Distributed Probabilistic and Quantum Strategies

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***Abstract*—Building on a new definition and characterization of probabilistic event structures, a general definition of distributed probabilistic strategies is proposed. Probabilistic strategies are shown to compose, with probabilistic copy-cat strategies as identities. A higher-order probabilistic process language reminiscent of Milner's CCS is interpreted within probabilistic strategies. Probabilistic games extend to games with payoff and games of imperfect information. W.r.t. new definitions of quantum event structures, it is shown how consistent parts of a quantum event structure are automatically probabilistic event structures, and so possess a probability measure. This gives a non-traditional take on the consistent-histories approach to quantum theory. The paper concludes with an extension to quantum strategies.**

I. Introduction

Concurrent strategies [1] are being investigated as a possible foundation for a generalized domain theory, in which concurrent games and strategies take over the roles of domains and continuous functions. One motivation is to broaden the range of applicability of denotational semantics. Hence it is important to see how concurrent strategies can be adapted to quantitative semantics, to probabilistic and quantum strategies.

Just as event structures can be thought of as models of distributed computation so are probabilistic event structures models of probabilistic distributed processes. Existing definitions of probabilistic event structures [2], [3] are not general enough to ascribe probabilities to the results of the sometimes partial interaction between strategies. This paper first provides a new workable definition of probabilistic event structures, extending existing definitions, and characterised as event structures together with a continuous valuation on their domain of configurations. Probabilistic event structures possess a probabilistic measure on their configurations. Technically, probabilistic event structures are defined via 'drop functions' expressing the probability drops across general intervals of configurations of the event structure; 'drop functions' provide a useful mathematical handle on probabilistic event structures and strategies.

This prepares the ground for a general definition of distributed probabilistic strategies, based on event structures. A probabilistic strategy for Player is a concurrent strategy whose behaviour is described by a probabilistic event structure when projected to just the Player moves. Probabilistic strategies are shown to compose—here 'drop functions' come into their own—with probabilistic copy-cat strategies as identities. The result of a play between Player and Opponent in a game will be a probabilistic event structure.

As an illustration of their expressive power, probabilistic strategies are shown to interpret a higher-order probabilistic process language reminiscent of Milner's CCS. Probabilistic strategies are easily extended to games with payoff and games of imperfect information. Their definition has been partly inspired by the work of Danos and Harmer on probabilistic HO games [4], and in an informal sense the definition here extends theirs from the sequential setting. (A formal connection must await the relation between concurrent games and HO games, being developed within concurrent games with symmetry [5].)

A novel application is to a new definition of quantum event structures and strategies. A (simple) quantum event structure is an event structure in which the events are interpreted as projection or unitary operators on a Hilbert space, so that concurrent events are associated with commuting operators; a configuration of the event structure is thought of as a partial-order history of the observations of a quantum experiment. Interestingly order-consistent families of configurations of a quantum event structure automatically determine a probabilistic event structures, and so possess a probability distribution.

This gives a non-traditional take on the consistent-histories approach to quantum theory, which provides consistency/decoherence conditions on histories to pick out those subfamilies of histories over which it is meaningful to place a probability distribution. (It is well-known that care needs to be exercised in what one takes to be the possible alternative results of a quantum experiment.) The approach via quantum event structures bypasses the consistency/decoherence conditions usually invoked [6]—those conditions appear to be too sensitive to what one considers the initial and final events of a finite history.

In a quantum game Player and Opponent interact to jointly create a probabilistic distributed experiment on a quantum system. Accordingly a quantum strategy is taken to be a distributed probabilistic strategy on a quantum event structure. The definition accords with work on quantum games [7]. There are similarities with the work of Delbecque [8], itself based on probabilistic HO games [4].

Full proofs can be found in [9].

II. Event structures

A. Event structures and configurations

An *event structure* comprises $(E, \mathrm{Con}, \leq)$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$
$$Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}$$
$$X \in \mathrm{Con} \ \& \ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Often we shall be concerned with just the finite configurations of an event structure. We write $\mathcal{C}(E)$ for the *finite* configurations of an event structure E.

We say an event structure is *elementary* when the consistency relation consists of all finite subsets of events. Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$ is a configuration.

Notation 1. Let E be an event structure. We use $x \longrightarrow\!\!\!\!\subset y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.* $x \subsetneq y$ in $\mathcal{C}^\infty(E)$ with nothing in between, and $x \overset{e}{\longrightarrow\!\!\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \overset{e}{\longrightarrow\!\!\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{\longrightarrow\!\!\!\!\subset} y$ for some y.

B. Maps and operations on event structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and

$$e_1, e_2 \in x \ \&\ f(e_1) = f(e_2) \text{ (with both defined) } \implies e_1 = e_2.$$

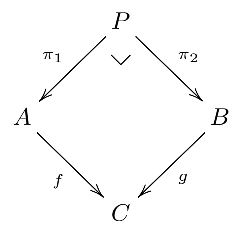
The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Partial maps of event structures compose as partial functions, with identity maps given by identity functions. We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. A total map of event structures which preserves causal dependency is called *rigid*.

1) Products: The category of event structures with partial maps has *products* $A \times B$ with projections π_1 to A and π_2 to B. The effect is to introduce arbitrary synchronisations between events of A and events of B in the manner of process algebra.

2) Pullbacks: Synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. The *restriction* of an event structure E to a subset of events R, written $E \upharpoonright R$, is the event structure with events $E' = \{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency induced by E. We obtain *pullbacks* as a special case. Let $f : A \to C$ and $g : B \to C$ be maps of event structures. Defining P to be

$$A \times B \upharpoonright \{p \in A \times B \mid f\pi_1(p) = g\pi_2(p)\}$$

we obtain a pullback square



in the category of event structures. When f and g are total the same construction gives the pullback in the category of event structures with *total* maps.

3) Projection: Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$. A partial map $f : E \to E'$ of event structures factors into a composition of a partial and total map

$$E \to E{\downarrow}V \to E'$$

where: $V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}$ is the domain of definition of f; the partial map $E \to E{\downarrow}V$ acts as identity on V as is undefined otherwise; and the total map $E{\downarrow}V \to E'$ acts as f.

4) Prefixes and sums: The category of event structures has coproducts given as sums; a coproduct $\sum_{i \in I} E_i$ is obtained as the disjoint juxtaposition of an indexed collection of event structures, making events in distinct components inconsistent. In practice, components of a sum are often prefixed by an event. The prefix of an event structure A, written $\bullet.A$, comprises the event structure in which all the events of A are made to causally depend on an event $\bullet$.

III. Probabilistic event structures

A probabilistic event structure comprises an event structure $(E, \leq, \mathrm{Con})$ together with a continuous valuation on its Scott open sets of configurations, *i.e.* a function w from the open subsets of configurations $\mathcal{C}^\infty(E)$ to $[0, 1]$ which is:
(normalized) $w(\mathcal{C}^\infty(E)) = 1$ *(strict)* $w(\emptyset) = 0$;
(monotone) $U \subseteq V \implies w(U) \leq w(V)$;
(modular) $w(U \cup V) + w(U \cap V) = w(U) + w(V)$;
(continuous) $w(\bigcup_{i \in I} U_i) = \sup_{i \in I} w(U_i)$, for *directed* unions.

Continuous valuations play a central role in probabilistic powerdomains [10]. Continuous valuations are determined by their restrictions to basic open sets $\widehat{x} =_{\mathrm{def}} \{y \in \mathcal{C}^\infty(E) \mid x \subseteq y\}$, for x a finite configuration. The intuition: $w(U)$ is the probability of the resulting configuration being in the open set U. Indeed, continuous valuations extend to unique probabilistic measures on the Borel sets.

This description of a probabilistic event structure extends the definitions in [3]. It turns out to be equivalent to a more workable definition, which relates more directly to the configurations of E, that we develop now.

A. General intervals and drop functions

Throughout this section assume E is an event structure event structure and $v : \mathcal{C}(E) \to \mathbb{R}$. Extend $\mathcal{C}(E)$ to a lattice $\mathcal{C}(E)^\top$ by adjoining an extra top element $\top$. Write its order as $x \sqsubseteq y$ and its finite join operations as $x \vee y$ and $\bigvee_{i \in I} x_i$. Extend v to $v^\top : \mathcal{C}(E)^\top \to \mathbb{R}$ by taking $v^\top(T) = 0$.

We are concerned with drops in value across a general intervals $[y; x_1, \cdots, x_n]$, where $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in$\mathcal{C}(E)^\top$. The interval is thought of as specifying the set of configurations $\widehat{y} \setminus (\widehat{x}_1 \cup \cdots \cup \widehat{x}_n)$.

Define the *drop functions* $d_v^{(n)}[y; x_1, \cdots, x_n] \in \mathbb{R}$ for $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)^\top$, by taking $d_v^{(0)}[y;] =_{\text{def}} v^\top(y)$ and $d_v^{(n)}[y; x_1, \cdots, x_n] =_{\text{def}}$

$$d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] .$$

Proposition 2. *Let $n \in \omega$. For $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_{\emptyset \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) .$$

It will be important that drops across general intervals can be reduced to sums of drops across intervals based on coverings, as explained in the next two results.

Lemma 3. *Let $n \geq 1$. Let $y, x_1, \cdots, x_n, x'_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Assume $x_n \sqsubseteq x'_n$. Then, $d_v^{(n)}[y; x_1, \cdots, x'_n] =$*

$$d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n] .$$

Corollary 4. *Let $\mathcal{C}(E)$ be an event structure and $v : \mathcal{C}(E) \to [0,1]$. Let $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n]$ is expressible as a sum of terms $d_v^{(k)}[u; w_1, \cdots, w_k]$ where $y \subseteq u \longrightarrow\!\!\subset w_i$ in $\mathcal{C}(E)$ and $w_i \subseteq x_1 \cup \cdots \cup x_n$, for all i with $1 \leq i \leq k$. [The set $x_1 \cup \cdots \cup x_n$ need not be in $\mathcal{C}(E)$.]*

B. Probabilistic event structures

A probabilistic event structure is an event structure associated with $[0,1]$-valuation on configurations such that no general interval has a negative drop.

Definition 5. Let E be an event structure. A *configuration-valuation* is function $v : \mathcal{C}(E) \to [0,1]$ such that $v(\emptyset) = 1$ and which satisfies the "drop condition:"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ with $y \subseteq x_1, \cdots, x_n$. A *probabilistic event structure* comprises an event structure E together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.

By Corollary 4, in showing we have a probabilistic event structure it suffices to verify the "drop condition" only for covering intervals.

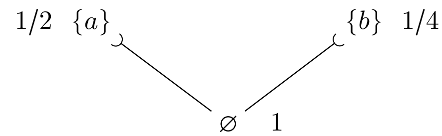
Theorem 6. *Let E be an event structure. A configuration valuation on E extends uniquely to a continuous valuation on the open sets of $\mathcal{C}^\infty(E)$. Conversely, a continuous valuation on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation on E.*

The above theorem holds (with the same proof) in greater generality, for Scott domains. Now, by [11], Corollary 4.3:

Theorem 7. *Let E be an event structure. A configuration-valuation v on $\mathcal{C}(E)$ extends to a unique probability measure μ_v on the Borel subsets of $\mathcal{C}^\infty(E)$.*

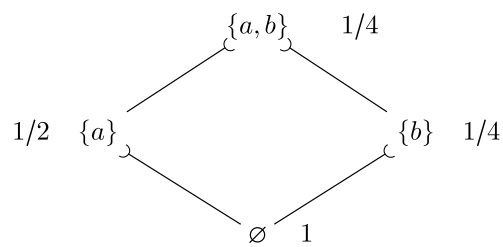
In the following examples, in all but the last, all subsets of configurations are Borel so we can describe probability measures by their values on singleton sets of configurations.

Example 8. Consider the event structure comprising two conflicting events a and b. It has configurations and configuration valuation as shown.



Accordingly, the probability of ending at the configuration $\{a\}$ is $1/2$ and at $\{b\}$, $1/4$. There is also a probability of $1/4$ of terminating at the configuration $\emptyset$, associated with the drop $1 - 1/2 - 1/4$ across the interval $[\emptyset; \{a\}, \{b\}]$. □

Example 9. Now the event structure comprises two concurrent events a and b. It has configurations and configuration valuation:



The probability of ending at the configuration $\{a, b\}$ is $1/4$; that of terminating at $\{a\}$ the drop $1/2 - 1/4 = 1/4$; that of terminating at $\{b\}$ the drop $1/4 - 1/4 = 0$ showing that $\{b\}$ is only a transient configuration; while the probability of terminating at $\emptyset$ is the drop $1 - 1/2 - 1/4 + 1/4 = 1/2$. □

Example 10. Consider the event structure with events $\mathbb{N}^+$ with causal dependency $n \leq n+1$, with all finite subsets consistent. Consider the ensuing probability distributions w.r.t. the following configuration-valuations:
(i) $v_0(x) = 1$ for all $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 1 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$, and 0 to $\emptyset$ and all other singleton sets of configurations.
(ii) $v_1(\emptyset) = v_1(\{1\}) = 1$ and $v_1(x) = 0$ for all other $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 0 to $\emptyset$ and probability 1 to the singleton set $\{1\}$, and 0 to all other singleton sets of configurations.
(iii) $v_2(\emptyset) = 1$ and $v_2(\{1, \cdots, n\}) = (1/2)^n$ for all $n \in \mathbb{N}^+$. The resulting probability distribution assigns probability $1/2$ to $\emptyset$ and $(1/2)^{n+1}$ to each singleton $\{\{1, \cdots, n\}\}$ and 0 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$. □

When x a finite configuration has $v(x) > 0$ and $\mu_v(\{x\}) = 0$ we can understand x as being a transient configuration on the way to a final result with probability $v(x)$. In general, there is a simple expression for the probability of terminating at a finite configuration.

Proposition 11. *Let E, v be a probabilistic event structure. For any finite configuration $y \in \mathcal{C}(E)$, the singleton set $\{y\}$ is a Borel subset with probability measure*

$$\mu_v(\{y\}) = \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \&\ y \subsetneq x_1, \cdots, x_n\}\,.$$

Example 12. A non-example: Consider the event structure comprising events $[0,1]$ where the only non-empty consistent sets are singletons. The valuation on its open sets extending the Lebesgue measure on $[0,1]$ is not continuous. So this example lies outside the present definition of probabilistic event structure. □

Remark. There is some redundancy in the definition of purely probabilistic event structures, in that there are two different ways to say, for example, that events e_1 and e_2 do not occur together at a finite configuration y where $y \overset{e_1}{\longrightarrow\!\!\!\subset} x_1$ and $y \overset{e_2}{\longrightarrow\!\!\!\subset} x_2$: either through $\{e_1, e_2\} \notin \mathrm{Con}$; or via the configuration-valuation v through $v(x_1 \cup x_2) = 0$. However, when we mix probability with nondeterminism, as we do in the next section, we shall make use of both order-consistency and the valuation.

IV. Probabilistic strategies

We show how concurrent strategies can be extended with probabilities, first reviewing the needed results from [1].

A. Strategies

1) Event structures with polarity: Both games and strategies in a game are represented in terms of event structures with polarity, which comprise (E, pol) where E is an event structure with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity + (Player) or − (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarities.

The *dual*, $E^\perp$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities. Let A and B be event structures with polarity. The operation $A \| B$, of *simple parallel composition*, juxtaposes disjoint copies of A and B, maintaining their causal dependency and specifying a finite subset of events as consistent if it restricts to consistent subsets of A and B. Polarities are unchanged. The empty game $\varnothing$ is the unit of $\|$.

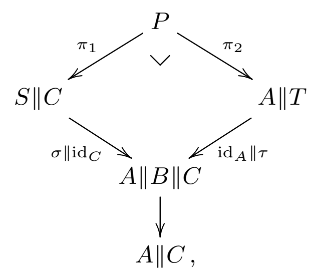
2) Pre-strategies: Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy in A* represents a nondeterministic play of the game and is defined to be a total map $\sigma : S \to A$ from an event structure with polarity S.

A map between pre-strategies, from $\sigma : S \to A$ and $\tau : T \to A$, is a map $f : S \to T$ such that $\sigma = \tau f$. Accordingly, $\sigma \cong \tau$ when there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$.

Let A and B be event structures with polarity. A *pre-strategy from A to B* is a pre-strategy in $A^\perp \| B$. Write $\sigma : A \twoheadrightarrow B$ to express that σ is a pre-strategy from A to B. Note that a pre-strategy σ *in* a game A, *e.g.* $\sigma : S \to A$, coincides with a pre-strategy *from* the empty game $\varnothing$ *to* the game A, *i.e.* $\sigma : \varnothing \twoheadrightarrow A$.

Strategies are defined to be those pre-strategies which "compose well with copy-cat."

3) Composing pre-strategies: We can present the composition of pre-strategies via pullbacks. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in the category of event structures as shown



where the map $A \| B \| C \to A \| C$ is undefined on B and acts as identity on A and C. The partial map from P to $A \| C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P{\downarrow}V$, where V is the set of events of P at which the map $P \to A \| C$ is defined, and a total map $P{\downarrow}V \to A \| C$. The resulting total map gives us the composition $\tau \odot \sigma : T \odot S =_{\mathrm{def}} P{\downarrow}V \to A^\perp \| C$ once we reinstate polarities.

In $T \odot S$ we have hidden the synchronization events over B due to the instantiation of Opponent moves of T in B by Player moves of S, and *vice versa*. Later we shall also be concerned with the event structure P, composition before hiding, which we shall denote more descriptively by $T \circledast S$.

4) Concurrent copy-cat: The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}_A \to A^\perp \| A$. It is based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity. For $c \in A^\perp \| A$ we use $\bar{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}_A$ to comprise the event structure with polarity $A^\perp \| A$ together with the extra causal dependencies generated by $\bar{c} \le_{\mathbb{C}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$. The *copy-cat* pre-strategy $\gamma_A : A \twoheadrightarrow A$ is defined to be the map $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ where γ_A is the identity on the common set of events.

5) Strategies: The main result of [1] is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. Innocence restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game. A pre-strategy σ is *receptive* iff $\sigma x \overset{a}{-\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{-\!\!\subset}$ & $\sigma(s) = a$. It is *innocent* iff $s \rightarrowtriangle s'$ & $(pol(s) = +$ or $pol(s') = -)$ implies $\sigma(s) \rightarrowtriangle \sigma(s')$. The main result of [1] is that $\gamma_B \odot \sigma \odot \gamma_A \cong \sigma$ iff σ is receptive and innocent. Copy-cats $\gamma_A : A \nrightarrow A$ are receptive and innocent.

A *strategy* is a pre-strategy which is receptive and innocent. We obtain a bicategory in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \nrightarrow B$ and 2-cells are total maps of pre-strategies with vertical composition the usual composition of such maps. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

An event structure with polarity S is *deterministic* iff any finite set of moves is consistent when it causally depends only on a consistent set of Opponent moves. Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic. Copy-cat strategies γ_A are deterministic iff the game A is

race-free: for all $x \in \mathcal{C}(A)$ such that $x \overset{a}{-\!\!\subset}$ and $x \overset{a'}{-\!\!\subset}$ with $pol(a) = -$ and $pol(a') = +$, we have $x \cup \{a, a'\} \in \mathcal{C}(A)$.

We obtain a sub-bicategory of deterministic strategies between race-free games—in fact equivalent to an order-enriched category [1].

Strategies inherit a duality from pre-strategies. A pre-strategy $\sigma : A \nrightarrow B$ corresponds to a dual pre-strategy $\sigma^\perp : B^\perp \nrightarrow A^\perp$, arising from the correspondence between pre-strategies $\sigma : S \to A^\perp \| B$ and $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$.

B. Probabilistic strategies

Without information about the stochastic rates of Player and Opponent we cannot hope to ascribe probabilities to outcomes of play in the presence of races, *i.e.* immediate conficts between moves of opposite polarities. Our results on probabilistic strategies will depend on restricting to games which are race-free.

It will be convenient, more generally, to define a probabilistic event structure in which some events are distinguished as Opponent events (where the other events may be Player events or "neutral" events due to synchronizations between Player and Opponent moves). Events which are not Opponent events we shall call p-events. For configurations x, y we shall write $x \subseteq^p y$ if $x \subseteq y$ and $y \setminus x$ contains no Opponent events; we write $x {-\!\!\subset^p} y$ when $x {-\!\!\subset} y$ and $x \subseteq^p y$; we similarly write *e.g.* $x \subseteq^- y$, respectively $x \subseteq^+ y$, if $x \subseteq y$ and $y \setminus x$ comprises solely Opponent, respectively Player, events. We can now extend the notion of configuration-valuation to the situation where events carry polarities.

Definition 13. Let E be an event structure in which a specified subset of events are Opponent events. A *configuration-valuation* is a function $v : \mathcal{C}(E) \to [0,1]$ for which $v(\varnothing) = 1$,

$$x \subseteq^- y \implies v(x) = v(y) \tag{1}$$

for all $x, y \in \mathcal{C}(E)$, and satisfies the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0 \tag{2}$$

for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ with $y \subseteq^p x_1, \cdots, x_n$.

A *probabilistic event structure with polarity* comprises E an event structure with polarity together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.

As indicated above, the extra generality in the definition of a configuration-valuation is to cater for a situation later in which we shall ascribe probabilities not only to results of Player moves but also to events arising as synchronizations between Player and Opponent moves. As earlier, by Corollary 4, it suffices to verify the "drop condition" for p-covering intervals.

Definition 14. Let A be a race-free event structure with polarity. A *probabilistic strategy* v, σ *in* A comprises S, v, a probabilistic event structure with polarity, and a strategy $\sigma : S \to A$. [It follows that S will also be race-free.]

Let A and B be a race-free event structures with polarity. A *probabilistic strategy from* A *to* B is a probabilistic strategy in $A^\perp \| B$.

The existing literature is often concerned with strategies which always progress, which we can express in our very general context by insisting non-$\subseteq^p$-maximal finite configurations of the game are transient—*cf.* Proposition 11:

Definition 15. Say a probabilistic strategy v and $\sigma : S \to A$ is *total* when $\inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \&\ y \subsetneq^+ x_1, \cdots, x_n\} \neq 0$ implies σy is $\subseteq^+$-maximal, for all $y \in \mathcal{C}(S)$.

We extend the usual composition of strategies to probabilistic strategies. Assume probabilistic strategies $\sigma : S \to A^\perp \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^\perp \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$. We first tentatively define their composition before hiding, taking $v : \mathcal{C}(T \circledast S) \to [0,1]$ to be

$$v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$$

for $x \in \mathcal{C}(T \circledast S)$. This is a configuration-valuation because:

Lemma 16. *Let* $v : \mathcal{C}(T \circledast S) \to [0,1]$ *be defined as above. Then,* $v(\varnothing) = 0$. *If* $x \subseteq^- y$ *in* $\mathcal{C}(T \circledast S)$ *then* $v(x) = v(y)$.

Let $y, x_1, \cdots, x_n \in \mathcal{C}(T \circledast S)$ *with* $y {-\!\!\subset^p} x_1, \cdots, x_n$. *Assume that* $\pi_1 y {-\!\!\subset^+} \pi_1 x_i$ *when* $1 \leq i \leq m$ *and* $\pi_2 y {-\!\!\subset^+} \pi_2 x_i$ *when* $m+1 \leq i \leq n$. *Then in* $\mathcal{C}(T \circledast S), v$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] =$$
$$d_{v_S}^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_{v_T}^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n]\,.$$

Hence the drop function for v being non-negative is reduced to the drop functions for v_S and v_T being non-negative.

Corollary 17. *The assignment* $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$ *to* $x \in \mathcal{C}(T \circledast S)$ *yields a configuration-valuation on* $\mathcal{C}(T \circledast S)$.

Example 18. The assumption that games are race-free is needed for Corollary 17. Consider the composition of strategies $\sigma : \varnothing \nrightarrow B$ and $\tau : B \nrightarrow \varnothing$ where B is the game comprising the two moves ⊕ and ⊖ in conflict with each other—a game with a race. Suppose σ assigns probability 1 to playing ⊕ and τ assigns probability 1 to playing ⊖, in the dual game. Then the "drop condition" required for the corollary fails. □

We can now complete the definition of the composition of probabilistic strategies. Note that for $x \in \mathcal{C}(T \odot S)$ its down-closure $[x] \in \mathcal{C}(T * S)$.

Lemma 19. *Let A, B and C be race-free event structure with polarity. Let $\sigma : S \to A^\perp \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^\perp \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$ be probabilistic strategies. Assigning $v_S(\pi_1[x]) \times v_T(\pi_2[x])$ to $x \in \mathcal{C}(T \odot S)$ yields a configuration-valuation on $T \odot S$ which with $\tau \odot \sigma : T \odot S \to A^\perp \| C$ forms a probabilistic strategy from A to C.*

A copy-cat strategy is easily turned into a probabilistic strategy, as is any deterministic strategy:

Lemma 20. *Let S be a deterministic event structure with polarity. Defining $v_S : \mathcal{C}(S) \to [0,1]$ to satisfy $v_S(x) = 1$ for all $x \in \mathcal{C}(S)$, we obtain a probabilistic event structure with polarity.*

Definition 21. We say a probabilistic event structure with polarity is *deterministic* when its configuration valuation assigns 1 to every finite configuration (it will necessarily also be deterministic as an event structure with polarity). We say a probabilistic strategy $\sigma : S \to A$ with configuration-valuation v on $\mathcal{C}(S)$ is *deterministic* when the probabilistic event structure S, v is deterministic. (Deterministic strategies are also total.)

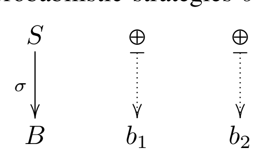
Recall that race-freedom of a game A ensures that $\mathbb{C}_A$ is deterministic [1]. Hence as a direct corollary of Lemma 20:

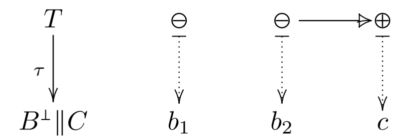
Corollary 22. *Let A be a race-free game. The copy-cat strategy from A to A comprising $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ with configuration-valuation $v_{\mathbb{C}_A} : \mathcal{C}(\mathbb{C}_A) \to [0,1]$ satisfying $v_{\mathbb{C}_A}(x) = 1$, for all $x \in \mathcal{C}(\mathbb{C}_A)$, forms a probabilistic strategy.*

Combining the results of this section:

Theorem 23. *Race-free games with probabilistic strategies with composition and copy-cat defined as in Lemma 19 and Corollary 22 inherit the structure of a bicategory from that of games with strategies.*

Example 24. Let A be the empty game $\varnothing$, B be the game consisting of two concurrent +ve events b_1 and b_2, and C the game with a single +ve event c. We illustrate the composition of two probabilistic strategies $\sigma : \varnothing \longrightarrow B$ and $\tau : B \longrightarrow C$.





The strategy σ plays b_1 with probability $2/3$ and b_2 with probability $1/3$ (and plays both with probability 0). The strategy τ does nothing if just b_1 is played and plays the single +ve event c of C with probabilty $1/2$ if b_2 is played. Their composition yields the strategy $\tau \odot \sigma : \varnothing \longrightarrow C$ which plays c with probability $1/6$, so has a $5/6$ chance of doing nothing.

The example illustrates how through probability we can track the presence of terminal configurations within a set of results despite their not being $\subseteq$-maximal. The empty configuration is such a terminal configuration; it could be the final result of the composition as could the configuration $\{c\}$. Such terminal but incomplete results can appear in a composition of strategies through the strategies being partial, in that one or both strategies do not respond in all cases—the example above. Such partial strategies can appear as the composition of two strategies through the occurrence of deadlocks because the two strategies impose incompatible causal dependencies on moves in game at which they interact. □

C. Probabilistic processes

As an indication of the expressivity of probabilistic strategies we show how they straightforwardly include a simple language of probabilistic processes, reminiscent of a higher-order CCS. For this section only, write $\sigma : A$ to mean σ is a probabilistic strategy in game A. Probabilistic strategies are closed under the following operations.

Synchronized composition $\sigma \odot \tau : A \| C$, if $\sigma : A \| B$ and $\tau : B^\perp \| C$. Hiding is achieved automatically in a synchronized composition directly based on the composition of strategies.

Simple parallel composition $\sigma \| \tau : A \| B$, if $\sigma : A$ and $\tau : B$—a special case of synchronized composition via the identification of $\sigma \| \tau$ with $\tau \odot \sigma$, in which $\sigma : A^\perp \longrightarrow \varnothing$ and $\tau : \varnothing \longrightarrow B$.

Input prefixing $\sum_{i \in I} \ominus.\sigma_i : \sum_{i \in I} \ominus.A_i$, if $\sigma_i : A_i$, for $i \in I$, where I is countable.

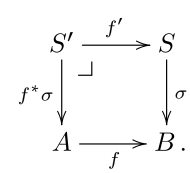
Output prefixing $\sum_{i \in I} p_i \oplus .\sigma_i : \sum_{i \in I} \oplus.A_i$, if $\sigma_i : A_i$, for $i \in I$, where I is countable, and $p_i \in [0,1]$ for $i \in I$ with $\sum_{i \in I} p_i \le 1$. If $\sum_{i \in I} p_i < 1$, there is non-zero probability of terminating without any action. By design $(\sum_{i \in I} \oplus.A_i)^\perp = \sum_{i \in I} \ominus.A_i^\perp$.

Relabelling, the composition $f\sigma : B$, if $\sigma : A$ and $f : A \to B$ is itself a strategy, *i.e.* total, receptive and innocent.

Pullback $f^*\sigma : A$, if $\sigma : B$ and $f : A \to B$ is a map of event structures which preserves +-conflict, *i.e.* is defined on all +ve events and satisfies

$$x \stackrel{a_1}{\longrightarrow\!\subset} x_1 \ \& \ x \stackrel{a_2}{\longrightarrow\!\subset} x_2 \ \& \ pol(a_1) = pol(a_2) = + \ \& \ x_1 \not\uparrow x_2$$
$$\implies f x_1 \not\uparrow f x_2 \,.$$

The strategy $f^*\sigma$ is got by the pullback



Then, the map f' also preserves +-conflict. The configuration valuation $v_{S'}$ on S' is defined from that v_S on S by taking $v_S(x) = v_S(f'x)$, for $x \in \mathcal{C}(f^*S)$. If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$. Pullback along $f : A \to B$ may introduce causal links, present in A but not in B.
Abstraction $\lambda x : A.\sigma : A \multimap B$. Because probabilistic strategies form a monoidal-closed bicategory, with tensor $A \| B$ and function space $A \multimap B =_{\text{def}} A^{\perp} \| B$, they support an (affine) λ-calculus with λ-abstractions, which in this context permits process-passing as in [12].
Recursive probabilistic processes can be dealt with along standard lines [13].

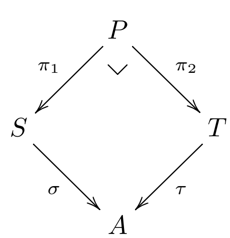
The types as they stand are somewhat inflexible. For example, that maps of event structures are locally injective means that simple labelling of events as in say CCS, where two concurrent events can carry the same label, cannot be directly captured through typing. These limitations can be remedied by introducing monads T and new types of the form $T(A)$, though doing this in sufficient generality would involve the introduction of symmetry to games—see Section IV-D.

D. Extensions:Imperfect information, payoff and symmety

As they stand the games here are so-called games of *perfect-information*. In games of *imperfect information* some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out. It is straightforward to extend concurrent games to games with imperfect information in way that respects the operations of the bicategory of games [14] and does not disturb the addition of probability. A fixed preorder of *levels* $(\Lambda, \leq)$ is pre-supposed. The levels are to be thought of as levels of access, or permission. A Λ-game comprises a game A with a *level function* $l : A \to \Lambda$ such that if $a \leq_A a'$ then $l(a) \leq l(a')$ for all $a, a' \in A$. A probabilistic Λ-strategy in the Λ-game is a probabilistic strategy $v_S, \sigma : S \to A$ for which if $s \leq_S s'$ then $l\sigma(s) \leq l\sigma(s')$ for all $s, s' \in S$.

One interpretation of Λ, relevant to the treatment of quantum strategies which follows, is as space-time with $\lambda \leq \lambda'$ meaning there is a causal curve from λ to λ'.

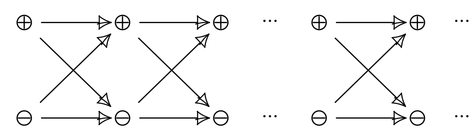
We can add *payoff* to a game A as a Borel measurable function $X : \mathcal{C}^{\infty}(A) \to \mathbb{R}$. Given a probabilistic strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^{\perp}$ we obtain



with valuation $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$, for $x \in \mathcal{C}(P)$, on the pullback P—a probabilistic event structure, with probability measure $\mu_{\sigma,\tau}$. Define $f =_{\text{def}} \sigma\pi_1 = \tau\pi_2$. The *expected pay-off* is obtained as as the Lebesgue integral

$$\mathbf{E}_{\sigma,\tau}(X) = \int X(f(x))\ d\mu_{\sigma,\tau}(x)\,.$$

In particular, *Blackwell games* [15] become a special case of probabilistic Λ-games with payoff. For Blackwell games an appropriate choice of Λ is the infinite elementary event structure:



A Blackwell game is given by A, a race-free concurrent game with payoff X, for which there is a (necessarily unique) polarity-preserving rigid map from A to Λ—this map becomes the level function. Moves in A occur in rounds comprising a choice of move for Opponent and a choice of move for Player made concurrently. Traditionally, in Blackwell games a strategy (for Player) is a total Λ-strategy in such a Λ-game—strategies are restricted to those assigning *total* probability distributions at each round.

The reduction of [16] appears to generalize, and, if so, would show the Borel determinacy of bounded-concurrent probabilistic games with imperfect information from that of Blackwell games [15].

There are several reasons to consider symmetry in games, situations where distinct plays are essentially similar to one another. Symmetry can help in the analysis of games, by for instance reducing the number of cases to consider. Symmetry can also help compensate for the overly-concrete nature of event structures in representing games; many useful operations on games which are not monads or comonads w.r.t. strategies become so *up to symmetry* [17], [5] and this leads to richer type systems, for example. Symmetry on an event structure can be captured through an *isomorphism family* which expresses when one finite configuration of the event structure is essentially the same as another [17]. It is a straightforward matter to ensure that configuration-valuations, attributing probability, respect the isomorphism family. The addition of symmetry to games meshes well with the introduction of probability. This should enable a formal connection with the probabilistic games of Danos and Harmer [4] which are based on HO games—allowing copying, so whose relation with concurrent games requires suitable (co)monads to exist, so symmetry.

V. QUANTUM STRATEGIES

A more novel application is to a definition of quantum event structures and strategies. We first explore a definition of quantum event structure in which events are associated with projection or unitary operators. It is shown how this structure induces configuration-valuations, and hence probability measures, on compatible parts of the domain of configurations of the event structure. We conclude with a brief exploration of quantum games and strategies. A quantum game is taken to be a quantum event structure in which events carry polarities and a strategy in a quantum game as a probabilistic strategy in its event structure.

A. Simple quantum event structures

Throughout let $\mathcal{H}$ be a separable Hilbert space over the complex numbers. For operators A, B on $\mathcal{H}$ we write $[A, B] =_{\text{def}} AB - BA$.

Definition 25. A *(simple) quantum event structure* (over $\mathcal{H}$) comprises an event structure $(E, \leq, \text{Con})$ together with an assignment Q_e of projection or unitary operators on $\mathcal{H}$ to events $e \in E$ such that for all $x \in \mathcal{C}(E), e_1, e_2 \in E$ for which $x \stackrel{e_1}{\longrightarrow\!\!\subset} x_1$ and $x \stackrel{e_2}{\longrightarrow\!\!\subset} x_2$,

$$x_1 \uparrow x_2 \implies [Q_{e_1}, Q_{e_2}] = 0\,,$$

i.e. the two events occur concurrently at x implies their associated operators commute. Say the quantum event structure is *strong* when

$$x_1 \uparrow x_2 \iff [Q_{e_1}, Q_{e_2}] = 0\,,$$

i.e. the two events occur concurrently at x *iff* their associated operators commute.

Interpretation. We regard $w \in \mathcal{C}^\infty(E)$ as a quantum experiment. The experiment specifies which unitary and projection operators to apply and in which order. The order being partial permits commuting operators to be applied concurrently, independently of each other, perhaps in a distributed fashion. Accordingly each finite configuration is associated with a unique operator.

Definition 26. Given a finite configuration, $x \in \mathcal{C}(E)$, define the operator A_x to be the composition $Q_{e_n} Q_{e_{n-1}} \cdots Q_{e_2} Q_{e_1}$ for some covering chain

$$\varnothing \stackrel{e_1}{\longrightarrow\!\!\subset} x_1 \stackrel{e_2}{\longrightarrow\!\!\subset} x_2 \cdots \stackrel{e_n}{\longrightarrow\!\!\subset} x_n = x$$

in $\mathcal{C}(E)$. This is well-defined as for any two covering chains up to x the sequences of events are Mazurkiewicz trace equivalent, *i.e.* obtainable, one from the other, by successively interchanging concurrent events. In particular $A_\varnothing$ is the identity operator on $\mathcal{H}$.

Proposition 27. *In a strong quantum event structure* $(E, \leq, \text{Con})$ *with assignment of operators* Q *the consistency predicate* Con *is determined in a pairwise fashion, i.e. for any finite subset of events* X,

$$X \in \text{Con} \iff \forall e_1, e_2 \in X.\ \{e_1, e_2\} \in \text{Con}\,.$$

Writing $e_1 \# e_2 \iff_{\text{def}} \{e_1, e_2\} \notin \text{Con}$,

$$e_1 \# e_2 \iff \exists e_1' \leq e_1, e_2' \leq e_2.$$
$$[e_1] \cup [e_2) \in \text{Con}\ \&\ [e_1) \cup [e_2] \in \text{Con}\ \&\ [P_{e_1}, P_{e_2}] \neq 0\,.$$

Example 28. In the quantum event structure E with assignment of projection operators P_e to events e, assume the event structure E comprises solely concurrent events. In other words, no event causally depends on any other and any two events are concurrent. This is an example of a strong quantum event structure. Each projection operator P_e commutes with every other $P_{e'}$. Therefore the eigenvectors of all the projection operators P_e extend to an orthonormal basis of $\mathcal{H}$. Each projection operator corresponds to that subset of basis vectors it fixes. Under this correspondence, a composition of projection operators is associated with the intersection of the sets of fixed basis vectors. In other words, for any finite configuration x, the operator A_x is the projection operator which fixes precisely those basis vectors which are fixed by all the P_e, for $e \in x$. □

Example 29. Consider an event structure consisting of two events e_1, e_2 incomparable under $\leq$ with $\{e_, e_2\} \notin \text{Con}$. Only assignments of operators to e_1, e_2 for which $[Q_{e_1}, Q_{e_2}] \neq 0$ will yield a *strong* quantum event structure. □

Example 30. Consider an event structure consisting of two events for which $e_1 \leq e_2$. Any assignment of projection operators to e_1, e_2 will yield a strong quantum event structure. □

Example 31. Let (M, L, I) be a Mazurkiewicz trace language consisting of an alphabet L with independence relation I and subset of strings $M \subseteq L^*$, so M is closed under prefixes and I-closed in the sense that if $sabt \in M$ and aIb then $sbat \in M$. Assume an assignment of projection and unitary operators Q_a to symbols $a \in \Sigma$ such that

$$a\,I\,b \implies [Q_a, Q_b] = 0\,.$$

Then, M determines a quantum event structure: as shown in [18], M determines an event structure with events e associated with the minimal ways a symbol, say a, appears in a string in M—then the operator assigned to e is Q_a. If we assume that

$$sa \in M\ \&\ sb \in M\ \&\ a\,I\,b \implies sab \in M\,.$$

and an assignment of operators Q_a to symbols $a \in \Sigma$ such that

$$a\,I\,b \iff a \neq b\ \&\ [Q_a, Q_b] = 0\,,$$

then M determines a *strong* quantum event structure. □

In particular, the unitary and projection operators of $\mathcal{H}$ intrinsically have the form a Mazurkiewicz trace language, and in turn a strong quantum event structure.

Proposition 32. *Take as Mazurkiewicz trace language that with alphabet comprising (names for) all the unitary and projection operators on* $\mathcal{H}$ *with all strings of such and with independence relation*

$$A\ I\ B \iff A \neq B\ \&\ [A, B] = 0\,,$$

between operators A and B. The Mazurkiewicz trace language determines a strong quantum event structure, associated with the Hilbert space $\mathcal{H}$.

B. From quantum to probabilistic

Consider a quantum event structure with an initial state given by a density operator ρ on $\mathcal{H}$. While it does not make sense to attribute a probability distribution globally, over the whole space of configurations $\mathcal{C}^\infty(E)$, there is a sensible probability distribution on compatible configurations of the event structure. Below, by an unnormalized density operator we mean a positive, self-adjoint operators with trace less than or equal to one.

Theorem 33. *Let E, Q be an simple quantum event structure with initial state a density operator ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalized density operator $\rho_x =_{\rm def} A_x \rho A_x^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\rm def} \mathrm{Tr}(\rho_x) = \mathrm{Tr}(A_x^\dagger A_x \rho)$. For any $w \in \mathcal{C}^\infty(E)$, the function v restricts to a configuration-valuation v_w on finite configurations in the family of configurations $\mathcal{F}_w =_{\rm def} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}$; hence v_w extends to a unique probability measure q_w on $\mathcal{F}_w$.*

Interpretation. We regard $w \in \mathcal{C}^\infty(E)$ as a quantum experiment. The experiment can end in an element of $\mathcal{F}_w$ with chance given by the probability measure got from the configuration-valuation v_w. To say an experiment ends or results in $w' \in \mathcal{F}_w$ means it succeeds in the confirmation, observation or test associated with w', but goes no further.

In particular, we may take w to be a maximal configuration, obtaining a maximal part of the space configurations over which it is sensible to attribute a probability distribution. Compatible parts of the domain of configurations of a quantum event structure with an initial state carry an intrinsic probability distribution. With the reading of configurations as histories the theorem is reminiscent of the consistent/decoherent histories view of quantum computation. Note however that the consistency/decoherence conditions traditional in that approach have been replaced here, in the case of quantum event structures, by compatibility w.r.t. the inclusion order on configurations, and that compatibility respects traditional quantum notions of commuting observables.

Example 34. Let E comprise the quantum event structure with two concurrent events e_0 and e_1 associated with projectors P_0 and P_1, where necessarily $[P_0, P_1] = 0$. Assume an initial state $|\psi\rangle\langle\psi|$. The configuration $\{e_0, e_1\}$ is associated with the following probability distribution. The probability that e_0 succeeds is $||P_0|\psi\rangle||^2$, that e_1 succeeds $||P_1|\psi\rangle||^2$, and that both succeed is $||P_1 P_0|\psi\rangle||^2$.

In the case where P_0 and P_1 commute because $P_0 P_1 = P_1 P_0 = 0$, the events e_0 and e_1 are mutually exclusive. There is probability zero of both events e_0 and e_1 succeeding, probability $||P_0|\psi\rangle||^2$ of e_0 succeeding, $||P_1|\psi\rangle||^2$ of e_1 succeeding, and probability $1 - ||P_0|\psi\rangle||^2 - ||P_1|\psi\rangle||^2$ of getting stuck at the empty configuration where neither event succeeds.

A special case of this is the measurement of a qubit in state ψ, the measurement of 0 where $P_0 = |0\rangle\langle 0|$, and the measurement of 1 where $P_1 = |1\rangle\langle 1|$, though here $||P_0|\psi\rangle||^2 + ||P_1|\psi\rangle||^2 = 1$, as a measurement of the qubit will determine a result of either 0 or 1. □

Example 35. The measurement of two entangled qubits. Suppose $\psi = 1/\sqrt{2}\,|00\rangle + 1/\sqrt{2}\,|11\rangle$. Let $L_0 = \sum_y |0y\rangle\langle 0y|$, $L_1 = \sum_y |1y\rangle\langle 1y|$, $R_0 = \sum_x |x0\rangle\langle x0|$ and $R_1 = \sum_x |x1\rangle\langle x1|$ be commuting projectors associated with concurrent events l_0, l_1, r_0 and r_1 respectively; for instance l_0 is the event of measuring 0 in the left qubit. The configuration valuation will assign $v(\{l_0\}) = v(\{l_1\}) = v(\{r_0\}) = v(\{r_1\}) = v(\{l_0, r_0\}) = v(\{l_1, r_1\}) = 1/2$ and $v(\{l_0, r_1\}) = v(\{l_1, r_0\}) = 0$. □

Example 36. Let E comprise the event structure with three events e_1, e_2, e_3 with trivial causal dependency and consistency relation generated by taking $\{e_1, e_2\} \in \mathrm{Con}$ and $\{e_2, e_3\} \in \mathrm{Con}$—so $\{e_1, e_3\} \notin \mathrm{Con}$. To be a quantum event structure we must have $[Q_{e_1}, Q_{e_2}] = 0$, $[Q_{e_2}, Q_{e_3}] = 0$ and, to be strong, that $[Q_{e_1}, Q_{e_3}] \neq 0$. The maximal configurations are $\{e_1, e_2\}$ and $\{e_2, e_3\}$. Assume an initial state $|\psi\rangle\langle\psi|$. The first maximal configuration is associated with a probability distribution where e_1 occurs with probability $||Q_{e_1}|\psi\rangle||^2$ and e_2 occurs with probability $||Q_{e_2}|\psi\rangle||^2$. The second maximal configuration is associated with a probability distribution where e_2 occurs with probability $||Q_{e_2}|\psi\rangle||^2$ and e_3 occurs with probability $||Q_{e_3}|\psi\rangle||^2$. □

C. General quantum event structures

Simple quantum event structures are a special case of a more general though less intuitive concept:

Definition 37. A *(general) quantum event structure* comprises an event structure $(E, \leq, \mathrm{Con})$ together with a functor Q from the partial-order $(\mathcal{C}(E), \subseteq)$ (regarded as a category) to the monoid of 1-bounded operators on $\mathcal{H}$ (regarded as a one-object category) which satisfy

$$\mathrm{id}_{\mathcal{H}} - \sum_{\emptyset \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} Q(y, \bigcup_{i \in I} x_i)^\dagger Q(y, \bigcup_{i \in I} x_i)$$

is a positive operator, for all $y \subseteq x_1, \cdots, x_n$ with $\{x_1, \cdots, x_n\}\uparrow$.

A simple quantum event structure E with assignment $e \mapsto Q_e$ of unitary or projection operators to events e, determines a general quantum event structure E, Q for which $Q(x, y) = Q_e$ when $x \overset{e}{\longrightarrow\!\!\subset} y$. The earlier Theorem 33 extends.

Theorem 38. *Let E, Q be a general quantum event structure with initial state a density operator ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalized density operator $\rho_x =_{\rm def} Q(\emptyset, x)\, \rho\, Q(\emptyset, x)^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\rm def} \mathrm{Tr}(\rho_x)$. For any $w \in \mathcal{C}^\infty(E)$, the function v restricts to a configuration-valuation v_w on finite configurations in the family of configurations $\mathcal{F}_w =_{\rm def} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}$; hence v_w extends to a unique probability measure q_w on the Borel sets of $\mathcal{F}_w$.*

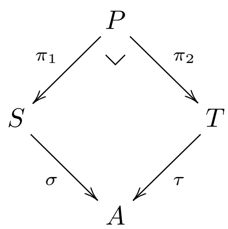
We would like a result showing how to realize a general quantum event structure from a simple quantum event structure by projection, possibly with tracing-out.

D. Quantum strategies

We define a *quantum game* to comprise A, pol, Q, ρ where A, pol is a race-free event structure with polarity, A, Q is a quantum event structure and ρ is a density operator.

A *strategy* in a quantum game A, pol, Q, ρ comprises a probabilistic strategy in A, so a strategy $\sigma : S \to A$ together with configuration-valuation v on $\mathcal{C}(S)$.

Given a strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ in a quantum game A, Q, ρ we obtain a probabilistic event structure P via pull-back, *viz.*



with a configuration-valuation $v(x) =_{\rm def} v_S\pi_1(x) \times v_T\pi_2(x)$ on finite configurations $x \in \mathcal{C}(P)$. This induces a probabilistic measure μ on the event structure P. We can interpret P as the probabilistic experiment which results from the interaction of the strategy σ and the counter-strategy τ.

We now investigate the probability the interaction of σ with τ produces a result in a Borel subset U of of $\mathcal{C}^\infty(A)$, that the probabilistic experiment the interaction induces succeeds in U.

First note that P becomes a quantum event structure via the map $f =_{\rm def} \sigma\pi_1 = \tau\pi_2$ to the quantum event structure A: the assignment of operators is given by the composition of Q with f. By Theorems 33 and 38, w.r.t. any $x \in \mathcal{C}^\infty(P)$, we obtain a probability measure q_x on $\mathcal{F}_x =_{\rm def} \{x' \in \mathcal{C}^\infty(P) \mid x' \subseteq x\}$. Write f_x for the restriction of f to $\mathcal{F}_x$. The expression

$$q_x(f_x^{-1}U)$$

gives the probability of obtaining a result in U conditional on $x \in \mathcal{C}^\infty(P)$. The probability of a result in $U \subseteq \mathcal{C}^\infty(A)$ is given by the Lebesgue integral

$$\int q_x(f_x^{-1}U)\, d\mu(x)\,.$$

We examine some special cases.

Consider the case where σ and τ are deterministic, with configuration valuations assigning one to each finite configuration. Then, P will also be deterministic in the sense that all its finite subsets will be consistent. It will thus have a single maximal configuration $w \in \mathcal{C}^\infty(P)$. The configuration-valuation v will assign one to each finite configuration of P. In this case the probability measure on Borel subsets V of $\mathcal{C}^\infty(P)$ is simple to describe:

$$\mu(V) = \begin{cases} 1 & \text{if } w \in V\,, \\ 0 & \text{otherwise,} \end{cases}$$

leading to

$$\int q_x(f_x^{-1}U)\, d\mu(x) = q_w(f^{-1}U)\,.$$

Consider now the case where Opponent initially offers $n \in \{1, \cdots, N\}$ mutually-inconsistent alternatives to Player and resumes with a deterministic strategy. Suppose too that initially Player chooses amongst the alternatives probabilistically, choosing option n with probability p_n, and then resumes deterministically. This will result in an event structure P taking the form of a prefixed sum $\sum_{1\le n\le N} e_n.P_n$ in which all the events of P_n causally depend on event e_n. In this situation,

$$\int q_x(f_x^{-1}U)\, d\mu(x) = \sum_{1\le n\le N} p_n \cdot q_{w_n}(f_n^{-1}U)\,,$$

where w_n is the maximal configuration of $e_n.P_n$ and $f_n : e_n.P_n \to A$ is the restriction of f, for $1 \le n \le N$.

In this way one can present traditional quantum games such as quantum-coin tossing or game presentations of quantum algorithms such as Grover's algorithm, though, of course, here the major cleverness is in the choice of operators [7]. It would be helpful to have a syntax for describing quantum strategies, perhaps based on the ideas of Section IV-C.

It remains to consider operations on quantum games. The previous simple parallel composition of games, *viz.* $A\|B$, is perhaps inappropriate for quantum games as it would appear to exclude moves introducing entanglement between the two games. A more apt parallel composition might obtain by basing games directly on Hilbert spaces with parallel composition as tensor; quantum games result by Proposition 32.

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