Borel Determinacy of Concurrent Games

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***Abstract*—Just as traditional games are represented by trees, so distributed/concurrent games are represented by event structures. We show the determinacy of such concurrent games with Borel sets of configurations as winning conditions, provided the games are race-free and bounded-concurrent. Both restrictions are shown necessary. The determinacy proof proceeds via a reduction to the determinacy of tree games, and the determinacy of these in turn reduces to the determinacy of Gale-Stewart games.**

***Keywords*- Concurrent games; Non-deterministic strategies; Winning conditions; Borel Determinacy; Event structures.**

I. Introduction

In mathematical logic, the study of determinacy problems (the existence of winning strategies) dates back, at least, to Zermelo's work [13] on finite games; he showed that all two-player perfect-information games of finite length are determined. Since then, more complex games and determinacy results have been studied, *e.g.* for games with plays of infinite length. A research line that began in the 1950s with the work of Gale and Stewart [5] on open games culminated with the work of Martin [6] who showed that all two-player zero-sum sequential games with perfect information in which the plays (winning sets) of the game form a Borel set are determined.

In computer science, determinacy results have most often been used rather than investigated, especially in verification. For instance, frequently a game with winning conditions is used to represent a given decision or verification problem, e.g. satisfiability, equivalence, or model-checking; winning strategies encode solutions to the problems being represented by the game. The determinacy of the games ensures that in all cases there is a solution to the problem under consideration. Determinacy is a desirable property from an algorithmic viewpoint as it guarantees the existence of winning strategies.

An important common feature of the games mentioned above is that they are generally represented as trees. As a consequence, the plays of such games form total orders—the branches/paths of such trees. The games we consider in this paper are not restricted to games represented by trees. Instead, they are played on games represented by event structures. Event structures [9] are the concurrency analogue of trees. Just as transitions systems unfold to trees, so Petri nets unfold to event structures. Plays are now partial orders of moves.

The concurrent games we consider are an extension of those introduced in [10]. Games there can be thought of as highly-interactive, distributed games between Player (thought of as a team of players) and Opponent (a team of opponents). The games model, as first introduced in [10], was extended with winning conditions in [4]. There a determinacy result was given for well-founded games (*i.e.* where only *finite* plays are possible) provided they are *race-free*, *i.e.* neither player could interfere with the moves available to the other.

Here we extend the result of [4] by providing a much more general determinacy theorem. We consider concurrent games in which plays may be *infinite* and where the winning set of configurations form a Borel set. We show such games are determined provided they have *bounded concurrency* and are race-free. Bounded concurrency expresses that no event played by one of the players can be concurrent with (independent of) infinitely many events played by the other player—a condition trivially satisfied when all plays are finite. We show in what sense bounded concurrency and race-freedom are necessary restrictions for Borel determinacy. Our determinacy proof follows by a reduction to the determinacy of Gale-Stewart games with Borel winning conditions, shown by Martin [6].

Related work: Determinacy problems have been studied for more than a century: *e.g.* for finite games [13]; open games [5]; Borel games [6]; or Blackwell games [7], to mention a few particularly relevant in computer science. Whereas the determinacy theorem in [4] is a concurrent generalisation of Zermelo's determinacy theorem for finite games, the determinacy theorem in this paper generalises the Borel determinacy theorem for infinite games—from trees to event structures, *i.e.* from total to partial orders of moves.

There are other concurrent games for which it is known whether they are determined or not. For instance, the games in [1], [8] only allow deterministic strategies—the reason why they are not determined, *cf.* [4]; Games in logic and formal verification such as those in [2], [3] are not determined with respect to pure strategies. However, in games where the players are allowed to use mixed (*i.e.* probabilistic) strategies, *e.g.* as in [3], are known to be determined up to some real value of accuracy; this follows from the determinacy of Blackwell games [7]—a class of imperfect-information games. Since our games require additional structure in order to model imperfect-information [11] and indeed probability, the determinacy result in this paper does not apply directly to Blackwell games, and so neither to the concurrent games in [3]—though see [12].

Structure of the paper: Section II presents concurrent games based on event structures and Section III introduces event structure analogues of tree games and Gale-Stewart games. In Section IV race-freedom and bounded concurrency are defined and a proof that they are necessary restrictions for the determinacy of concurrent games with Borel winning conditions is presented. Section V contains the main result: the Borel determinacy theorem. Section VI concludes.

II. Concurrent games on event structures

An *event structure* comprises $(E, \le, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\le$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy four axioms:

$$\{e' \mid e' \le e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$
$$Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}$$
$$X \in \mathrm{Con} \;\&\; e \le e' \in X \implies X \cup \{e\} \in \mathrm{Con}.$$

The *configurations* of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \le e \in x \implies e' \in x$.

We write $\mathcal{C}^\infty(E)$ for the set of *finite and infinite* configurations of the event structure E.

Two events e_1, e_2 which are both consistent and incomparable with respect to causal dependency in an event structure are regarded as *concurrent*, written $e_1\, co\, e_2$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \le e'$ and no event in between plays an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \le e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$ then $[X] \in \mathrm{Con}$. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{C}^\infty(E)$, i.e. $x \subset y$ with nothing in between, and $x \stackrel{e}{-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \stackrel{e}{-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \stackrel{e}{-\!\!\subset} y$ for some y.

Let E and E' be event structures. A *(partial) map* of event structures is a partial function on events $f : E \to E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$ (with both defined) then $e_1 = e_2$. The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Partial maps of event structures compose as partial functions, with identity maps given by identity functions. We say that the map is *total* if the function f is total.

The category of event structures is rich in useful constructions on processes. For instance, it has products and pullbacks (both forms of synchronised composition) and coproducts (nondeterministic sums). In particular, pullbacks will be used to define the composition of *strategies* on event structures.

Event structures with polarity: Both a game and a strategy in a game are represented with event structures with polarity, comprising an event structure E together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity + (Player) or − (Opponent) to its events; the events correspond to moves. Maps of event structures with polarity, are maps of event structures which preserve polarities. An event structure with polarity E is *deterministic* iff

$$\forall X \subseteq_{\mathrm{fin}} E.\ Neg[X] \in \mathrm{Con}_E \implies X \in \mathrm{Con}_E\,,$$

where $Neg[X] =_{\mathrm{def}} \{e' \in E \mid pol(e') = - \;\&\; \exists e \in E.\ e' \le e\}$.

The *dual*, $E^\perp$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities. This operation is useful when reasoning about games (it interchanges the players' roles in a game).

Games and strategies: Let A be an event structure with polarity, thought of as a game; its events stand for the possible moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey its constraints.

A *strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S, which is both *receptive* and *innocent*. Receptivity ensures an openness to all possible moves of Opponent. Innocence, on the other hand, restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game. Formally:

Receptivity: A map σ is *receptive* iff
$\sigma x \stackrel{a}{-\!\!\subset} \;\&\; pol_A(a) = - \Rightarrow \exists! s \in S.\ x \stackrel{s}{-\!\!\subset} \;\&\; \sigma(s) = a\,.$

Innocence: A map σ is *innocent* iff
$s \rightarrowtriangle s' \;\&\; (pol(s) = +$ or $pol(s') = -)$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

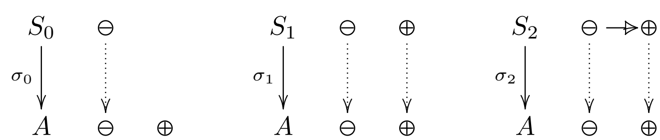
a) Composing strategies: The composition of strategies can be defined via pullbacks. Suppose that $\sigma : S \to A$ is a strategy in a game A. A counter-strategy is a strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,

$$\begin{array}{ccc} P & \xrightarrow{\Pi_2} & T \\ {\scriptstyle \Pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow[\sigma]{} & A\,, \end{array}$$

to obtain the event structure P resulting from the interaction $\tau \odot \sigma$ of σ and τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}(P)$. A maximal z images to a configuration $\sigma \Pi_1 z = \tau \Pi_2 z$ in $\mathcal{C}(A)$. Define the set of *results* of playing σ against τ to be

$$\langle \sigma, \tau \rangle =_{\mathrm{def}} \{\sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}(P)\}\,.$$

Example 1. Let $\sigma_i : S_i \to A$ be a strategy in $A = \oplus\; co\; \ominus$



Likewise, there are three counter-strategies $\tau_j : T_i \to A^\perp$ for Opponent—the unique dual strategies obtained in $A^\perp$. The results of playing each σ_i against each τ_j are as follows:

$$\langle \sigma_i, \tau_j \rangle = \begin{cases} \{\varnothing\} & \text{if } i \in \{0,2\} \;\&\; j \in \{0,2\}, \\ \{\{\oplus\}\} & \text{if } i = 1 \;\&\; j = 0, \\ \{\{\ominus\}\} & \text{if } i = 0 \;\&\; j = 1, \\ \{\{\oplus, \ominus\}\} & \text{otherwise.} \end{cases}$$

Note that Player/Opponent can try to force *some* plays to happen sequentially by adding causal dependencies, e.g. when using σ_2/τ_2. This situation may lead to a deadlock since, when using σ_2/τ_2, Player/Opponent would stay waiting for Opponent/Player to play first—which may never happen. □

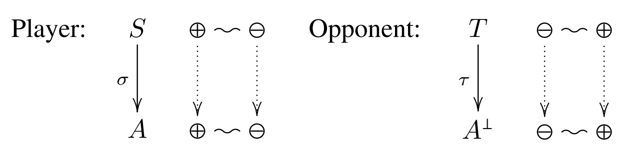
Determinacy and winning conditions: A *game with winning conditions* [4] comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}(A)$ consists of the *winning configurations* for Player. Define the *losing conditions* to be $L = \mathcal{C}(A) \smallsetminus W$. The dual $G^\perp$ of a game with winning conditions $G = (A, W)$ is defined to be $G^\perp = (A^\perp, L)$, a game where the roles of Player and Opponent are reversed, and consequently that of winning and losing conditions too.

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all $\oplus$-maximal configurations $x \in \mathcal{C}(S)$—a configuration x is $\oplus$-maximal if whenever $x \overset{s}{\longrightarrow\!\subset}$ then the event s has −ve polarity.

Equivalently, a strategy σ for Player is winning if when played against any counter-strategy τ of Opponent, the final result is a win for Player. It can be shown [4] that a strategy σ is a winning for Player iff all the results of the interaction $\langle \sigma, \tau \rangle$ lie within W, for any counter-strategy τ of Opponent.

A game with winning conditions is *determined* when either Player or Opponent has a winning strategy in the game.

Example 2. Consider the game A with two inconsistent events $\oplus$ and $\ominus$ with the obvious polarities and winning conditions $W = \{\{\oplus\}\}$. In the game (A, W) no strategy for either player dominates all other counter-strategies of the other player. In particular, let σ be the unique map of event structures that contains $\oplus$ and τ a particular counter-strategy for Opponent:



This game is not determined since neither $\langle \sigma, \tau \rangle \subseteq W$ nor $\langle \sigma, \tau \rangle \subseteq L$, i.e. because $\{\{\oplus\}, \{\ominus\}\} \subseteq \langle \sigma, \tau \rangle$. □

III. Tree games and Gale–Stewart games

We introduce tree games as a special case of concurrent games, traditional Gale–Stewart games as a variant, and show how to reduce the determinacy of tree games to that of Gale–Stewart games. Via Martin's theorem for the determinacy of Gale–Stewart games with Borel winning conditions we show that tree games with Borel winning conditions are determined.

A. Tree games

Definition 3. Say E, an event structure with polarity, is *tree-like* iff it is race-free, has empty concurrency relation (so $\leq_E$ forms a forest) and is such that polarities alternate along branches, *i.e.* if $e \rightarrowtriangle e'$ then $pol_E(e) \neq pol_E(e')$.

A *tree game* is (E, W), a concurrent game with winning conditions in which E is tree-like. □

Proposition 4. *Let E be a tree-like event structure with polarity. Then, its configurations $\mathcal{C}^\infty(E)$ form trees w.r.t. $\subseteq$. Its root is the empty configuration $\varnothing$. Its (maximal) branches may be finite or infinite; finite sub-branches correspond to finite configurations of E; infinite branches correspond to infinite configurations of E. Its arcs, associated with $x \overset{e}{\longrightarrow\!\subset} x'$, are in 1-1 correspondence with events $e \in E$. The events e associated with initial arcs $\varnothing \overset{e}{\longrightarrow\!\subset} x$ all have the same polarity. In a branch*

$$\varnothing \overset{e_1}{\longrightarrow\!\subset} x_1 \overset{e_2}{\longrightarrow\!\subset} x_2 \overset{e_3}{\longrightarrow\!\subset} \cdots \overset{e_i}{\longrightarrow\!\subset} x_i \overset{e_{i+1}}{\longrightarrow\!\subset} \cdots$$

the polarities of the events $e_1, e_2, \ldots, e_i, \ldots$ alternate.

Proposition 4 gives the precise sense in which the terms 'arc,' 'sub-branch' and 'branch' are synonyms for the terms 'events,' 'configurations' and 'maximal configurations' when an event structure with polarity is tree-like.

Definition 5. We say a non-empty tree game (E, W) has polarity + or − depending on whether its initial events are +ve (positive) or −ve (negative). It is convenient to adopt the convention that the empty game $(\varnothing, \varnothing)$ has polarity +, and the empty game $(\varnothing, \{\varnothing\})$ has polarity −. □

Observe that:

Proposition 6. *Let $f : S \to A$ be a total map of event structures with polarity and let A be tree-like. Then, it follows that S is also tree-like and that the map f is innocent. The map f is a strategy if and only if it is receptive.*

B. Gale–Stewart games

We shall present Gale–Stewart games as a slight variant of tree games, a variant in which all maximal configurations of the tree game are infinite, and where Player and Opponent must play to a maximal, infinite configuration.

Definition 7. A *Gale–Stewart* game (G, V) comprises

- G, a tree-like event structure with polarity for which all maximal configurations are infinite, and
- V, a subset of infinite configurations—the *winning* configurations of the game (for Player).

A *winning strategy* in (G, V) is $\sigma : S \to G$, a deterministic strategy such that $\sigma x \in V$ for all maximal x in $\mathcal{C}^\infty(S)$. □

This is not how a Gale–Stewart game and a winning strategy in a Gale–Stewart game are traditionally defined. However, because σ is deterministic it is injective as a map on configurations, so corresponds to the subfamily of configurations $T = \{\sigma x \mid x \in \mathcal{C}^\infty(S)\}$ of $\mathcal{C}^\infty(G)$. The family T forms a subtree of the tree of configurations of G. Its properties, given below, reconcile our definition with the traditional one.

Proposition 8. *A winning strategy in a Gale–Stewart game (G, V) is a non-empty subset $T \subseteq \mathcal{C}^\infty(G)$ such that*

(i) $\forall x, y \in \mathcal{C}^\infty(G).\ \ y \subseteq x \in T \implies y \in T$,
(ii) $\forall x, y \in \mathcal{C}(G).\ \ x \in T \ \&\ x \overset{-}{\longrightarrow\!\subset} y \implies y \in T$,
(iii) $\forall x, y_1, y_2 \in T.\ x \overset{+}{\longrightarrow\!\subset} y_1 \ \&\ x \overset{+}{\longrightarrow\!\subset} y_2 \implies y_1 = y_2$, *and*
(iv) *all $\subseteq$-maximal members of T are infinite and in V.*

A Gale–Stewart game (G, V) has a *dual* game $(G, V)^* =_{\text{def}} (G^\perp, V^*)$, where V^* is the set of all maximal configurations in $\mathcal{C}^\infty(G)$ not in V. A winning strategy for Opponent in (G, V) is a winning strategy (for Player) in the dual game $(G, V)^*$.

For any event structure A there is a topology on $\mathcal{C}^\infty(A)$ given by the Scott open subsets. The $\subseteq$-maximal configurations in $\mathcal{C}^\infty(A)$ inherit a sub-topology from that on $\mathcal{C}^\infty(A)$. The Borel subsets of a topological space are those subsets of configurations in the sigma-algebra generated by the Scott open subsets. Donald Martin proved in [6] that Gale–Stewart games (G, V) are determined, *i.e.* there is either a winning strategy for Player or a winning strategy for Opponent, when V is a Borel subset of the maximal configurations of $\mathcal{C}^\infty(A)$.

C. Determinacy of tree games

We show the determinacy of tree games with Borel winning conditions through a reduction of the determinacy of tree games to the determinacy of Gale–Stewart games.

Let (E, W) be a tree game. We construct a Gale–Stewart game $\text{GS}(E, W) = (G, V)$ and a partial map $\textit{proj} : G \to E$. The events of G are built as sequences of events in E together with two new symbols δ^- and δ^+ decreed to have polarity $-$ and $+$, respectively; the symbols δ^- and δ^+ represent delay moves by Opponent and Player, respectively.

Precisely, an event of G is a non-empty finite sequence

$$[e_1, \cdots, e_k]$$

of symbols from $E \cup \{\delta^-, \delta^+\}$ where: e_1 has the same polarity as (E, W); polarities alternate along the sequence; and for all subsequences $[e_1, \cdots, e_i]$, with $i \le k$,

$$\{e_1, \cdots, e_i\} \cap E \in \mathcal{C}(E) .$$

The immediate causal dependency relation of G is given by

$$[e_1, \cdots, e_k] \le_G [e_1, \cdots, e_k, e_{k+1}]$$

and consistency by compatibility w.r.t. $\le_G$. Events $[e_1, \cdots, e_k]$ of G have the same polarity as their last entry e_k. Note that G is tree-like and that the only maximal configurations are infinite (because of the possibility of delay moves).

The map $\textit{proj} : G \to E$ takes an event $[e_1, \cdots, e_k]$ of G to e_k if $e_k \in E$, and is undefined otherwise. The set V consists of all those configurations x of G for which $\textit{proj}\, x \in W$.

We have built a Gale–Stewart game $\text{GS}(E, W) = (G, V)$. The construction respects the duality on games. The following lemma follows from the definition of the operation GS.

Lemma 9. *Letting (E, W) be a tree game, we have that*

$$\text{GS}((E, W)^\perp) = (\text{GS}(E, W))^* .$$

Suppose $\sigma : S \to G$ is a winning strategy for (G, V). The composite

$$S \xrightarrow{\sigma} G \xrightarrow{\textit{proj}} E \qquad (F1)$$

is a partial map of event structures with polarity. Letting $D \subseteq S$ be the subset of events on which $\textit{proj} \circ \sigma$ is defined, the map $\textit{proj} \circ \sigma$ factors as

$$S \longrightarrow S{\downarrow}D \xrightarrow{\sigma_0} E \qquad (F2)$$

where: the first partial map acts like the identity on events in D and is undefined otherwise—it sends a configuration $x \in \mathcal{C}^\infty(S)$ to $x \cap D \in \mathcal{C}^\infty(S{\downarrow}D)$; and σ_0 is the total map that acts like σ on D. We shall show that σ_0 is a (possibly *nondeterministic*) winning strategy for (E, W).

Lemma 10. *The map σ_0 is a winning strategy for (E, W).*

Proof: Write $S_0 =_{\text{def}} S{\downarrow}D$. By Proposition 6, for the map $\sigma_0 : S_0 \to E$ to be a strategy we only require its receptivity. From the construction of G and $\textit{proj}$,

$$\textit{proj}\, x \,{-}\!\!\subset y \text{ in } \mathcal{C}(E) \implies \exists! x' \in \mathcal{C}(G).\ x \,{-}\!\!\subset x' \ \&\ \textit{proj}\, x' = y .$$

This (with the receptivity of σ) entails the receptivity of σ_0.

To show σ_0 is winning, suppose z is $\oplus$-maximal in $\mathcal{C}^\infty(S)_0$; we require $\sigma_0 z \in W$. We show this by exhibiting an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$. Then, according to the factorisation $(F2)$, $x \mapsto z \mapsto \sigma_0 z$, so we have $\sigma_0 z = \textit{proj}\, \sigma x$. The configuration x being infinite ensures $\sigma x \in V$ because σ is winning in the Gale–Stewart game (G, V). By definition, $\sigma x \in V$ implies $\textit{proj}\, \sigma x \in W$, so $\sigma_0 z \in W$.

It remains to exhibit an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$. When z is infinite this is readily achieved by defining $x =_{\text{def}} [z]_S \in \mathcal{C}^\infty(S)$. Suppose z is finite. Define $x_0 =_{\text{def}} [z]_S \in \mathcal{C}(S)$, ensuring $x_0 \cap D = z$. We inductively build an infinite chain

$$x_0 \overset{s_1}{-\!\!\subset} x_1 \overset{s_2}{-\!\!\subset} \cdots \overset{s_n}{-\!\!\subset} x_n \overset{s_{n+1}}{-\!\!\subset} \cdots$$

in $\mathcal{C}(S)$ where all the events s_n are 'delay' moves not in D. Then $x_n \cap D = z$ for all $n \in \mathbb{N}$. By the definition of a winning strategies in Gale–Stewart games, no x_n can be $\subseteq$-maximal in $\mathcal{C}(S)$. For each Opponent move s_n choose to delay—as we may do by the receptivity of σ. For each Player move s_n we have no choice as only a delay move is possible—otherwise we would contradict the $\oplus$-maximality assumed of z. Taking $x =_{\text{def}} \bigcup_n x_n$ produces an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$, as required. ■

Corollary 11. *Let H be a tree game. If the game* $\text{GS}(H)$ *has a winning strategy, then H has a winning strategy.*

Theorem 12. *Tree games with Borel winning conditions are determined.*

Proof: Assume (E, W) is a tree game and W is Borel. Construct $\text{GS}(E, W) = (G, V)$ as above. The function $\textit{proj}$, acting as $x \mapsto \textit{proj}\, x$ on configurations, is a Scott-continuous function from $\mathcal{C}^\infty(G) \to \mathcal{C}^\infty(E)$. It restricts to a continuous function from the subspace of maximal configurations in $\mathcal{C}^\infty(G)$. Hence V, as the inverse image of W under this restricted function, is Borel. By Martin's Borel-determinacy theorem [6], the game (G, V) is determined.

First, suppose $\text{GS}(E, W)$ has a winning strategy for Player. By Corollary 11 we obtain a winning strategy for (E, W). Suppose, on the other hand, $\text{GS}(E, W)$ has a winning strategy for Opponent, *i.e.* Player wins $\text{GS}(E, W)^*$. By Lemma 9, $\text{GS}((E, W)^\perp) = \text{GS}(E, W)^*$ has a winning strategy for Player. By Corollary 11, $(E, W)^\perp$ has a winning strategy, *i.e.* there is a winning strategy for Opponent in (E, W). ■

IV. Race-freedom and bounded-concurrency

Not all games are determined, cf. Example 2. However, a determinacy theorem holds for well-founded games (games where all configurations are finite) which satisfy a property called *race-freedom*:

$$x \overset{a}{\longrightarrow\!\!\subset} \;\&\; x \overset{a'}{\longrightarrow\!\!\subset} \;\&\; pol(a) \neq pol(a') \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\mathbf{Race-freedom})$$

Note that the game in Example 2 is not *race-free*. However, for well-founded race-free games the following holds:

Theorem 13 (from [4])**.** *Let A be a well-founded game. Then (A, W) is determined for all W iff A is race-free.*

It is easy to believe that a nondeterministic winning strategy always has a winning deterministic sub-strategy. This is not so and determinacy does not hold for well-founded race-free games if we restrict to deterministic strategies, cf. [4]. Another observation made in [4] is that being race-free is not in itself sufficient to ensure determinacy when infinite behaviour is allowed, *i.e.* when A is not well-founded.

Example 14. Let A be the event structure with polarity consisting of one positive event $\oplus$ which is concurrent with an infinite chain of alternating negative and positive events (and let $i \in \mathbb{N}$), *i.e.* for each i we have both $\oplus$ *co* $\oplus_i$ and $\oplus$ *co* $\ominus_i$:

$$A = \qquad \oplus \qquad \ominus_1 \longrightarrow\!\!\!\!\!\!\triangleright \oplus_1 \longrightarrow\!\!\!\!\!\!\triangleright \ominus_2 \longrightarrow\!\!\!\!\!\!\triangleright \oplus_2 \longrightarrow\!\!\!\!\!\!\triangleright \cdots$$

and winning conditions (for Player) given by

$$W = \{\varnothing, \{\ominus_1, \oplus_1\}, ..., \{\ominus_1, \oplus_1, ..., \ominus_i, \oplus_i\}, ..., A\}.$$

Intuitively, Player wins if (i) no event is played, or (ii) the event $\oplus$ is not played and the play is finite and finishes in some $\oplus_i$, or (iii) all of the events in A are played. Otherwise, Opponent wins the game—recall that $L = \mathcal{C}(A) \setminus W$.

Player does not have a winning strategy because Opponent has an infinite family of strategies which cannot all be dominated by a single strategy of Player. Let $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$ be strategies for Opponent, with $i \in \mathbb{N}$, such that

$$\begin{array}{lll} T_\infty^\perp & =_{\text{def}} & A \text{ , and} \\ T_i^\perp & =_{\text{def}} & A \setminus \{e' \in A \mid \ominus_i \le e' \text{ for some finite } i\}. \end{array}$$

Any strategy for Player that plays $\oplus$ is dominated by some strategy τ_i for Opponent; likewise, any strategy for Player that does not play $\oplus$ and plays only finitely many positive events $\oplus_i$ is also dominated by some strategy τ_i for Opponent. moreover, a strategy for Player that does not play $\oplus$ and plays all of the events $\oplus_i$ in A is dominated by τ_∞. Then, Player does not have a winning strategy in this game.

Similarly, Opponent does not have a winning strategy in A because Player has two strategies that cannot be both dominated by any strategy for Opponent. Let $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$ be strategies for Player such that

$$\begin{array}{lll} S_{\overline{\oplus}} & =_{\text{def}} & A \setminus \{\oplus\} \text{ , and} \\ S_\oplus & =_{\text{def}} & A \end{array}$$

On the one hand, any strategy for Opponent that plays only finitely many (possibly zero) negative events $\ominus_i$ is dominated by $\sigma_{\overline{\oplus}}$; on the other hand, any strategy for Opponent that plays all of the negative events $\ominus_i$ in A is dominated by $\sigma_\oplus$. Thus, neither player has a winning strategy in this game! □

An issue when building a winning strategy for Player is that $\oplus$ cannot causally depend on *infinitely many* events. A natural property to be required of a concurrent game in order that it be determined is that an event is not concurrent with infinitely many events of the opposite polarity. This property is called *bounded-concurrency*:

$$\begin{array}{l} \forall y \in \mathcal{C}^\infty(A).\ \forall e \in y. \\ \qquad \{e' \in y \mid e \ co\ e' \ \&\ pol(e) \neq pol(e')\} \text{ is finite.} \end{array} \qquad (\mathbf{Bounded-concurrency})$$

Bounded concurrency, as we will show, is in fact a necessary restriction for determinacy w.r.t. Borel winning conditions.

Notation 15. We shall write $max_+(y', y)$ if and only if y' is $\oplus$-maximal in y, *i.e.* $y' \overset{e}{\longrightarrow\!\!\subset} \;\&\; pol(e) = + \implies e \notin y$; in a dual way, we write $\overline{max}_+(y', y)$ if and only if y' is not $\oplus$-maximal in y. We also use max_- when $pol(e) = -$ instead. □

In order to show that if A—a race-free event structure with polarity—is not bounded-concurrent, then there are Borel winning conditions W so that the game (A, W) is not determined, we shall use the following general schema (a set of rules) for defining the winning conditions/sets of the game.

Since A is not bounded-concurrent, there is $y \in \mathcal{C}^\infty(A)$ and $e \in y$ such that e is concurrent with infinitely many events $e_i \in y$ of opposite polarity. W.l.o.g. assume that $pol(e) = +$ and based on y define W using the following rules (let $y' \in \mathcal{C}(A)$):

1) $y' \supseteq y \implies y' \in W$;
2) $y' \subset y \ \&\ e \in y' \implies y' \in L$;
3) $y' \subset y \ \&\ e \notin y' \ \&$
   $max_+(y', y \setminus \{e\}) \ \&\ \overline{max}_-(y', y \setminus \{e\}) \implies y' \in W$;
4) $y' \subset y \ \&\ e \notin y' \ \&$
   $\overline{max}_+(y', y \setminus \{e\})$ or $max_-(y', y \setminus \{e\}) \implies y' \in L$;
5) $y' \not\subset y \ \&\ y' \not\supseteq y \ \&\ (y' \cap y) \subset^- y' \implies y' \in W$;
6) $y' \not\subset y \ \&\ y' \not\supseteq y \ \&\ (y' \cap y) \subset^+ y' \implies y' \in L$;
7) otherwise assign any polarity to y'.

The rules assign a winner to every configuration. Moreover, no y' is assigned as winning to both Player and Opponent: the implications' antecedents are pair-wise mutually exclusive.[1]

Lemma 16. *Let (A, W) be a race-free game. If A is not bounded-concurrent then there is W, a Borel subset of $\mathcal{C}^\infty(A)$, such that the game (A, W) is not determined.*

Proof: (Sketch) Define W using the set of rules given above. W.l.o.g. assume y is minimal in the sense that if $y = y_1 \uplus y_2$ then either $Neg[y_1]$ is finite or $Neg[y_2]$ is finite.

Player does not have a winning strategy. Consider the following infinite family of strategies for Opponent, namely $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$ (for $i \in \mathbb{N}$ and recall that for each $e_i \in y$ we have that e *co* e_i), such that:

$$\begin{array}{lll} T_\infty^\perp & =_{\text{def}} & \{e' \in A \mid e' \in y \ \vee\ pol(e) = +\} \text{ , and} \\ T_i^\perp & =_{\text{def}} & \{e' \in A \mid e' \in y \setminus \{e_i\} \ \vee\ pol(e') = +\}. \end{array}$$

[1] Note that the winning conditions W in Example 14 are a particular instance of the use of this set of winning rules—use rules 1 and 3.

Then each strategy τ for Opponent only plays negative events contained in y; and, each strategy τ_i does not play a −ve event e_i which is concurrent with e. For a contradiction, suppose Player has a winning strategy $\sigma : S \to A$.

Because of the definition of τ_∞ we know that for all $y' \in \langle\sigma,\tau_\infty\rangle$, we have that $y' \supseteq y$ (Player only wins using rule 1 as rules 3 and 5 do not apply). Then $Pos[y] \subseteq \sigma S$.

We also have that σ dominates τ_i, for every $i \in \mathbb{N}$. As every τ_i does not play some −ve event in y then Player cannot win using rule 1 when playing against every τ_i. And, as each τ_i never plays −ve events not in y then Player cannot win using rule 5 either. Thus, Player can only win using rule 3; thus

$$\forall \tau_i.\ y' \in \langle\sigma,\tau_i\rangle \Longrightarrow e \notin y'.$$

But we know that there is $s_e \in S$ such that $\sigma(s_e) = e$. Since $[e]$ is finite then $[s_e]$ is finite too. And because $Neg[y]$ is infinite, then there are infinitely many τ_i such that

$$\exists y' \in \langle\sigma,\tau_i\rangle.\ y' \subset y \ \&\ e \in y',$$

i.e. infinitely many τ_i with which Opponent wins using rule 2; contradiction. Then, σ is not a winning strategy.

Opponent does not have a winning strategy either. Consider $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$, two Player strategies where:

$$\begin{aligned} S_\oplus &=_{\text{def}} \{e' \in A \mid e' \in y \ \lor\ pol(e) = -\}\ \text{, and} \\ S_{\overline{\oplus}} &=_{\text{def}} \{e' \in A \mid e' \in y \setminus \{e\} \ \lor\ pol(e) = -\}. \end{aligned}$$

Thus, $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ only play +ve events in y; moreover, $\sigma_\oplus$ plays $\oplus$-maximally in y—hence in $y \setminus \{e\}$ too—and $\sigma_{\overline{\oplus}}$ plays $\oplus$-maximally in $y \setminus \{e\}$. And, while $\sigma_\oplus$ plays e as long as Opponent plays $Neg[e]$, the strategy $\sigma_{\overline{\oplus}}$ never plays e.

Again, for a contradiction, suppose Opponent has a strategy $\tau : T \to A^\perp$ that is winning. Because of the definitions of $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ and the set of winning rules there are two ways how τ can win (see rules 2 and 4), namely when:

(i) $y' \subset y \ \&\ e \in y'$, or
(ii) $y' \subset y \ \&\ e \notin y' \ \&\ max_-(y', y \setminus \{e\})$.

The first observation is that since both $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ play $\oplus$-maximally in $y \setminus \{e\}$, then every result y' of playing τ against either $\sigma_\oplus$ or $\sigma_{\overline{\oplus}}$ satisfies that $max_+(y', y \setminus \{e\})$. The second observation is that since $y \setminus y' \neq \varnothing$ and $max_+(y', y \setminus \{e\})$, then it follows that for all $e' \in y$ such that $y' \xrightarrow{e'}\subset$ we have that $pol(e') = + \Longrightarrow e' = e$ and $e \in y' \Longrightarrow pol(e') = -$. Let $y' \in \langle\sigma_\oplus,\tau\rangle$. Since $max_+(y', y)$ then $pol(y \setminus y') \subseteq \{-\}$.

Either $e \notin y'$ or $e \in y'$. The former is impossible. Then, y' satisfies (i). As y' is $\oplus$-maximal in y, then τ does not play at least one −ve event e_i in A which does not causally depend on e. Then, $[e_i)$ is a sub-configuration of some $y'_i \in \langle\sigma_{\overline{\oplus}},\tau\rangle$ because e_i is not in conflict with any event in y and $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ produce the same results—unless they contain e.

Now, let $y' \in \langle\sigma_{\overline{\oplus}},\tau\rangle$. In this case, $max_+(y', y \setminus \{e\})$ holds and hence $\forall e' \in y.\ y' \xrightarrow{e'}\subset \ \&\ pol(e') = + \Longrightarrow e' = e$.

Necessarily $e \notin y'$ and Opponent can only win using rule 4, *i.e.* y' satisfies (ii) above. This implies that $max_-(y', y \setminus \{e\})$ must hold and we know that $max_+(y', y \setminus \{e\})$ holds too. As y' is both $\oplus$-maximal and $\ominus$-maximal in $y \setminus \{e\}$ and $y \setminus y' \neq \varnothing$, then there is only one event that y' enables, namely e.

Since e is concurrent with infinitely many $e_i \in A$, then all such e_i must already be in y'—hence $Neg[y']$ is infinite. Recall that y is a *miminal* configuration in the sense that if $y = y_1 \uplus y_2$ then either $Neg[y_1]$ is finite or $Neg[y_2]$ is finite. Let $y_1 = y'$ and $y_2 = y \setminus y'$. Since $Neg[y_1]$ is infinite then $Neg[y_2]$ is finite. And the smallest such a set is $y_2 = \{e\}$.

Thus, $y \setminus y' = \{e\}$ and $Neg[y] \subseteq (\tau T)^\perp$, which leads to a contradiction. Again, note that the existence of a y'_i such that $y'_i \in \langle\sigma_{\overline{\oplus}},\tau\rangle.\ [e_i) \subseteq y'_i \ \&\ y'_i \xrightarrow{e_i}\subset$, with $pol(e_i) = -$, violates that $max_-(y', y \setminus \{e\})$—the reason why $Neg[y] \subseteq (\tau T)^\perp$ cannot hold (as well as why a y' satisfying (ii) is impossible). Thus, Opponent does not have a winning strategy either. ∎

V. Borel determinacy for concurrent games

We now construct a tree game $\mathrm{TG}(A,W)$ from a concurrent game (A,W). We can think of the events of $\mathrm{TG}(A,W)$ as corresponding to (non-empty) *rounds* of −ve (negative) or +ve (positive) events in the original concurrent game (A,W). When (A,W) is race-free and bounded-concurrent, a winning strategy for $\mathrm{TG}(A,W)$ will induce a winning strategy for (A,W). In this way we reduce determinacy of concurrent games to determinacy of tree games.

A. The tree game of a concurrent game

From a concurrent game (A,W) we construct a tree game

$$\mathrm{TG}(A,W) = (\mathit{TA}, \mathit{TW}).$$

The construction of TA depends on whether $\varnothing \in W$.

In the case where $\varnothing \in W$, define an alternating sequence of (A,W) to be a sequence

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2i} \subset^- x_{2i+1} \subset^+ x_{2i+2} \subset^- \cdots$$

of configurations in $\mathcal{C}^\infty(A)$—the sequence need not be maximal. Define the −ve events of $\mathrm{TG}(A,W)$ to be

$$[\varnothing, x_1, x_2, \ldots, x_{2k-2}, x_{2k-1}],$$

finite alternating sequences of the form

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2k-2} \subset^- x_{2k-1},$$

and the +ve events to be

$$[\varnothing, x_1, x_2, \ldots, x_{2k-1}, x_{2k}],$$

finite alternating sequences

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^- x_{2k-1} \subset^+ x_{2k},$$

where $k \geq 1$. The causal dependency relation on TA is given by the relation of initial sub-sequence, with a finite subset of events being consistent iff the events are all initial sub-sequences of a common alternating sequence.

It is easy to see that a configuration of TA corresponds to an alternating sequence, the −ve events of TA matching arcs $x_{2k-2} \subset^- x_{2k-1}$ and the +ve events arcs $x_{2k-1} \subset^+ x_{2k}$. As such, we say a configuration $y \in \mathcal{C}^\infty(TA)$ is winning, and in TW, iff y corresponds to an alternating sequence

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$$

for which $\bigcup_i x_i \in W$.

In the case where $\varnothing \in L$, we define an alternating sequence of (A, W) as a sequence

$$\varnothing \subset^+ x_1 \subset^- x_2 \subset^+ \cdots \subset^- x_{2i} \subset^+ x_{2i+1} \subset^- x_{2i+2} \subset^+ \cdots$$

of configurations in $\mathcal{C}^\infty(A)$. In this case, the −ve events of $\mathrm{TG}(A, W)$ are finite alternating sequences ending in x_{2k}, while the +ve events end in x_{2k-1}, for $k \geq 1$. The remaining parts of the definition proceed analogously.

We have constructed a tree game $\mathrm{TG}(A, W)$ from a game (A, W). The construction respects the duality on games.

Lemma 17. *Let (A, W) be a concurrent game.*

$$\mathrm{TG}((A, W)^\perp) = (\mathrm{TG}(A, W))^\perp .$$

Proposition 18. *Suppose (A, W) is a bounded-concurrent game. Maximal alternating sequences have one of two forms,*

(i) *finite:*

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots x_k ,$$

where x_i is finite for all $0 < i < k$ (where possibly x_k is infinite), or

(ii) *infinite:*

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots ,$$

where each x_i is finite.

Proof: Otherwise, taking the first infinite x_i, within configuration x_{i+1} there would be an event of $x_{i+1} \setminus x_i$ concurrent with infinitely many events of x_i of opposite polarity—contradicting the bounded-concurrency of A. ■

Example 19. Let (A, W) be the concurrent game with A as in Example 1 and $W = \{\varnothing, \{\oplus, \ominus\}\}$. Player has an obvious winning strategy: await Opponent's move and then make their move. Because $\varnothing \in W$, its tree game is

$$e_1 = [\varnothing, \{\ominus\}] \longrightarrow e_2 = [\varnothing, \{\ominus\}, \{\ominus, \oplus\}]$$

In the tree game only the empty sub-branch and the maximal branch are winning for Player. Its Gale–Stewart game has events which correspond to the non-empty subsequences of

$$(\delta^- \delta^+)^* e_1 (\delta^+ \delta^-)^* e_2 (\delta^- \delta^+)^*$$

and branches which comprise consecutive sequences of such. An infinite branch is winning for Player if it comprises solely delay events or contains both e_1 and e_2. Player has a winning strategy in the Gale–Stewart game: delay while Opponent delays and play e_2 when Opponent plays e_1. □

B. Borel determinacy of concurrent games

Now assume that the concurrent game (A, W) is race-free and bounded-concurrent. Suppose that $str : T \to TA$ is a (winning) strategy in the tree game $TG(A, W)$. Note that T is necessarily tree-like. We construct $\sigma_0 : S \to A$, a (winning) strategy in the original concurrent game (A, W). We construct S indirectly, from a prime-algebraic domain $\mathcal{Q}$, built as follows. For technical reasons, in the construction of $\mathcal{Q}$ it is convenient to assume—as can easily be arranged—that

$$A \cap (A \times T) = \varnothing .$$

Via str a sub-branch $\vec{t} = (t_1, \cdots, t_i, \cdots)$ of T determines a *tagged alternating sequence*

$$\varnothing \ \cdots \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

where $str(t_i) = [\varnothing, \ldots, x_{i-1}, x_i]$. (the arc t_i is associated with a round extending x_{i-1} to x_i in the original game.)

Define $q(\vec{t})$ to be the partial order comprising events

$$\bigcup \{(x_i \setminus x_{i-1}) \mid t_i \text{ is a −ve arc of } \vec{t}\} \cup$$
$$\bigcup \{(x_i \setminus x_{i-1}) \times \{t_i\} \mid t_i \text{ is a +ve arc of } \vec{t}\}$$

—so a copy of the events $\bigcup_i x_i$ but with +ve events tagged by the +ve arc of T at which they occur[2]—with order a copy of that $\bigcup_i x_i$ inherits from A with additional causal dependencies pairs from (with x_{i-1}^- the set of −ve events in x_{i-1})

$$x_{i-1}^- \ \times \ ((x_i \setminus x_{i-1}) \times \{t_i\})$$

—making the +ve events occur after the −ve events which precede them in the alternating sequence.

Define the partial order $\mathcal{Q}$ as follows. Its elements are posets q, not necessarily finite, for which there is a rigid inclusion $q \hookrightarrow q(t_1, t_2, \cdots, t_i, \cdots)$, for some sub-branch $(t_1, t_2, \cdots, t_i, \cdots)$ of T. The order on $\mathcal{Q}$ is that of rigid inclusion. Define the function $\sigma : \mathcal{Q} \to \mathcal{C}^\infty(A)$ by taking

$$\sigma q \ = \ \{a \in A \mid a \text{ is −ve } \& \ a \in q\} \cup \{a \in A \mid \exists t \in T. \ a \text{ is +ve } \& \ (a, t) \in q\}$$

for $q \in \mathcal{Q}$. We should check that σq is indeed a configuration of A. Clearly, $\sigma q(\vec{t}) = \bigcup_{i \in I} x_i$ where

$$\varnothing \ \cdots \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

is the tagged alternating sequence determined by

$$\vec{t} =_{\mathrm{def}} (t_1, \cdots, t_i, \cdots).$$

Any q for which there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ will be sent to a sub-configuration of $\bigcup_i x_i$.

Proposition 20. *Let $(t_1, \cdots, t_i, \cdots)$ be a sub-branch of T, so corresponding to some $\{t_1, \cdots, t_i, \cdots\} \in \mathcal{C}^\infty(T)$. Then,*

$$str\{t_1, \cdots, t_i, \cdots\} \in TW \iff \sigma q(t_1, \cdots, t_i, \cdots) \in W .$$

[2]It is so that the two components remain disjoint under tagging that we make the technical assumption above.

Proof: Let $\vec{t} =_{\rm def} (t_1, \cdots, t_i, \cdots)$. Then, we have that $str(t_i) = [\varnothing, \ldots, x_{i-1}, x_i]$ for some

$$\varnothing \ \cdots \subset^- x_{i-1} \subset^+ x_i \subset^- \cdots,$$

an alternating sequence of (A, W). Directly from the definitions of TW, $q(\vec{t})$ and σ, we have that

$$str\{\vec{t}\} \in TW \iff \bigcup_i x_i \in W$$
$$\iff \sigma q(\vec{t}) \in W\,.$$

■

We shall also make use of the following proposition.

Proposition 21. *For all $q, q' \in \mathcal{Q}$, whenever there is an inclusion of the events of q in the events of q' there is a rigid inclusion $q \hookrightarrow q'$.*

Notation 22. Proposition 21 justifies writing $\subseteq$ for the order of $\mathcal{Q}$. We shall also write $q \subseteq^- q'$ when all the events in q' above those of q are −ve, and similarly $q \subseteq^+ q'$ when all the events in q' above those of q are +ve. We also write q^+ for the set of +ve events in q and q^- for the set of −ve ones. □

The following lemma is crucial and depends critically on (A, W) being race-free and bounded-concurrent.

Lemma 23. *The order $(\mathcal{Q}, \subseteq)$ is a prime algebraic domain in which the primes are precisely those (necessarily finite) partial orders with a maximum.*

Proof: Any compatible finite subset X of $\mathcal{Q}$ has a least upper bound: if all the members of X include rigidly in a common q then taking the union of their images in q, with order inherited from q, provides their least upper bound. Provided that $\mathcal{Q}$ has least upper bounds of directed subsets it will then be consistently complete with the additional property that every $q \in \mathcal{Q}$ is the least upper bound of the primes below it—this will make $\mathcal{Q}$ a prime algebraic domain.

To establish prime algebraicity it remains to show that $\mathcal{Q}$ has least upper bounds of directed sets.

Let S be a directed subset of $\mathcal{Q}$. The +ve events of orders $q \in S$ are tagged by +ve arcs of T. Because S is directed the +ve tags which appear throughout all $q \in S$ must determine a common sub-branch of T, *viz.*

$$\vec{t} =_{\rm def} (t_1, t_2, \cdots, t_i, \cdots)\,.$$

Every +ve arc of the sub-branch appears in some $q \in S$ and all −ve arcs are present only by virtue of preceding a +ve arc. The sub-branch $\vec{t}$ may be

(1) infinite and necessarily a full branch of T, if the elements of S together mention infinitely many tags;
(2) finite with $q(\vec{t})$ infinite, and necessarily finishing with a +ve arc;
(3) finite and non-empty with $q(\vec{t})$ finite, and necessarily finishing with a +ve arc; or
(4) empty with $\vec{t} = ()$.

(1) Consider the case where $\vec{t}$ forms an infinite branch of T. We shall argue that for all $q \in S$, there is a rigid inclusion

$$q \hookrightarrow q(\vec{t})\,.$$

Then, forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(\vec{t})$ we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(\vec{t})\,,$$

so a least upper bound of S in $\mathcal{Q}$.

Let $q \in S$. By Proposition 21, to establish the rigid inclusion $q \hookrightarrow q(\vec{t})$ it suffices to show the events of q are included in those of $q(\vec{t})$. From the nature of the sub-branch determined by S, we must have that all the +ve events of q are included in those of $q(\vec{t})$—all +ve events of q are tagged by a +ve arc of $\vec{t}$. Suppose, to obtain a contradiction, that there is some −ve event a of q not in $q(\vec{t})$. For every +ve arc t_i in $\vec{t}$ there is $q_i \in S$ with a +ve tagged event (a_i, t_i). Let

$$I \subseteq_{\rm fin} \{i \mid t_i \text{ is a +ve arc of } \vec{t}\}\,.$$

As S is directed, there is an upper bound in S of

$$\{q\} \cup \{q_i \mid i \in I\}\,.$$

It follows that $\{a\} \cup \{a_i \mid i \in I\} \in {\rm Con}_A$; and forming the down-closure in A of $\{a\} \cup \{a_i \mid t_i \text{ is a +ve arc in } \vec{t}\}$ we get

$$[\{a\} \cup \{a_i \mid t_i \text{ is a +ve arc in } \vec{t}\}] \in \mathcal{C}^\infty(A)\,.$$

Moreover it is a configuration which violates bounded concurrency—the −ve event a is concurrent with infinitely many of the +ve events a_i. From this contradiction we deduce that the events of q are included in the events of $q(\vec{t})$.

(2) Consider the case where $\vec{t}$ is a finite branch $(t_1, \cdots, t_k)$, where necessarily t_k is a +ve arc, and where $q(\vec{t})$ is infinite. By bounded-concurrency, all $q(t_1, \cdots, t_i)$, for $0 \le i < k$, are finite with only $q(\vec{t}) = q(t_1, \cdots, t_k)$ infinite.

Let $q \in S$. By Proposition 21, there is a rigid inclusion

$$q \hookrightarrow q(\vec{t})$$

by showing all the events of q are in $q(\vec{t})$. Again, all the +ve events of q are in $q(\vec{t})$. Suppose, to obtain a contradiction, that $b \in q$ with $b \notin q(\vec{t})$, so b has to be −ve. There is a member of S with an event tagged by t_k. Thus, using the directedness of S, there has to be $q_1 \in S$ with $q \subseteq q_1$ and where q_1 has an event tagged by t_k. Because of the extra dependencies introduced in the construction of $q(\vec{t})$, all the −ve events of $q(\vec{t})$ are included in q_1. Note in addition that

$$[q_1^+] \subseteq q(\vec{t})$$

because all the +ve events of q_1 are in $q(\vec{t})$. We deduce

$$[q_1^+] \subseteq^+ q(\vec{t})\,. \qquad (i)$$

Also,

$$[q_1^+] \subset^- q_1\,, \qquad (ii)$$

where the inclusion has to be strict because $b \in q_1 \setminus q(\vec{t})$. Consider the images of (i) and (ii) in $\mathcal{C}^\infty(A)$:

$$\sigma[q_1^+] \subseteq^+ \sigma q(\vec{t}) \ \text{ and } \ \sigma[q_1^+] \subset^- \sigma q_1\,.$$

As A is race-free, we obtain the configuration

$$x =_{\rm def} \sigma q(\vec{t}) \cup \sigma q_1 \in \mathcal{C}^\infty(A)$$

and the strict inclusion

$$\sigma q(\vec{t}) \subset^- x\,,$$

making x a configuration which contains the $-$ve event b concurrent with infinitely many +ve events—the images of those tagged by t_k. But this contradicts the bounded-concurrency of A. Hence all the events of q are in $q(\vec{t})$.

As in case (1) we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(\vec{t})\,,$$

and a least upper bound of S in $\mathcal{Q}$.

(3) The case where $\vec{t}$ is a non-empty finite branch $(t_1, \cdots, t_k)$ and $q(\vec{t})$ is finite. Again, t_k is necessarily a +ve arc. As S is directed, the set of events $\bigcup_{q\in S} \sigma q$ is a configuration in $\mathcal{C}^\infty(A)$. Again, all the +ve events of any $q \in S$ are in $q(\vec{t})$, from which it follows that as sets,

$$(\bigcup_{q\in S} \sigma q)^+ \subseteq \sigma q(\vec{t})\,.$$

Hence, the down-closure

$$[(\bigcup_{q\in S} \sigma q)^+]_A \subseteq \sigma q(\vec{t}) \ \text{ in } \mathcal{C}^\infty(A)\,. \qquad (iii)$$

There is $q_1 \in S$ with an event tagged by t_k. Because of the extra dependencies introduced in the construction of $q(\vec{t})$, all the $-$ve events of $q(\vec{t})$ are included in q_1. Consequently, all the $-$ve events of $\sigma q(\vec{t})$ are included in $\bigcup_{q\in S} \sigma q$. From this and (iii) we deduce

$$[(\bigcup_{q\in S} \sigma q)^+] \subseteq^+ \sigma q(\vec{t}) \ \text{ in } \mathcal{C}^\infty(A)\,. \qquad (iv)$$

Also, straightforwardly,

$$[(\bigcup_{q\in S} \sigma q)^+] \subseteq^- \bigcup_{q\in S} \sigma q \ \text{ in } \mathcal{C}^\infty(A)\,. \qquad (v)$$

From (iv) and (v), because A is race-free, we can define

$$y =_{\rm def} (\sigma q(\vec{t}) \cup \bigcup_{q\in S} \sigma q) \in \mathcal{C}^\infty(A)$$

for which

$$\sigma q(\vec{t}) \subseteq^- y \in \mathcal{C}^\infty(A)\,.$$

But by receptivity of the original strategy $str : T \to TA$, there is a unique extension of the branch $\vec{t} = (t_1, \cdots, t_k)$ to $(t_1, \cdots, t_k, t_{k+1})$ in T such that

$$\sigma q(t_1, \cdots, t_k, t_{k+1}) = y\,.$$

W.r.t. this extension, forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(t_1, \cdots, t_k, t_{k+1})$, we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(t_1, \cdots, t_k, t_{k+1})\,,$$

so a least upper bound of S in $\mathcal{Q}$.

(4) Finally, consider the case where $\vec{t} = ()$. Then all $q \in S$ consist purely of $-$ve events. As S is directed, $\bigcup_{q\in S} \sigma q \in \mathcal{C}^\infty(A)$. If $\bigcup_{q\in S} \sigma q = \varnothing$ we have $\bigcup S = q()$. Assume $\varnothing \neq \bigcup_{q\in S} \sigma q$.

First suppose $\varnothing \in W$. We can form the alternating sequence

$$\varnothing \subset^- \bigcup_{q\in S} \sigma q\,.$$

By the receptivity of $str : T \to TA$ there is a unique 1-arc branch (t_1) of T with $\bigcup_{q\in S} \sigma q = \sigma q(t_1)$. Then $\bigcup S = q(t_1)$.

Now suppose $\varnothing \notin W$. In this case all alternating sequences must begin $\varnothing \subset^+ x_1 \cdots$ and consequently all initial arcs of T must be +ve. We are assuming $\bigcup_{q\in S} \sigma q$ is non-empty so contains some non-empty q. There must therefore be a rigid inclusion $q \hookrightarrow q(\vec{u})$ for some non-empty sub-branch $\vec{u} = (u_1, \cdots)$. Via str the sub-branch $\vec{u}$ determines the alternating sequence $\varnothing \subset^+ x_1 \subset^- \cdots$. Noting $\varnothing \subset^- \bigcup_{q\in S} \sigma q$, because A is race-free there is $x_1 \cup \bigcup_{q\in S} \sigma q \in \mathcal{C}^\infty(A)$. Form the alternating sequence

$$\varnothing \subset^+ x_1 \subset^- x_1 \cup \bigcup_{q\in S} \sigma q\,.$$

From the receptivity of str there is a sub-branch (u_1, u_2') such that $x_1 \cup \bigcup_{q\in S} \sigma q = \sigma q(u_1, u_2')$; then $\bigcup S \hookrightarrow q(u_1, u_2')$. ■

Definition 24. Define S to be the event structure with polarity, with events the primes of $\mathcal{Q}$; causal dependency the restriction of the order on $\mathcal{Q}$; with a finite subset of events consistent if they include rigidly in a common element of $\mathcal{Q}$. The polarity of event of S is the polarity in A of its top element (recall the event is a prime in $\mathcal{Q}$). Define $\sigma_0 : S \to A$ to be the function which takes a prime with top element an untagged event $a \in A$ to a and top element a tagged event (a, t) to a. □

Then we have that

Lemma 25. *The function which takes $q \in \mathcal{Q}$ to the set of primes below q in $\mathcal{Q}$ gives an order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$. The function $\sigma_0 : S \to A$ is a strategy for which*

$$\begin{array}{ccc} \mathcal{Q} & \cong & \mathcal{C}^\infty(S) \\ {\scriptstyle\sigma}\downarrow & \swarrow {\scriptstyle\sigma_0} & \\ \mathcal{C}^\infty(A) & & \end{array}$$

commutes.

And a (possibly *nondeterministic*) strategy in a game (A, W) can be defined based on a strategy in $\mathrm{TG}(A, W)$. In particular, a winning strategy in a concurrent game can be built based on a winning strategy in its associated tree game.

Theorem 26. *Suppose that $str : T \to TA$ is a winning strategy in the tree game $\mathrm{TG}(A, W)$. Then $\sigma_0 : S \to A$ is a winning strategy in (A, W).*

Proof: For σ_0 to be winning we require that $\sigma_0 x \in W$ for any $\oplus$-maximal $x \in \mathcal{C}^\infty(S)$. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ we can carry out the proof in $\mathcal{Q}$ rather than $\mathcal{C}^\infty(S)$. For any q which is $\oplus$-maximal in $\mathcal{Q}$ (*i.e.* whenever $q \subseteq^+ q'$ in $\mathcal{Q}$ then $q = q'$) we require that $\sigma q \in W$.

Let q be $\oplus$-maximal in $\mathcal{Q}$. We show that $q = q(\vec{u})$ for some $\oplus$-maximal branch $\vec{u}$ of T. Certainly there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some sub-branch $\vec{t} = (t_1, \cdots, t_i, \cdots)$ of T. Let

$$\varnothing \ \cdots \ \overset{t_{i-1}}{\subset^-} \ x_{i-1} \ \overset{t_i}{\subset^+} \ x_i \ \overset{t_{i+1}}{\subset^-} \ \cdots$$

be the tagged sequence determined by $\vec{t}$.

Consider the case in which the set q^+ is infinite. There are two possibilities. Suppose first that

$$q^+ \cap ((x_i \setminus x_{i-1}) \times \{t_i\}) \neq \varnothing .$$

for infinitely many +ve t_i. Because of the extra causal dependencies introduced in the definition of $q(\vec{t})$, the set of −ve events $q(\vec{t})^-$ is included in q. Hence $q \subseteq^+ q(\vec{t})$. But q is $\oplus$-maximal, so $q = q(\vec{t})$. The second possibility is that $(\sigma q)^+ \subseteq x_k$ for some necessarily terminal configuration in the tagged alternating sequence, which now has to be of the form

$$\varnothing \ \cdots \ \overset{t_{i-1}}{\subset^-} \ x_{i-1} \ \overset{t_i}{\subset^+} \ x_i \ \overset{t_{i+1}}{\subset^-} \ \cdots \subset^+ x_k \,.$$

Because of the causal dependencies in $q(\vec{t})$, the set $q(\vec{t})^-$ is in q. Hence $q \subseteq^+ q(\vec{t})$, so $q = q(\vec{t})$ since q is $\oplus$-maximal.

Now consider the case where q^+ is finite. Then the set $(\sigma q)^+$, also finite, must be included in some x_k of the tagged alternating sequence, which we may assume is the earliest. Then t_k must be +ve. If $\sigma q \subseteq q(t_1, \cdots, t_k)$, then the set $q(t_1, \cdots, t_k)^-$ is included in q—again because of the causal dependencies there; and again $q \subseteq^+ q(t_1, \cdots, t_k)$ so $q = q(t_1, \cdots, t_k)$ as q is $\oplus$-maximal. Otherwise, $x_k \subset^- x_k \cup (\sigma q)$ and we can extend the alternating sequence to

$$\varnothing \cdots \ \subset^+ x_k \subset^- x_k \cup (\sigma q) \,.$$

From the receptivity of *str* there is a sub-branch $t_1, \ldots, t_k, t'_{k+1}$ of T which has this alternating sequence as image. Now $q \subseteq^+ q(t_1, \ldots, t_k, t'_{k+1})$ so $q = q(t_1, \ldots, t_k, t'_{k+1})$ from the $\oplus$-maximality of q.

Thus any $q \in \mathcal{Q}$ which is $\oplus$-maximal has the form $q = q(\vec{u})$ for some sub-branch $\vec{u}$ of T. Any extension of $\vec{u}$ by a +ve arc would yield a +ve extension of $q(\vec{u})$, contradicting the $\oplus$-maximality of q. Therefore $\vec{u}$ is $\oplus$-maximal, so its image $str\{\vec{u}\}$ is in TW, as *str* is a winning strategy in $\mathrm{TG}(A, W)$. But, by Proposition 20,

$$str\{\vec{u}\} \in TW \iff \sigma q(\vec{u}) \in W \,.$$

Hence, $\sigma q \in W$, as required. ∎

It immediately follows that

Corollary 27. *Let (A, W) be a race-free, bounded-concurrent game. If the tree game $\mathrm{TG}(A, W)$ has a winning strategy, then the concurrent game (A, W) has a winning strategy.*

Finally, we can establish the main result of the paper, a concurrent analogue of Martin's determinacy theorem.

Theorem 28 (Concurrent Borel determinacy)**.** *Any race-free, bounded-concurrent game (A, W), in which the set W is a Borel subset of $\mathcal{C}^\infty(A)$, is determined.*

Proof: Assuming that (A, W) is race-free and concurrent-bounded and that W is Borel, we obtain a tree game $\mathrm{TG}(A, W) = (TA, TW)$ in which TW is also Borel. To see that TW is Borel, recall that a configuration y of TA corresponds to an alternating sequence

$$\varnothing \cdots \ \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots ,$$

so determines $f(y) =_{\mathrm{def}} \bigcup_i x_i \in \mathcal{C}^\infty(A)$. Thus, this yields a Scott-continuous function $f : \mathcal{C}^\infty(TA) \to \mathcal{C}^\infty(A)$. The set TW is the inverse image $f^{-1}W$, so Borel. As the tree game $\mathrm{TG}(A, W)$ is determined—by Theorem 12—we obtain a winning strategy for Player or Opponent in the tree game.

Suppose first that $\mathrm{TG}(A, W)$ has a winning strategy (for Player). By Corollary 27 we obtain a winning strategy for (A, W). Suppose, on the other hand, that $\mathrm{TG}(A, W)$ has a winning strategy for Opponent, *i.e.* there is a winning strategy in the dual game $(\mathrm{TG}(A, W))^\perp$. By Lemma 17, $\mathrm{TG}((A, W)^\perp) = \mathrm{TG}(A, W)^\perp$ has a winning strategy. By Corollary 27, $(A, W)^\perp$ has a winning strategy, *i.e.* there is a winning strategy for Opponent in (A, W). ∎

VI. Conclusion

The determinacy result in this paper is, in a sense, the *strongest* one can hope to obtain (with respect to the descriptive complexity of the winning sets) for concurrent games on event structures—and hence on partial orders—since any generalisation of the winning sets would require an extension of the Borel determinacy theorem by Martin [6]—well known to be at the limits of traditional set theory.

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References

[1] S. Abramsky and P.-A. Melliès. Concurrent games and full completeness. In *LICS*, pages 431–442. IEEE Computer Society, 1999.

[2] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, 49(5):672–713, 2002.

[3] K. Chatterjee and T. A. Henzinger. A survey of stochastic ω-regular games. *Journal of Computer and System Sciences*, 78(2):394–413, 2012.

[4] P. Clairambault, J. Gutierrez, and G. Winskel. The winning ways of concurrent games. In *LICS*, pages 235–244, 2012.

[5] D. Gale and F. Stewart. Infinite games with perfect information. *Annals of Mathematical Studies*, 28:245–266, 1953.

[6] D. Martin. Borel determinacy. *Annals of Mathematics*, 102(2):363–371, 1975.

[7] D. Martin. The determinacy of Blackwell games. *Journal of Symbolic Logic*, 63(4):1565–1581, 1998.

[8] P.-A. Melliès and S. Mimram. Asynchronous games: Innocence without alternation. In *CONCUR*, volume 4703 of *LNCS*, pages 395–411. Springer, 2007.

[9] M. Nielsen and G. Winskel. Models for concurrency. In *Handbook of Logic in Computer Science*, pages 1–148. OUP, 1995.

[10] S. Rideau and G. Winskel. Concurrent strategies. In *LICS*, pages 409–418, 2011.

[11] G. Winskel. Winning, losing and drawing in concurrent games with perfect or imperfect information. In *Logic and Program Semantics*, volume 7230. LNCS Springer, 2012.

[12] G. Winskel. Distributed probabilistic and quantum strategies. Submitted LICS 13, 2013.

[13] E. Zermelo. On an application of set theory to the theory of the game of chess. In *Congress Mathematicians*, pages 501–504. CUP, 1913.

VII. Some proofs omitted or sketched in the main text due to space restrictions

Proposition 29 (Proposition 6)**.** *Let $f : S \to A$ be a total map of event structures with polarity and let A be tree-like. Then, it follows that S is also tree-like and that the map f is innocent. The map f is a strategy if and only if it is receptive.*

Proof: As f preserves the concurrency relation, being a map of event structures, S must be tree-like. Innocence of f now follows so that only its receptivity is required for it to be a strategy. ∎

Proposition 30 (Proposition 8)**.** *A winning strategy in a Gale–Stewart game (G, V) corresponds to a non-empty subset $T \subseteq \mathcal{C}^\infty(G)$ such that*

(i) $\forall x, y \in \mathcal{C}^\infty(G).\ \ y \subseteq x \in T \implies y \in T$,
(ii) $\forall x, y \in \mathcal{C}(G).\ \ x \in T \ \&\ x \stackrel{-}{\longrightarrow}\!\!\subset y \implies y \in T$,
(iii) $\forall x, y_1, y_2 \in T.\ x \stackrel{+}{\longrightarrow}\!\!\subset y_1 \ \&\ x \stackrel{+}{\longrightarrow}\!\!\subset y_2 \implies y_1 = y_2$, *and*
(iv) *all $\subseteq$-maximal members of T are infinite and in V.*

Proof: Given σ, a winning strategy in the Gale–Stewart game we define T as above. Then, (i) follows because σ is a map of event structures and G is tree-like; (ii) and (iii) follow from σ being receptive and deterministic; (iv) is a consequence of all winning configurations being infinite. Conversely, given T a subfamily of $\mathcal{C}^\infty(G)$ satisfying (i)-(iv) it is a relatively routine matter to construct a tree-like event structure S and map $\sigma : S \to G$ which is a winning strategy in (G, V). ∎

Lemma 31 (Lemma 16)**.** *Let (A, W) be a race-free game. If A is not bounded-concurrent then there is W, a Borel subset of $\mathcal{C}^\infty(A)$, such that the game (A, W) is not determined.*

Proof: Since A is not bounded-concurrent, there is $y \in \mathcal{C}^\infty(A)$ and $e \in y$ such that e is concurrent with infinitely many events $e_i \in y$ of opposite polarity. Without loss of generality assume that $pol(e) = +$ and that y is minimal in the sense that if $y = y_1 \uplus y_2$ then either $Neg[y_1]$ is finite or $Neg[y_2]$ is finite. Based on y define W using the following rules: let $y' \in \mathcal{C}(A)$, then

1) $y' \supseteq y \Longrightarrow y' \in W$;
2) $y' \subset y \ \&\ e \in y' \Longrightarrow y' \in L$;
3) $y' \subset y \ \&\ e \notin y' \ \&$
$max_+(y', y \setminus \{e\}) \ \&\ \overline{max}_-(y', y \setminus \{e\}) \Longrightarrow y' \in W$;
4) $y' \subset y \ \&\ e \notin y' \ \&$
$\overline{max}_+(y', y \setminus \{e\})$ or $max_-(y', y \setminus \{e\}) \Longrightarrow y' \in L$;
5) $y' \not\subset y \ \&\ y' \not\supseteq y \ \&\ (y' \cap y) \subset^- y' \Longrightarrow y' \in W$;
6) $y' \not\subset y \ \&\ y' \not\supseteq y \ \&\ (y' \cap y) \subset^+ y' \Longrightarrow y' \in L$;
7) otherwise assign any polarity to y'.

The rules assign a winner to every configuration (because of rule 7). Moreover, no y' is assigned as winning to both Player and Opponent: the antecedents of all implications are pair-wise mutually exclusive.

First, let us show that Player does not have a winning strategy. Consider the following infinite family of $\forall$-strategies, namely $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$ (for $i \in \mathbb{N}$ and recall that for each $e_i \in y$ we have that $e\ co\ e_i$), such that:

$$T_\infty^\perp =_{\mathrm{def}} \{e' \in A \mid e' \in y \ \vee\ pol(e) = +\} \text{ , and}$$
$$T_i^\perp =_{\mathrm{def}} \{e' \in A \mid e' \in y \setminus \{e_i\} \ \vee\ pol(e') = +\}.$$

Then each strategy τ for Opponent only plays negative events contained in y, i.e. $Neg[(\tau T)^\perp] \subseteq y$; and, each strategy τ_i does not play a −ve event e_i which is concurrent with e, i.e. $\forall \tau_i.\ \exists e_i \in Neg[y].\ e\ co\ e_i\ \&\ \forall z \in \mathcal{C}^\infty(T_i).\ e_i \notin (\tau_i z)^\perp$.

To get a contradiction, suppose Player has a winning strategy $\sigma : S \to A$.

Since σ is a winning strategy then σ dominates τ_∞, i.e. $y' \in \langle \sigma, \tau_\infty \rangle \Longrightarrow y' \in W$. Note that because of the definition of τ_∞ we know that for all $y' \in \langle \sigma, \tau_\infty \rangle$, we have that $y' \supseteq y$ (Player only wins using rule 1); rules 3 and 5 cannot be used to win the game, respectively, because (for 3) τ_∞ always plays $\ominus$-maximally in y—hence in $y \setminus \{e\}$ too—and (for 5) τ_∞ never plays −ve events not in y. Then

$$Pos[y] \subseteq \sigma S.$$

In other words, rules 3 and 5 cannot be used because:

- $\forall y_1, y_2 \subseteq y$.
  if $\exists z_1 \in T_\infty.\ (\tau_\infty z_1)^\perp = y_1 \ \&\ y_1 \subseteq^- y_2$
  then $\exists z_2 \in T_\infty.\ z_1 \subseteq^- z_2 \ \&\ (\tau_\infty z_2)^\perp = y_2$ (for 3).
- $Neg[(\tau_\infty T_\infty)^\perp] \subseteq y$ (for 5).

We also have that σ dominates τ_i, for every $i \in \mathbb{N}$. As every τ_i does not play some −ve event in y, i.e.

$$\forall \tau_i.\ \exists e_i \in Neg[y].\ \forall z \in T_i.\ e_i \notin (\tau_i z)^\perp,$$

then Player cannot win using rule 1 when playing against every τ_i. And, as for τ_∞, each τ_i never plays −ve events not in y (i.e. $Neg[(\tau_i T_i)^\perp] \subseteq y$); then Player cannot win using rule 5 either. As a consequence, Player can only win using rule 3.

Winning with rule 3 requires that

$$\forall \tau_i.\ y' \in \langle \sigma, \tau_i \rangle \Longrightarrow e \notin y'.$$

But we know that there is $s_e \in S$ such that $\sigma(s_e) = e$ (because $Pos[y] \subseteq \sigma S$). Since $[e]$ is finite then $[s_e]$ is finite too—hence $Neg[s_e]$ is also finite. And because $Neg[y]$ is infinite, then there are infinitely many τ_i such that

$$\exists y' \in \langle \sigma, \tau_i \rangle.\ y' \subset y \ \&\ e \in y',$$

i.e. infinitely many τ_i with which Opponent wins using rule 2—which contradicts that Player wins using rule 3 when playing against every τ_i (formally, a contradiction with the statement above, namely, that $\forall \tau_i.\ y' \in \langle \sigma, \tau_i \rangle \Longrightarrow e \notin y'$).

Then, we conclude that $\sigma : S \to A$ is not a winning strategy, i.e. that Player does not have a winning strategy in the concurrent game (A, W).

Now, we show that Opponent does not have a winning strategy either. Consider the following two strategies for Player, $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$, where:

$$S_\oplus =_{\mathrm{def}} \{e' \in A \mid e' \in y \ \vee\ pol(e) = -\} \text{ , and}$$
$$S_{\overline{\oplus}} =_{\mathrm{def}} \{e' \in A \mid e' \in y \setminus \{e\} \ \vee\ pol(e) = -\}.$$

Thus, $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ only play +ve events in y (i.e. $Pos[\sigma_\oplus S_\oplus] \subseteq y$ and $Pos[\sigma_{\overline{\oplus}} S_{\overline{\oplus}}] \subseteq y$); moreover, $\sigma_\oplus$ plays $\oplus$-maximally in y—hence in $y \setminus \{e\}$ too—and $\sigma_{\overline{\oplus}}$ plays $\oplus$-maximally in $y \setminus \{e\}$:

- (playing $\oplus$-maximal in y):
  $\forall y_1, y_2 \subseteq y$.
  if $\exists x_1 \in S_\oplus$. $\sigma_\oplus x_1 = y_1$ & $y_1 \subseteq^+ y_2$
  then $\exists x_2 \in S_\oplus$. $x_1 \subseteq^+ x_2$ & $\sigma_\oplus x_2 = y_2$;
- (playing $\oplus$-maximal in $y \setminus \{\oplus\}$):
  $\forall y_1, y_2 \subseteq y \setminus \{\oplus\}$.
  if $\exists x_1 \in S_{\overline{\oplus}}$. $\sigma_{\overline{\oplus}} x_1 = y_1$ & $y_1 \subseteq^+ y_2$
  then $\exists x_2 \in S_{\overline{\oplus}}$. $x_1 \subseteq^+ x_2$ & $\sigma_{\overline{\oplus}} x_2 = y_2$.

And, while $\sigma_\oplus$ plays e as long as Opponent plays $Neg[e]$, the strategy $\sigma_{\overline{\oplus}}$ never plays e.

Again, in order to get a contradiction, suppose that Opponent has a strategy $\tau : T \to A^\perp$ which is winning; in particular, so that τ dominates both $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$.

Because of the definitions of $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ and the set of winning rules there are two ways how τ can win (see rules 2 and 4), namely when:

(i) $y' \subset y$ & $e \in y'$, or
(ii) $y' \subset y$ & $e \notin y'$ & $max_-(y', y \setminus \{e\})$,

for any result y'.

The first observation is that since both $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ play $\oplus$-maximally in $y \setminus \{e\}$, then every result y' of playing τ against either $\sigma_\oplus$ or $\sigma_{\overline{\oplus}}$ satisfies that

$$max_+(y', y \setminus \{e\}).$$

The second observation is that since $y \setminus y' \neq \varnothing$ and $max_+(y', y \setminus \{e\})$, then it follows that for all $e' \in y$ such that $y' \overset{e'}{\longrightarrow\!\!\subset}$ we have that

$$pol(e') = + \Longrightarrow e' = e \quad \text{and} \quad e \in y' \Longrightarrow pol(e') = -.$$

Let $y' \in \langle \sigma_\oplus, \tau \rangle$. Since $max_+(y', y)$ holds (because $\sigma_\oplus$ plays $\oplus$-maximally in y—rather than only in $y \setminus \{e\}$) then it follows that $pol(y \setminus y') \subseteq \{-\}$, i.e. all events in the non-empty set $y \setminus y'$ have negative polarity. Formally, that

$$\forall e' \in y.\ y' \overset{e'}{\longrightarrow\!\!\subset} \Longrightarrow pol(e') = -.$$

Thus, there are two options: either $e \notin y'$ or $e \in y'$. The former is impossible because in such a case Opponent would have to win using rule 4, and hence y' would satisfy (ii), but y' fails to satisfy $max_-(y', y \setminus \{e\})$. Therefore, we have that $e \in y'$ and hence Opponent wins using rule 2, i.e. y' satisfies (i). Since y' is $\oplus$-maximal in y, we know, in particular, that τ does not play all negative events in A, that is, we have that

$$Neg[y] \nsubseteq (\tau T)^\perp,$$

as otherwise there would be a result where Player would win using rule 1.

Note, in particular, that such a negative event that τ does not play, say some $e_i \in A$, does not causally depend on e, i.e. $e \nleq e_i$. Then, the configuration $[e_i)$ will be a sub-configuration of some result $y_i' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle$, that is

$$\exists y_i' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle.\ [e_i) \subseteq y_i' \ \&\ y_i' \overset{e_i}{\longrightarrow\!\!\subset},$$

because e_i is not in conflict with any event in y and $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$ produce the same results (i.e., play in the same way) unless such results contain e.

Now, let $y' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle$. In this case, $max_+(y', y \setminus \{e\})$ holds (as $\sigma_{\overline{\oplus}}$ plays $\oplus$-maximally in $y \setminus \{e\}$) and hence $\forall e' \in y.\ y' \overset{e'}{\longrightarrow\!\!\subset}$ & $pol(e') = + \Longrightarrow e' = e$.

Necessarily $e \notin y'$ (because $\sigma_{\overline{\oplus}}$ does not play e) and Opponent can only win using rule 4, that is, so that y' satisfies (ii) above. This implies that $max_-(y', y \setminus \{e\})$ must hold and we know that $max_+(y', y \setminus \{e\})$ holds too. As y' is both $\oplus$-maximal and $\ominus$-maximal in $y \setminus \{e\}$ and $y \setminus y' \neq \varnothing$, then there is only one event that y' enables, namely e; formally

$$\exists e' \in y.\ y' \overset{e'}{\longrightarrow\!\!\subset} \ \&\ \forall e' \in y.\ y' \overset{e'}{\longrightarrow\!\!\subset} \Longrightarrow e' = e.$$

Since e is concurrent with infinitely many $e_i \in A$, then all such e_i must already be in y'—hence $Neg[y']$ is infinite. And recall that y is a *miminal* configuration in the sense that if $y = y_1 \uplus y_2$ then either $Neg[y_1]$ is finite or $Neg[y_2]$ is finite. Let $y_1 = y'$ and $y_2 = y \setminus y'$. Since $Neg[y_1]$ is infinite then $Neg[y_2]$ is finite. And the smallest such a set is $y_2 = \{e\}$—when $Neg[y_2] = \varnothing$.

Thus, it necessarily is the case that $y \setminus y' = \{e\}$ and hence that

$$Neg[y] \subseteq (\tau T)^\perp,$$

which leads to a contradiction. Again, note in particular that the existence of a configuration/result y_i' such that $y_i' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle$. $[e_i) \subseteq y_i'$ & $y_i' \overset{e_i}{\longrightarrow\!\!\subset}$, with $pol(e_i) = -$, violates that $max_-(y', y \setminus \{e\})$—the reason why $Neg[y] \subseteq (\tau T)^\perp$ cannot be true (as well as why a result y' satisfying (ii) is impossible).

As a consequence, Opponent does not have a strategy that dominates both $\sigma_\oplus$ and $\sigma_{\overline{\oplus}}$, i.e., Opponent does not have a winning strategy either.

Thus, we finally conclude that neither player has a winning strategy. ∎

Lemma 32 (Lemma 17)**.** *Let (A, W) be a concurrent game.*

$$\mathrm{TG}((A, W)^\perp) = (\mathrm{TG}(A, W))^\perp.$$

Proof: From the construction TG, because alternating sequences

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$$

in $\mathcal{C}^\infty(A)$ correspond to alternating sequences

$$\varnothing \cdots \subset^- x_i \subset^+ x_{i+1} \subset^- \cdots$$

in $\mathcal{C}^\infty(A^\perp)$. ∎

Proposition 33 (Proposition 21)**.** *For all $q, q' \in \mathcal{Q}$, whenever there is an inclusion of the events of q in the events of q' there is a rigid inclusion $q \hookrightarrow q'$.*

Proof: To see this, suppose the events of q are included in the events of q'. To establish the rigid inclusion $q \hookrightarrow q'$ we require that, for all $a \in q, b \in q'$,

$$b \rightarrowtriangle_q a \iff b \rightarrowtriangle_{q'} a. \qquad (\dagger)$$

However, in the construction of $q(t_1, t_2, \cdots, t_i, \cdots)$ the only immediate dependencies introduced beyond those of A are those of the form $b \rightarrowtriangle (a', t)$, of tagged +ve events on −ve rounds specified earlier in the branch on which the +ve arc t occurs. This property is inherited by q and q' in $\mathcal{Q}$. Thus in checking (†) we can restrict attention to the case where b is −ve and a is +ve and of the form (a', t) for some $a' \in A$ and arc t of T. The arc t determines a sub-branch $t_1, \cdots, t_k = t$ of T and a corresponding tagged alternating sequence

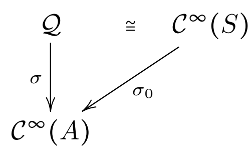
$$\varnothing \ \cdots \ \overset{t_{k-1}}{\subset^-} x_{k-1} \overset{t_k}{\subset^+} x_k \,.$$

So in this case,

$$\begin{aligned} b \multimap_q a \quad &\iff \quad b \text{ is } \leq_A\text{-maximal in } x_{k-1}^- \ \& \\ &\qquad\ \ a' \text{ is } \leq_A\text{-maximal in } x_k \setminus x_{k-1} \\ &\iff \quad b \multimap_{q'} a, \end{aligned}$$

which ensures (†), and the proposition. ■

Lemma 34 (Lemma 25)**.** *The function which takes $q \in \mathcal{Q}$ to the set of primes below q in $\mathcal{Q}$ gives an order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$. The function $\sigma_0 : S \to A$ is a strategy for which*



commutes.

Proof: The isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ is established in [M. Nielsen, G. Plotkin, G. Winskel. Petri Nets, Event Structures and Domains. Theor. Comput. Sci., 13:85–108, 1981]. The diagram is easily seen to commute. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ we can carry out the argument that σ_0 is a strategy in terms of $\mathcal{Q}$ and σ. Innocence follows because the only additional causal dependencies introduced in $q(\vec{t})$ are of +ve events on −ve events. To show receptivity, suppose $q \in \mathcal{Q}$ is finite and $\sigma q \subset^- y$ in $\mathcal{C}(A)$. There is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some $\vec{t} = (t_1, \cdots, t_i, \cdots)$, a sub-branch of T. Let

$$\varnothing \ \cdots \ \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

be the tagged sequence determined by $\vec{t}$.

First consider when $(\sigma q)^+ \neq \varnothing$. Suppose x_k is the earliest configuration at which $(\sigma q)^+ \subseteq x_k$. Then, t_k has to be +ve and

$$q^+ \cap ((x_k \setminus x_{k-1}) \times \{t_k\}) \neq \varnothing \,.$$

The latter entails

$$x_k^- \subseteq \sigma q$$

because of the extra causal dependencies introduced in the definition of $q(\vec{t})$. It follows that

$$(\sigma q) \cap x_k \subseteq^+ x_k \,.$$

Moreover, as $(\sigma q)^+ \subseteq x_k$, we deduce

$$(\sigma q) \cap x_k \subseteq^- \sigma q \subseteq^- y \,.$$

By race-freedom, $x_k \cup y \in \mathcal{C}(A)$ with

$$x_k \subseteq^- x_k \cup y \text{ in } \mathcal{C}(A) \,.$$

In fact $x_k \subset^- x_k \cup y$ as $x_k^- \subseteq \sigma q \subset^- y$. Now

$$\varnothing \cdots \ \subset^+ x_k \subset^- x_k \cup y$$

is seen to form an alternating sequence, so a sub-branch of TA. From the receptivity of str there is a unique sub-branch $t_1, \ldots, t_k, t'_{k+1}$ of T which has this alternating sequence as image. Take q' to be the down-closure of y in $q(t_1, \ldots, t_k, t'_{k+1})$. This gives the unique q' such that $q \subseteq q'$ and $\sigma q' = y$.

Now consider when $(\sigma q)^+ = \varnothing$. Then $\varnothing \subseteq^- \sigma q \subset^- y$.

In the case where $\varnothing \in W$ we may form the alternating sequence

$$\varnothing \subset^- y \,.$$

The receptivity of str ensures there is a unique 1-arc branch (u_1) of T such that $\sigma q(u_1) = y$.

In the case where $\varnothing \notin W$ we also have $\varnothing \notin TW$. In this case all alternating sequences must begin $\varnothing \subset^+ x_1 \cdots$ and consequently all initial arcs of T must be +ve. Also, the empty configuration (or branch) of T cannot be ⊕-maximal because its image under str is the empty configuration (or branch) of TW—impossible because str is a winning strategy. Thus there must be v_1, an initial, necessarily +ve arc of T. Via str the sub-branch (v_1) yields the alternating sequence $\varnothing \subset^+ x_1$, say. As A is race-free we obtain $x_1 \cup y \in \mathcal{C}^\infty(A)$ and the alternating sequence

$$\varnothing \subset^+ x_1 \subset^- x_1 \cup y \,.$$

From the receptivity of str there is a unique sub-branch (v_1, v_2) of T for which $\sigma q(v_1, v_2) = x_1 \cup y$. Take q' to be the down-closure of y in $q(v_1, v_2)$. This furnishes the unique q' such that $q \subseteq q'$ and $\sigma q' = y$.

We have shown the receptivity of σ, as required. ■