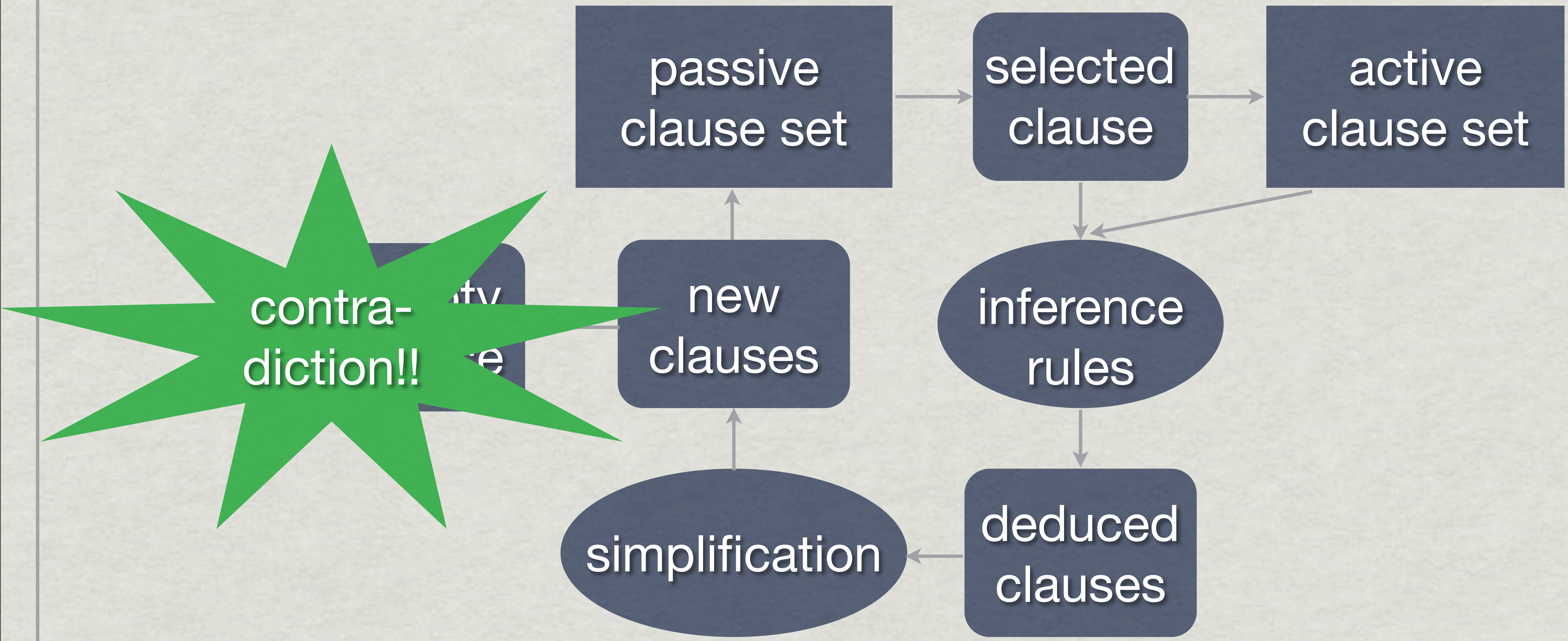
$$\neg R(0,1) \vee Q(1)$$

$$x \mapsto 0$$

$$y \mapsto 1$$

$$P(0) \vee Q(1)$$

resolution data flow



modifications to Metis

* algebraic literal deletion, via decision procedure
* algebraic redundancy test (subsumption)
* formula normalization and simplification
* modified Knuth-Bendix ordering
* “dividing out” products

algebraic literal deletion

- Our version of Metis keeps a list of all *ground, algebraic clauses* (+ − × ≤, no variables).
- Any literal that is inconsistent with those clauses can be *deleted*.
- Metis simplifies new clauses by calling QEPCAD to detect inconsistent literals.
- Deleting literals brings us closer to the empty clause!

literal deletion examples

- We delete $x^2+1 < 0$, as it has no real solutions.
- Knowing $xy > 1$, we delete the literal $x=0$.
- We take adjacent literals into account: in the clause $x^2 > 2 \vee x > 3$, we delete $x > 3$.

Specifically, QEPCAD finds $\exists x\, [x^2 \leq 2 \wedge x > 3]$ to be equivalent to FALSE.

algebraic subsumption

- If a new clause is an instance of another, it is *redundant* and should be DELETED.
- We apply this idea to *ground algebraic formulas,* deleting any that follow from existing facts.
- Example: knowing $x^2 > 4$ we can delete the clause $x < -1 \vee x > 2$.

QEPCAD: $\exists x\, [x^2 > 4 \wedge \neg(x < -1 \vee x > 2)]$

is equivalent to FALSE.

formula normalization

- How do we suppress redundant equivalent forms such as 2*x*+1, *x*+1+*x*, 2(*x*+1)−1? *Horner canonical form* is a recursive representation of polynomials.

$$a_n x^n + \cdots + a_1 x + a_0$$
$$= a_0 + x(a_1 + x(a_2 + \cdots x(a_{n-1} + x a_n))$$

The normalised formula is unique and reasonably compact.

normalization example

$$3xy^2 + 2x^2yz + zx + 3yz$$
$$= [y(z3)] + x([z1 + y(y3)] + x[y(z2)])$$

first variable

second variable

- The "variables" can be arbitrarily non-algebraic sub-expressions.
- Thus, formulas containing special functions can also be simplified, and the function *isolated*.

formula simplification

- Finally we simplify the output of the Horner transformation using laws like $0+z=z$ and $1\times z=z$.
- The maximal function term, say ln *E*, is isolated (if possible) on one side of an inequality.
- Formulas are converted to *rational functions*:

$$\left(\frac{x}{y}\right)\frac{1}{(x+\frac{1}{x})}=\frac{x^2}{y(x^2+1)}$$

choosing the best literal

$$x \leq 2 \vee \exp x \leq 2 \vee \frac{1}{x} \leq u$$

This is the critical one:
it is the most difficult!

And then this one
should be tackled next.

Knuth-Bendix ordering

- *Superposition* is a refinement of resolution, selecting the *largest* literals using an *ordering*.
- Since ln, exp, ... are complex, we give them high *weights*. This focuses the search on them.
- The Knuth-Bendix ordering (KBO) also counts *occurrences* of variables, so t is more complex than u if it contains more variables.

modified KBO

- Our bounds for $f(x)$ contain multiple occurrences of x, so standard KBO regards the bounds as worse than the functions themselves!
- Ludwig and Waldmann (2007) propose a modification of KBO that lets us say e.g. "$\ln(x)$ is more complex than 100 occurrences of x."
- This change greatly improves the is performance for our examples.

dividing out products

- The heuristics presented so far only isolate function occurrences that are *additive*.
- If a function is MULTIPLIED by an expression u, then we must divide both sides of the inequality by u.
- The outcome depends upon the sign of u.
- In general, u could be positive, negative or zero; its sign does not need to be fixed.

dividing out example

- Given a clause of the form $f(t) \cdot u \leq v \vee C$
- deduce the three clauses $f(t) \leq v/u \vee u \leq 0 \vee C$

$$0 \leq v \vee u \neq 0 \vee C$$

$$f(t) \geq v/u \vee u \geq 0 \vee C$$

- Numerous problems can only be solved using this form of inference.

notes on the axioms

- We omit general laws: transitivity is too prolific!
    - The decision procedure, QEPCAD, catches many instances of general laws.
    - We build *transitivity* into our bounding axioms.
- We use lgen(R,X,Y) to express both X≤Y (when R=0) and X<Y (when R=1).
- We identify $x<y$ with $\neg(y \leq x)$.

some exp lower bounds

Covers both $<$ and $\le$

Transitivity is built in: to show $Y<\exp(X)$, show $Y<1+X$.

```
cnf(exp_lower_taylor_1,axiom,
    ( ~ lgen(R,Y,1+X)
    | lgen(R,Y,exp(X)) )).

cnf(exp_lower_bound_cf2,axiom,
    ( ~ lgen(R, Y, (X^2 + 6*X + 12) /
                   (X^2 - 6*X + 12))
    | lgen(R,Y,exp(X)) )).
```

absolute value axioms

- Simply |X| = X if X≥0 and |X| = –X otherwise.
- It helps to give abs a high *weight*, discouraging the introduction of occurrences of abs.

```
cnf(abs_nonnegative,axiom,
   ( ~ 0 <= X
   | abs(X) = X )).

cnf(abs_negative,axiom,
   ( 0 <= X
   | abs(X) = -X )).
```

a few solved problems

problem	**seconds**
$\lvert x\rvert < 1 \Longrightarrow \lvert\ln(1+x)\rvert \le -\ln(1-\lvert x\rvert)$	0.153
$\lvert\exp(x)-1\rvert \le \exp(\lvert x\rvert)-1$	0.318
$-1 < x \Longrightarrow 2\lvert x\rvert/(2+x) \le \lvert\ln(1+x)\rvert$	4.266
$\lvert x\rvert < 1 \Longrightarrow \lvert\ln(1+x)\rvert \le \lvert x\rvert(1+\lvert x\rvert)/\lvert 1+x\rvert$	0.604
$0 < x \le \pi/2 \Longrightarrow 1/\sin^2 x < 1/x^2 + 1 - 4/\pi^2$	410

hybrid systems

- Many hybrid systems can be specified by systems of linear differential equations. (The HSOLVER Benchmark Database presents 18 examples.)
- We can solve these equations using Maple, typically yielding a problem involving the exponential function.
- MetiTarski can often solve these problems.

collision avoidance system

- differential equations for the velocity, acceleration and gap between two vehicles:

$$\dot{v} = a, \quad \dot{a} = -3a - 3(v - v_f) + gap - (v + 10), \quad g\dot{a}p = v_f - v$$

- solution for the gap (as a function of *t*):

$$gap = 12 - 14.2e^{-0.318t} + 3.24e^{-1.34t}\cos(1.16t) - 0.154e^{-1.34t}\sin(1.16t)$$

- MetiTarski can prove that the gap is positive!

some limitations

- No range reduction: proofs about exp(20) or sin(3000) are likely to fail.
- Not everything can be proved using upper and lower bounds. Adding laws like exp(X+Y) = exp(X)exp(Y) greatly increases the search space.
- Problems can have only a few variables or QEPCAD will never terminate.

example of a limitation

- We can prove this theorem if we replace 1/2 by 100/201. Approximating π by a fraction loses information.

$$0 < \mathrm{x} < 1/2 \implies \cos(\pi x) > 1 - 2x$$

related work?

- SPASS+T and SPASS(T) combine the SPASS prover with various decision procedures.
- Ratschan's RSOLVER solves quantified inequality constraints over the real numbers using constraint programming methods.
- There are many attempts to add quantification to *SMT solvers*, which solve propositional assertions involving linear arithmetic, etc.

final remarks

- By combining a resolution prover with a decision procedure, we can solve many hard problems.
- The system works by *deduction* and outputs *proofs* that could be checked independently.
- A similar architecture would probably perform well using other decision procedures.

acknowledgements

- Assistance from C. W. Brown, A. Cuyt, I. Grant, J. Harrison, J. Hurd, D. Lester, C. Muñoz, U. Waldmann, etc.
- The research was supported by the Engineering and Physical Sciences Research Council [grant number EP/C013409/1].