A modular presentation of modal logics in a logical framework (abstract)

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Keywords: modal logics, logical frameworks, proof theory

Logical Frameworks such as the Edinburgh LF [5] and Isabelle [6] have been proposed as a solution to the problem of the explosion of logics and specialized provers for them. However, it is also acknowledged that this solution is not perfect: these frameworks are best suited for encoding 'well-behaved' natural deduction formalisms whose metatheory does not deviate too far from the metatheory of the framework logic. Modal logics, in particular, are considered difficult to implement in a clean direct way (e.g. [2, §4.4.1] and [4]): The deduction theorem, 'if by adding A as an axiom we can prove B, then we can prove $A \rightarrow B$ without A', fails in modal logic, since semantically we only have that 'if, for any world w, A is true in w implies that B is true in w, then, for any world w, $A \rightarrow B$ is true in w'. Thus, a naive embedding of a modal logic in a logical framework captures the wrong consequence relation. Encodings in both the LF and Isabelle have been proposed, but they have been either based on Hilbert-style presentations (and thus difficult to use in practice) or quite specialized, and their correctness (i.e. faithfulness and adequacy) is subtle.

Motivated by this problem, we propose a natural deduction presentation of modal logics that is well-suited for manipulating within a formal meta-theory. Our starting point is the view of a logic as a Labelled Deductive System (LDS) proposed by Gabbay [3], among others. We use this approach to lift semantic information in the syntax by pairing formulae with labels: Instead of considering the formula A, we consider the labelled formula $w:A$, where the label w is a variable denoting a possible world in the Kripke model $M = (W, R, V)$. In order to be able to explicitly reason about the accessibility relation R, we introduce a second kind of formulae, relational formulae of the form $w\ R\ w'$. This allows us to give a proof-theoretic statement of the deduction theorem which is the analogue of the semantic version. The same mechanism yields a direct formalization of modal operators (e.g. $\vdash w:\Box A$ iff $\forall w' \in W(\vdash w\ R\ w' \rightarrow w':A)$), given that we can capture the behavior of R.

Our presentation proceedes by first introducing a labelled natural deduction presentation of the (basic) modal logic K, where no assumptions are made on the relations holding between possible worlds. After, using the above observations, we present a large and well known family of propositional modal logics (a subclass of the logics characterized by the generalized Geach axiom schema, including K, KD, T, B, $S4$, $S4.2$, $S5$) by adding rules capturing the properties of the corresponding model. That is, we present a modal logic parameterized over the behavior of R, which we separately present as a simple (Horn) theory of one binary relation. This allows us to specify particular modal logics by modifying this separate theory. We have implemented our work in Isabelle and the result is a simple, usable, and completely modular natural deduction implementation of these logics.

Our contributions are several. First, we show that the LDS approach can be specialized to yield a simple implementation of natural deduction presentations of propositional modal logics within logical framework based theorem provers (our presentations differ from Gabbay's proposals which cannot be directly so implemented). Second, since all logics are produced by extensions of the (Horn) theory of R, we show how this provides a natural hierarchy of logics, inheriting theorems and derived rules. This has important practical applications for the organization and construction of complex theories on a computer. Third, we use the parameterized theories to provide, once and for all, the correctness of the encodings. That is, we show the parameterized soundness and completeness (with respect to a Kripke-style semantics) of our parameterized logics, and then argue that they are faithfully and adequately embedded in a higher-order metalogic. Moreover, these theorems show that our implementation not only properly captures modal provability within our hierarchy, but also the appropriate consequence relations [1]. The use of explicit labels leads to simple proofs of these properties, but they are substantially modified compared to the standard ones. For example, to show completeness we must provide a new kind of canonical model construction that accounts for the explicit formalization of labels and of the accessibility relation in the proof system. Finally, although not formally quantifiable, our experience shows that proof construction using our presentation is natural and intuitive. This implementation is currently in use at the University of Saarbrücken for teaching students modal logic.

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