Descriptive complexity of linear algebra

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UNIVERSITY OF CAMBRIDGE

Logical Approaches to Barriers in Computing & Complexity II

Isaac Newton Institute

2012

Overview

Study definability of natural problems in linear algebra and expressiveness of logics with algebraic operators.

- Background & motivation
- Descriptive complexity of problems in linear algebra
- Logics with matrix-rank operators
- Pebble games for rank logics & the Weisfeiler-Lehman method

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A logic for **NP**

ESO — Existential second-order logic

Second-order variables existentially quantified, followed by a first-order formula:

$$\exists R_1, \ldots, R_k \,.\, \varphi(R_1, \ldots, R_k)$$

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Is there a logic for PTIME?

A logic for **PTIME**?

Fixed-point logic captures PTIME on ordered structures

FP is first-order logic with an inflationary fixed-point operator.

A property P of ordered structures can be *decided* in PTIME if and only if P can be *defined* by a sentence of FP.

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Ordered structure: Vocabulary contains a binary symbol "$\leqslant$" interpreted as a total ordering of the vertices.

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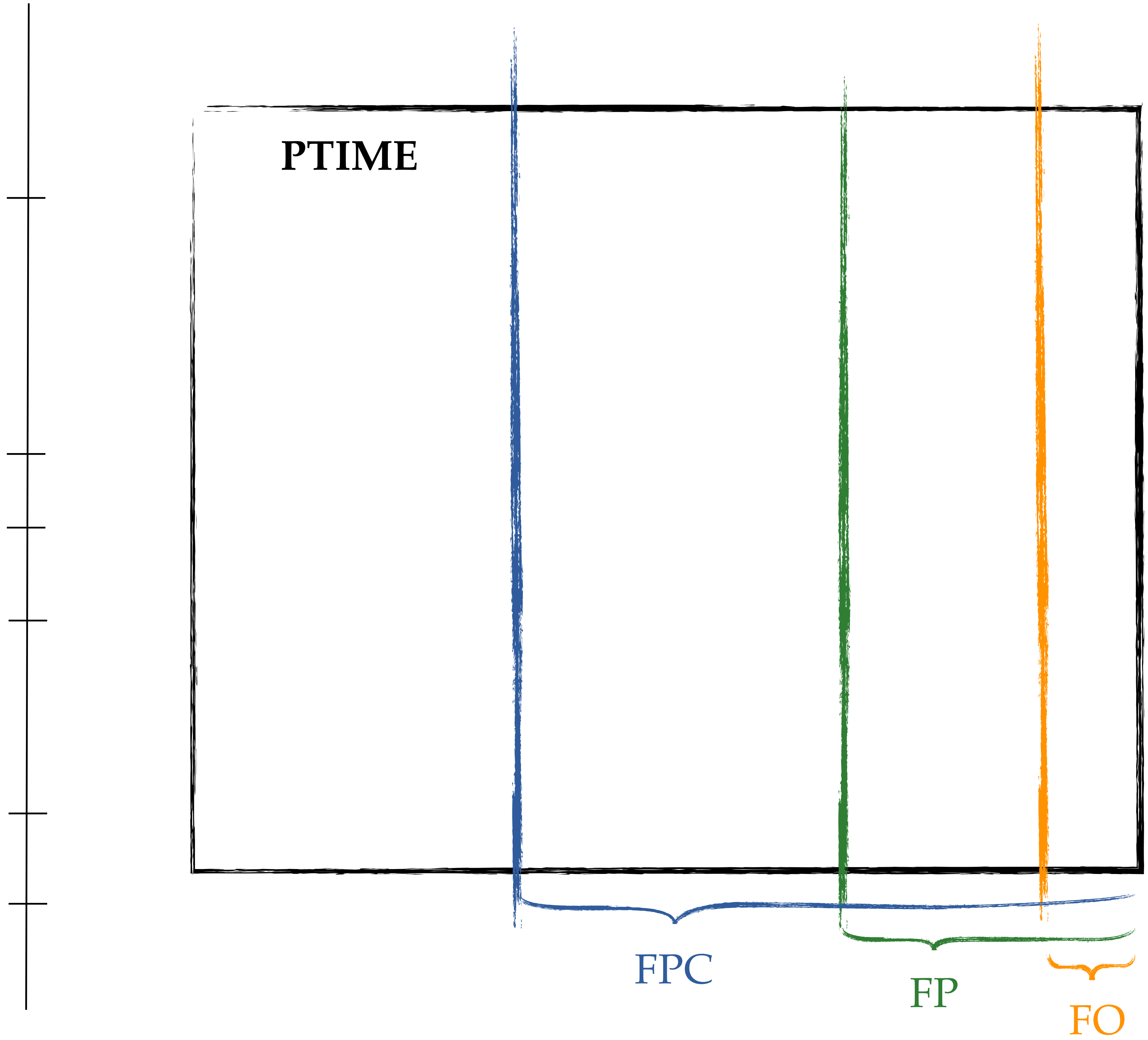
- On unordered structures, FP cannot even express if a graph has an even or odd number of vertices.
- *Fixed-point logic with counting* (FPC) is FP together with terms that count the number of solutions to formulas.

FPC captures **PTIME** on...



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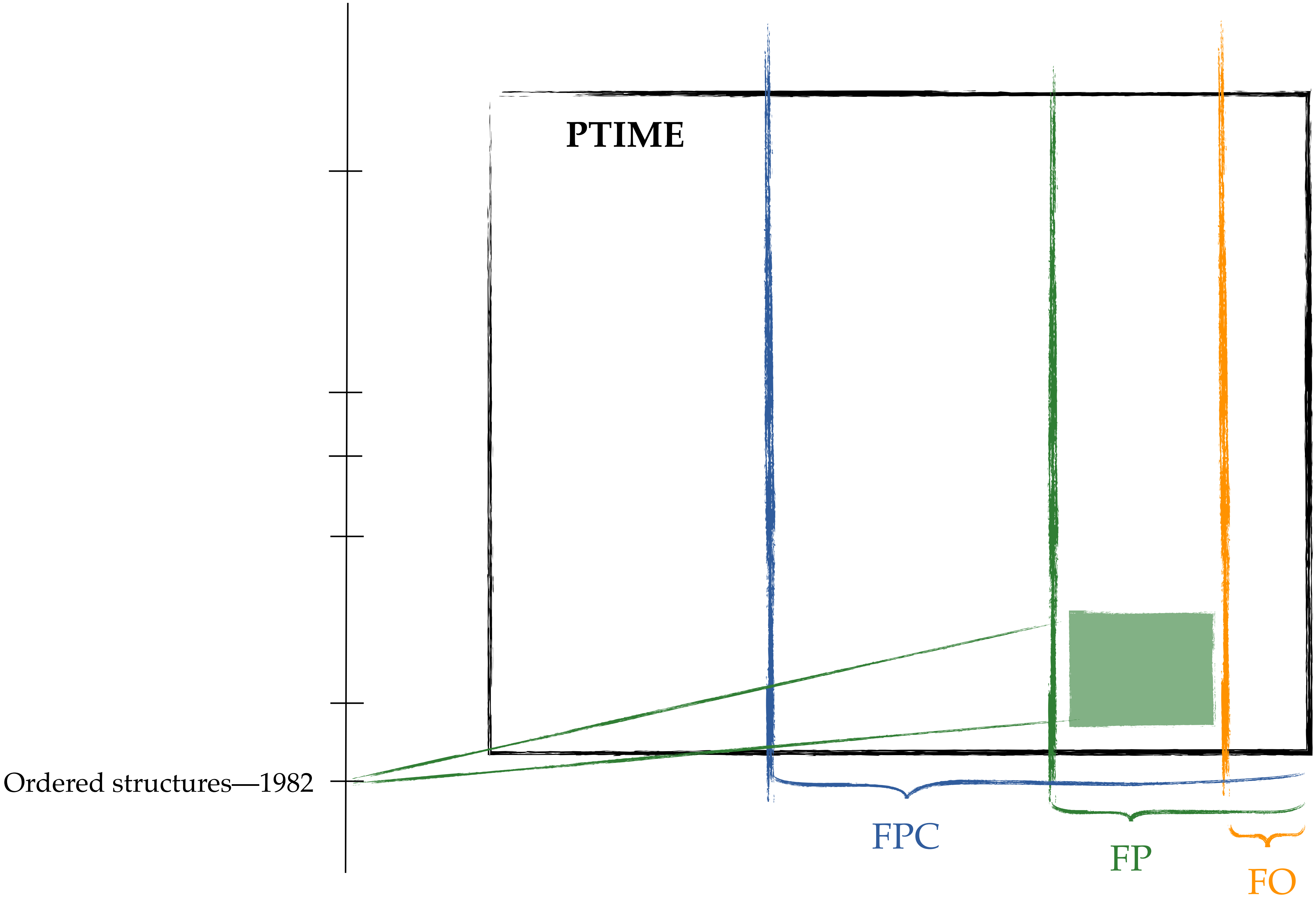

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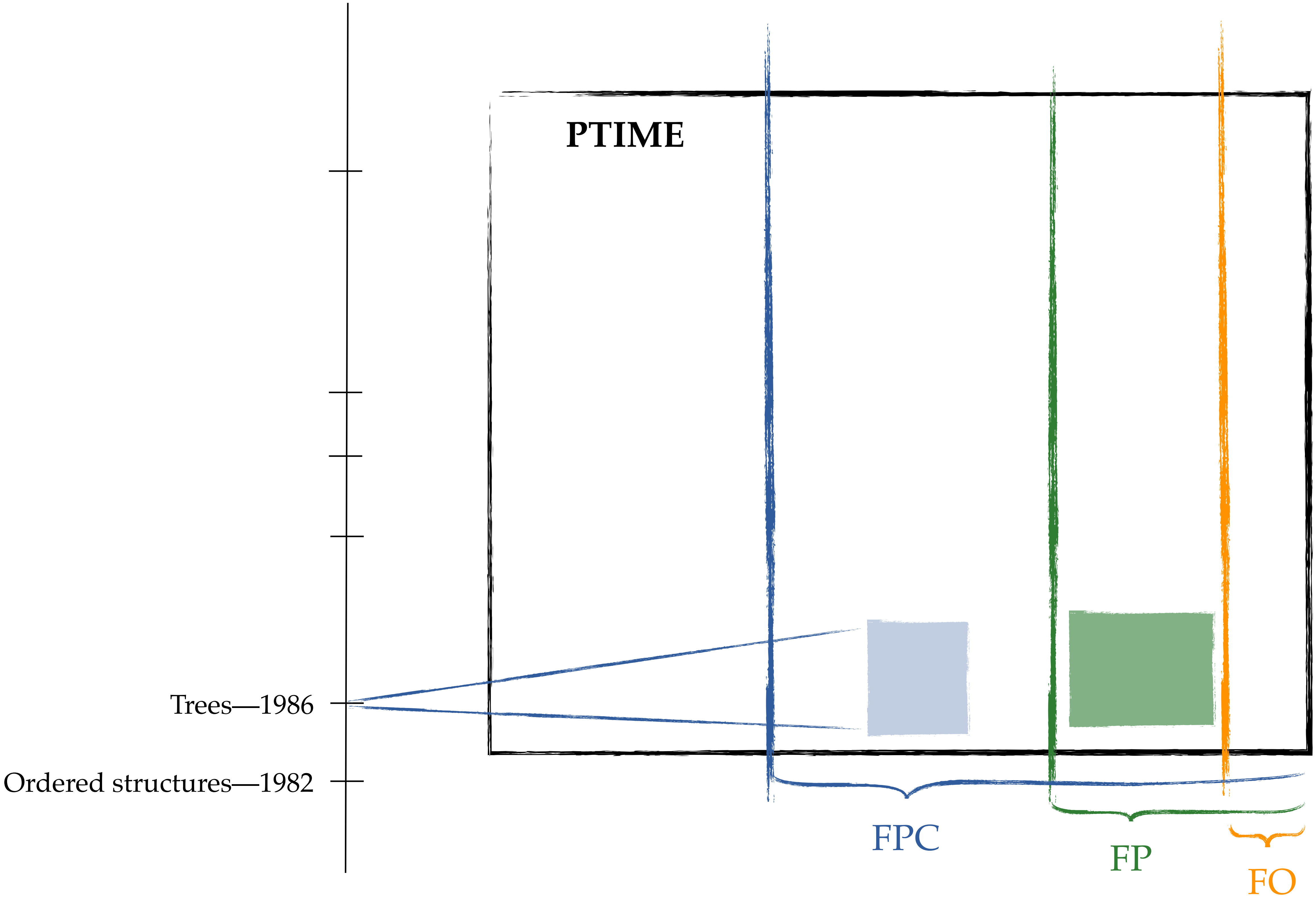

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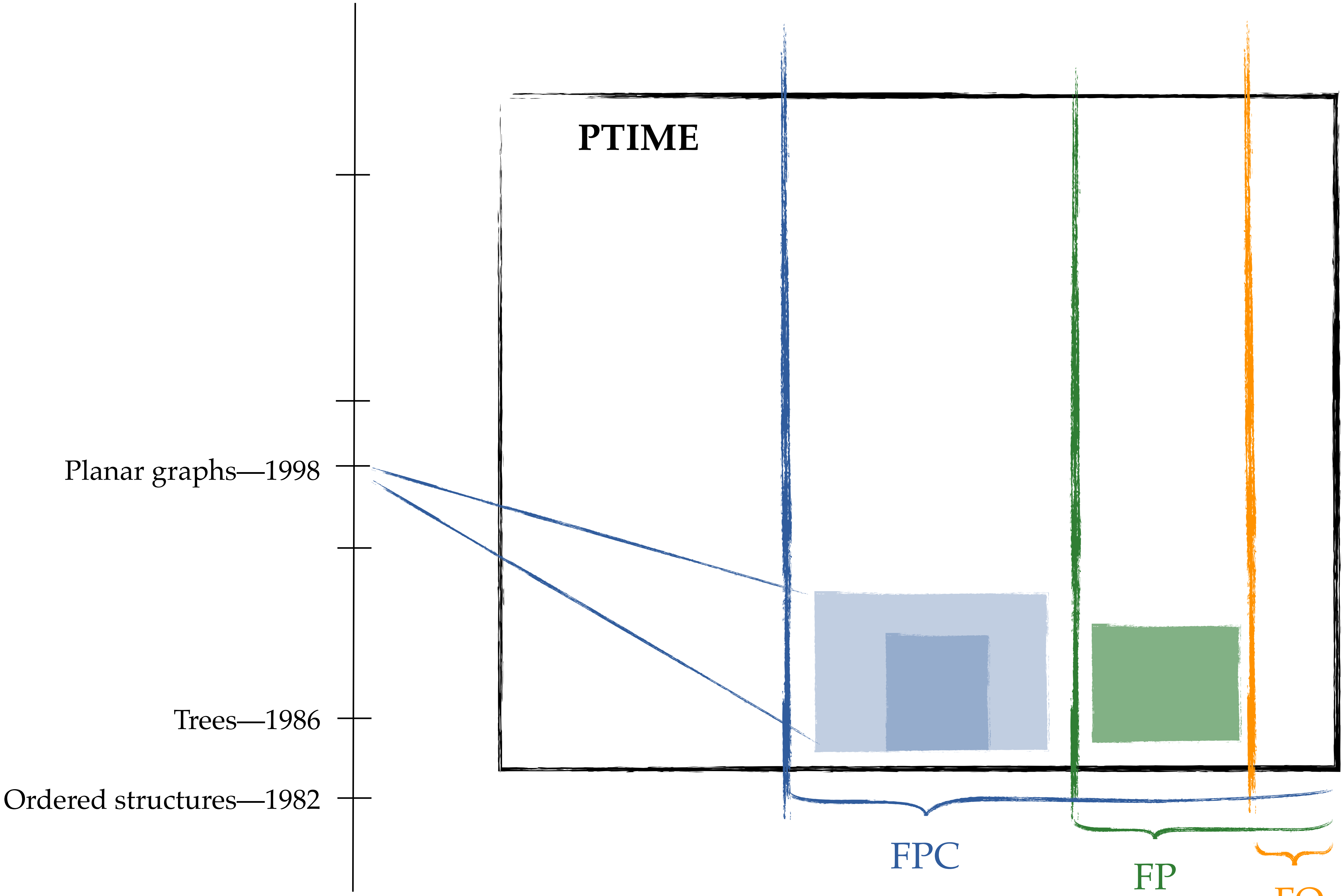

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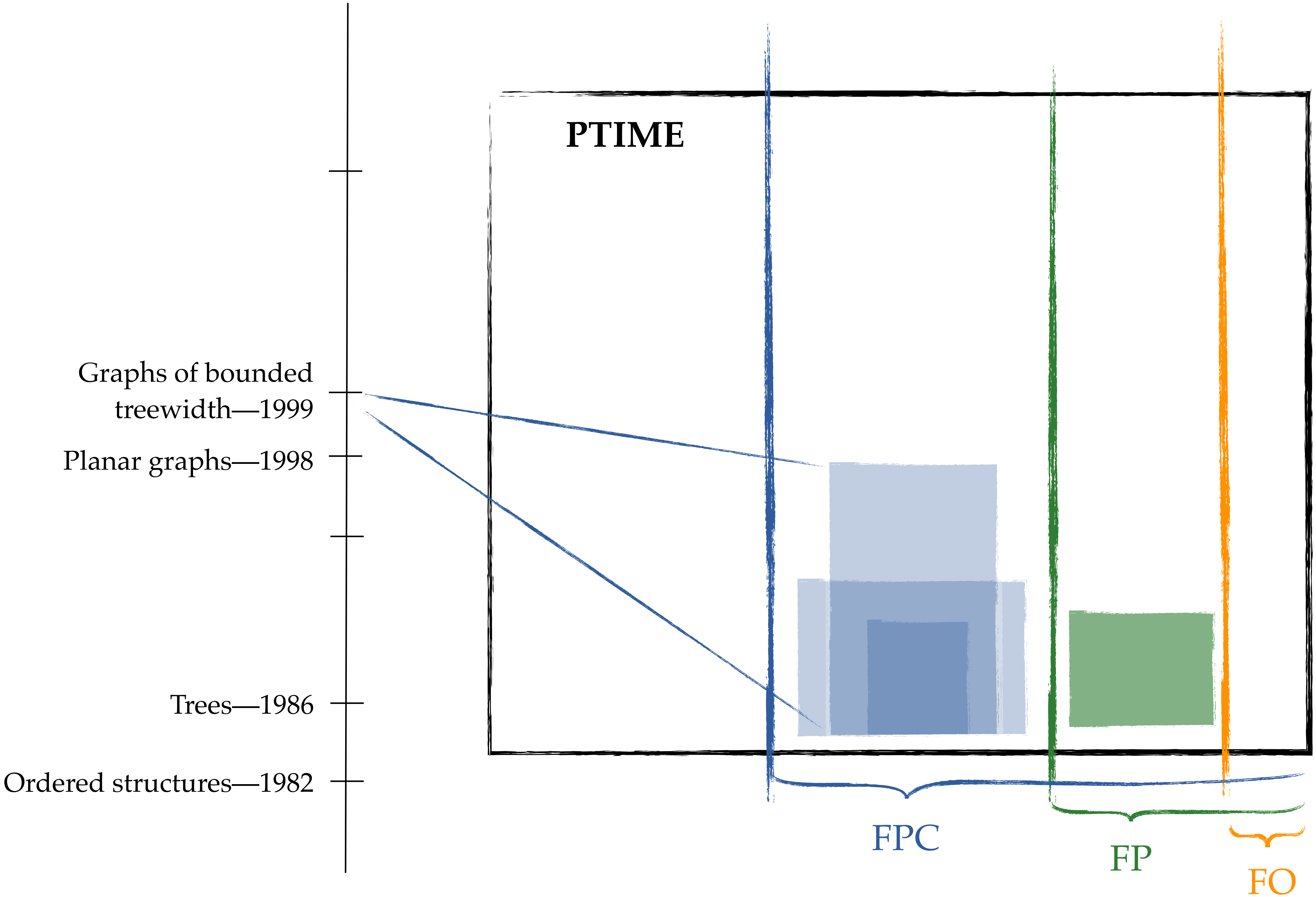

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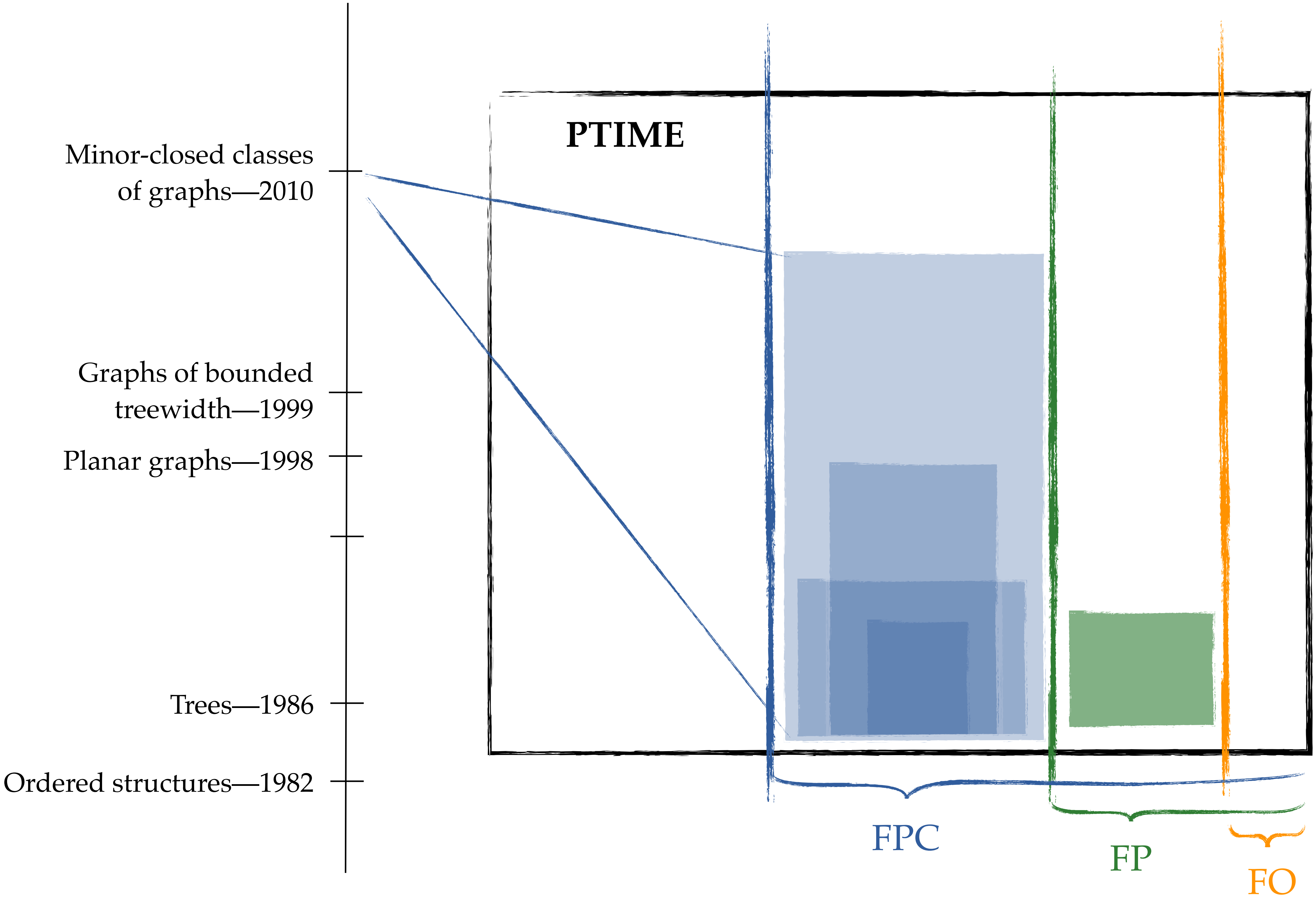

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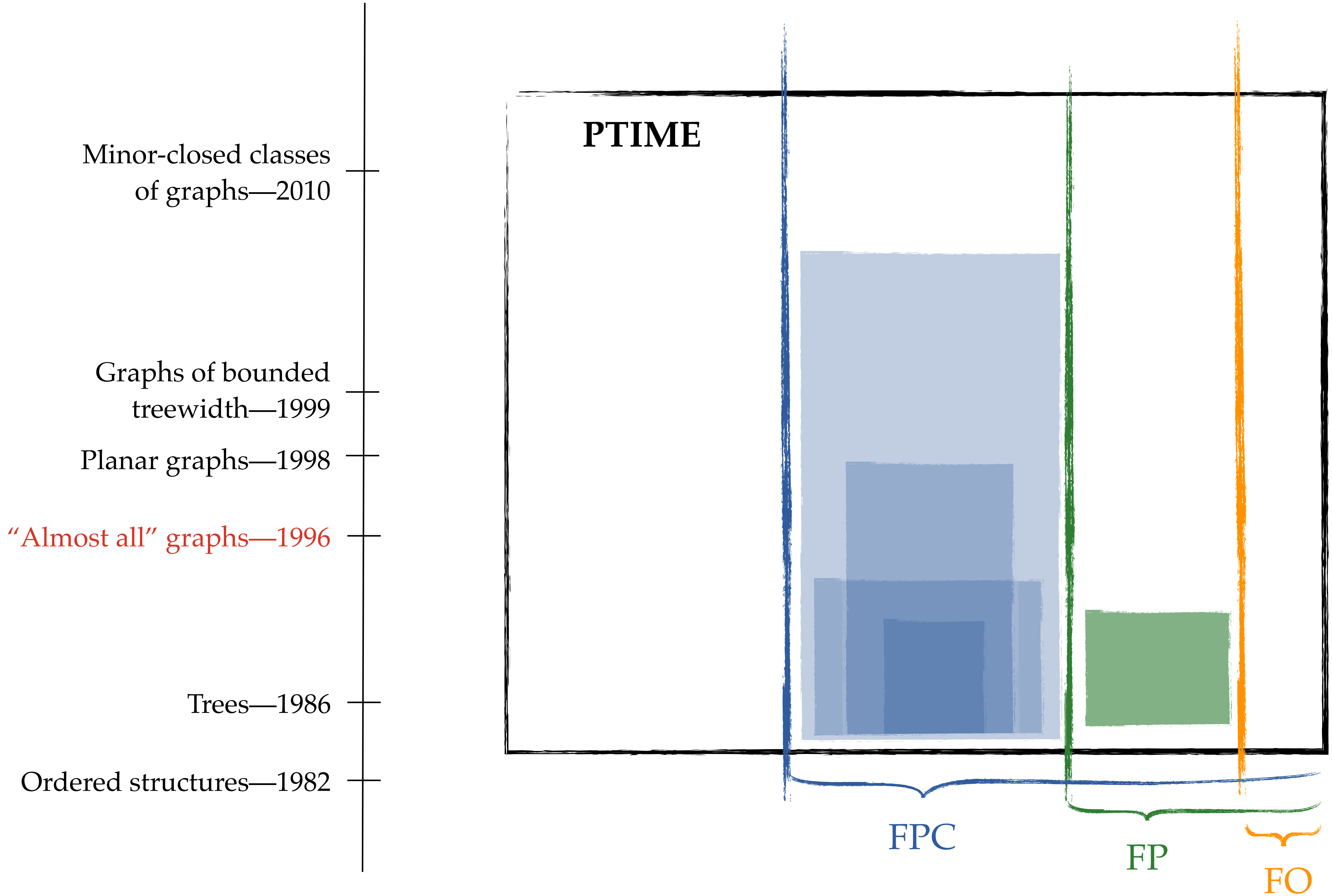
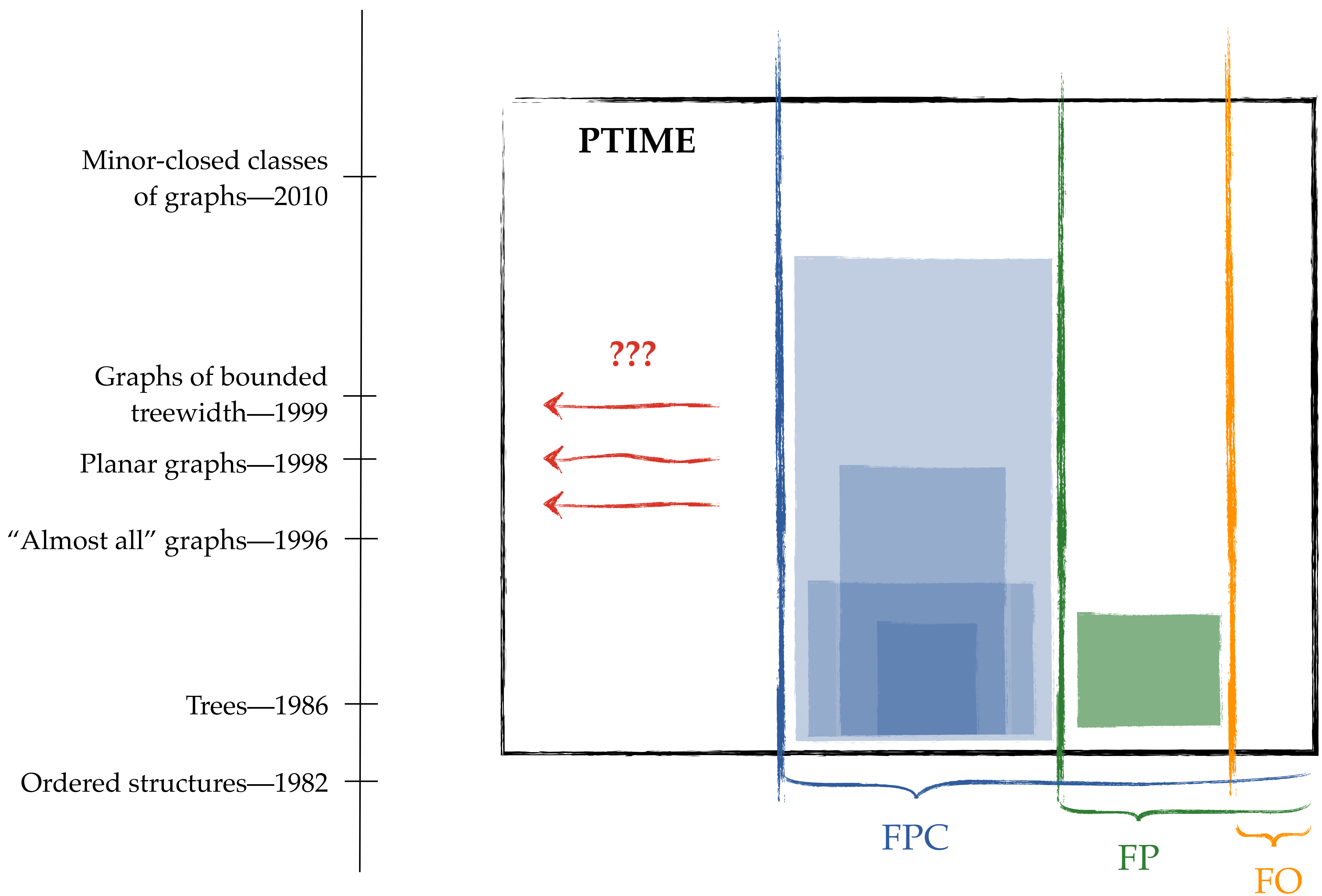


FPC captures **PTIME** on... all graphs?



Proving non-definability in FPC

C^k — first-order logic with variables x_1, ..., x_k and counting quantifiers of the form $\exists^{\geq i} x \,.\, \varphi$

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G and H agree on all sentences of C^k	iff	Duplicator has a winning strategy in the k-pebble bijection game on G and H

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- Duplicator wins the k-pebble game on G_k and H_k.

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Facts

- For each k, we can decide the winner of the k-pebble game in polynomial time.
- Close connection with a family of algorithms for graph isomorphism: Weisfeiler-Lehman method.

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There is a polynomial-time decidable property of finite graphs that is not definable in FPC.

Cai, Fürer and Immerman (1992)

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More recently: See which problems in linear algebra can be expressed in FPC

Descriptive complexity of problems in linear algebra

The usual notion of a matrix

$A = (a_{ij})$ — an m-by-n rectangular array of elements

	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
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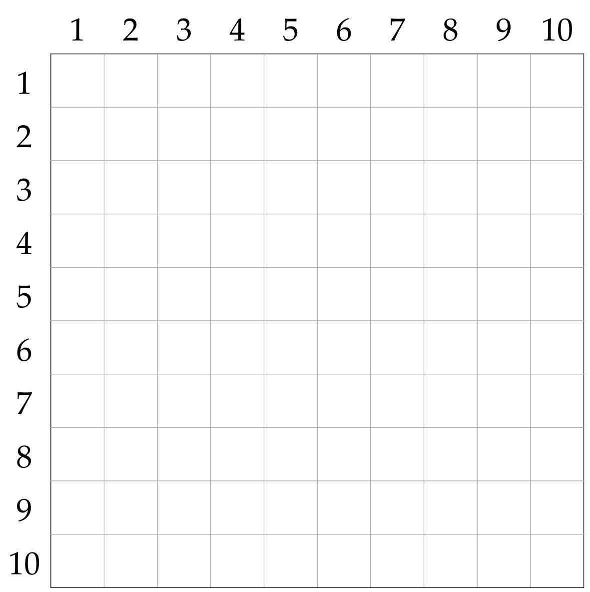

Recall: Over ordered structures FP (and hence FPC) can define *all* polynomial-time properties.

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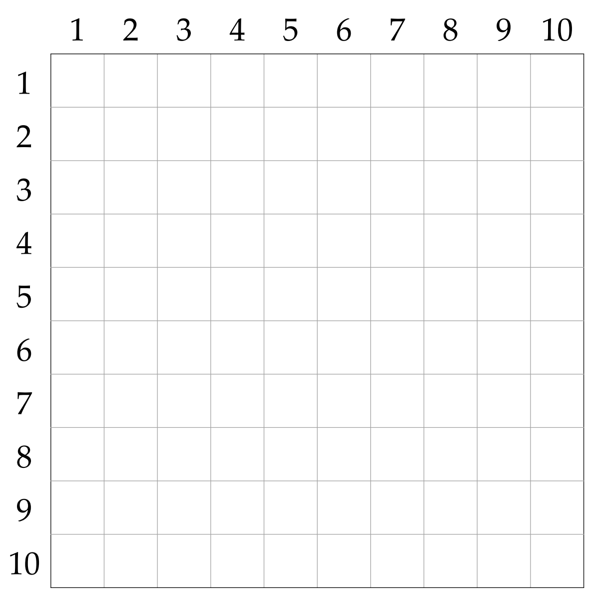

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rows and columns ordered

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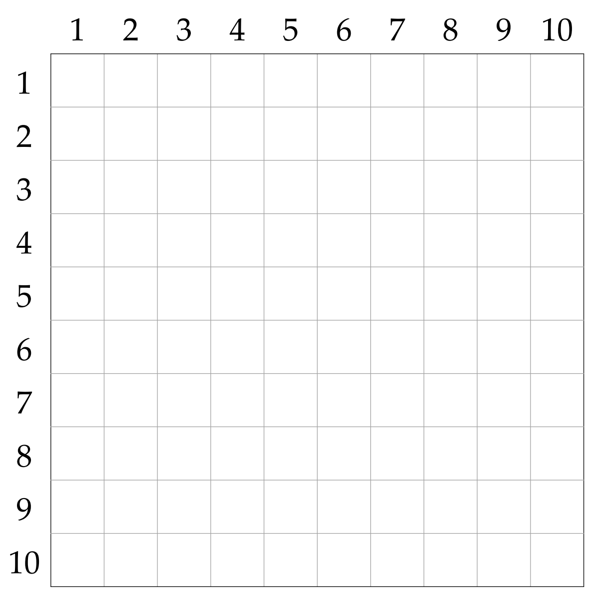

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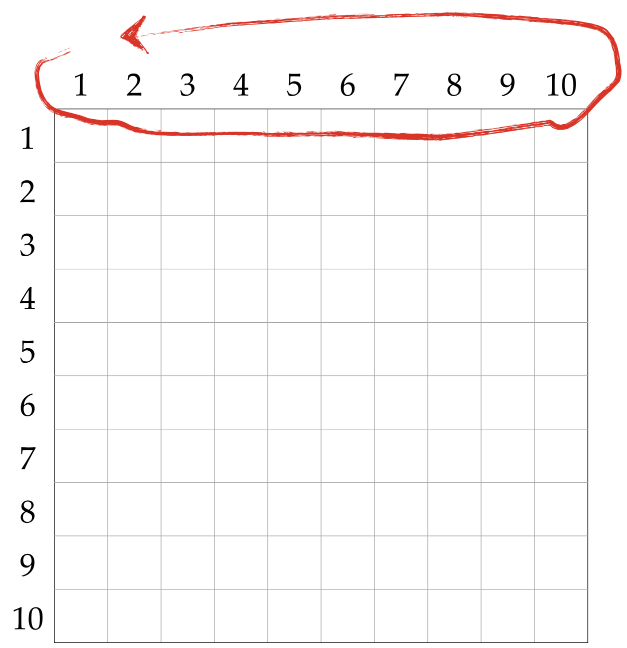

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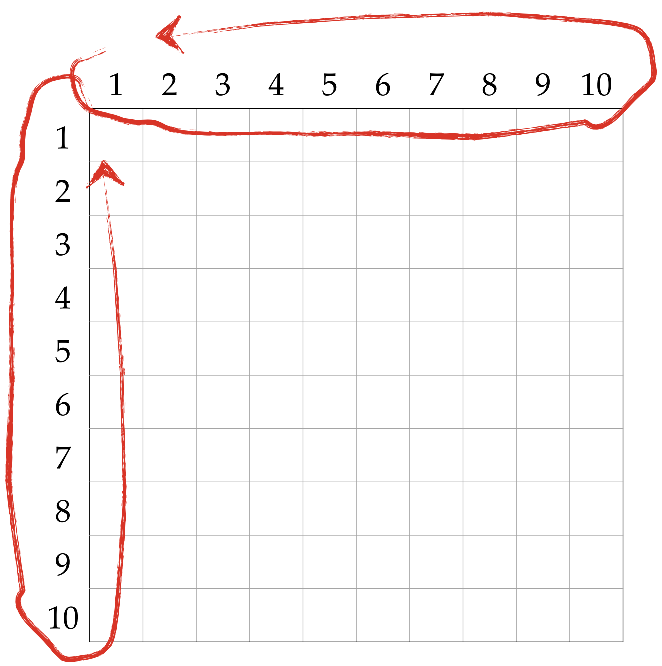

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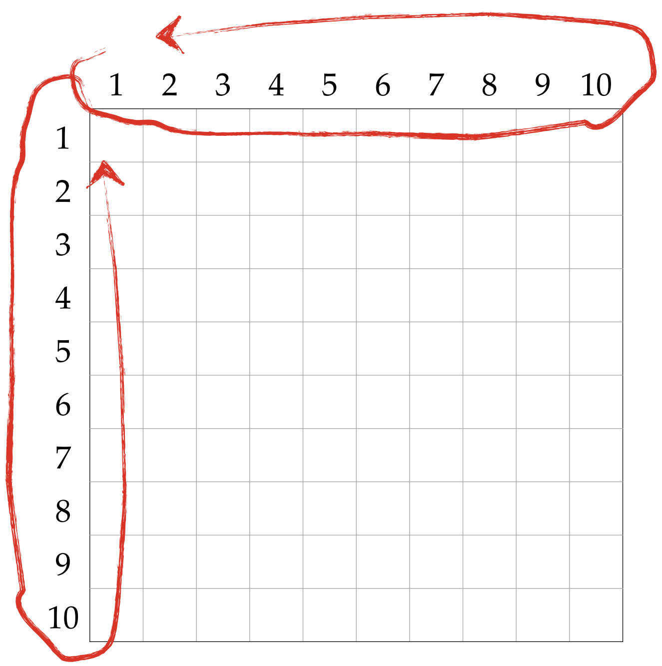

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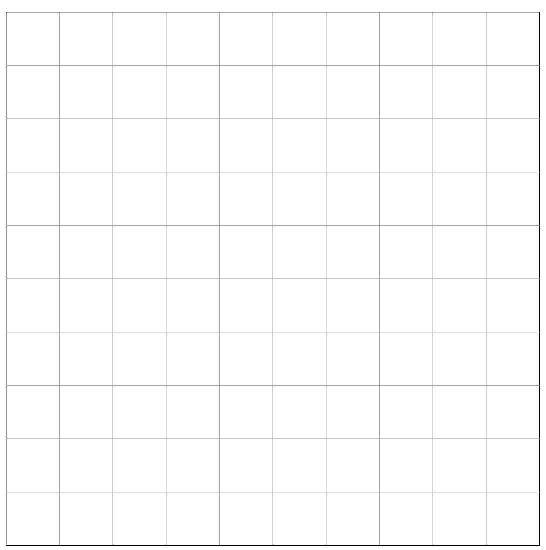
(rank, determinant, etc.)

Unordered matrices

I, J — finite and non-empty sets

D — a group, a ring or a field



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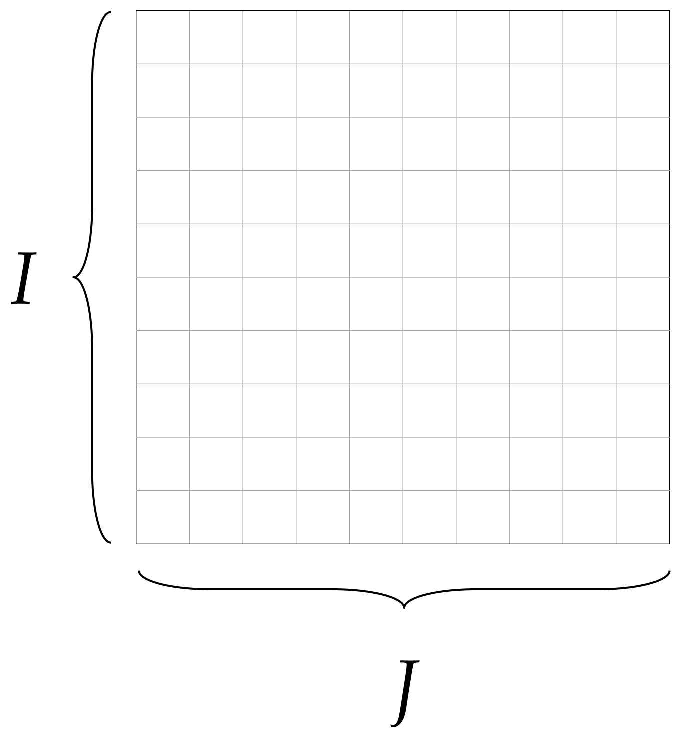

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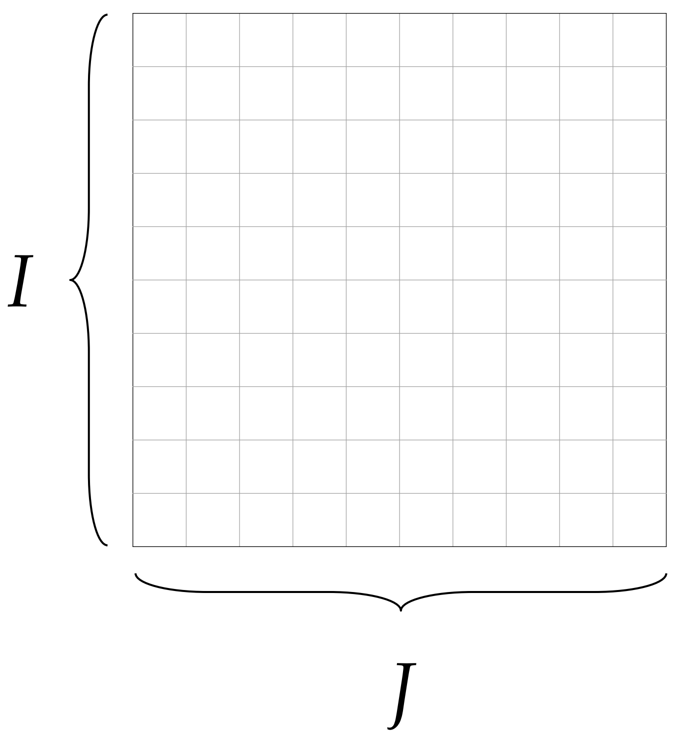

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"an *I*-by-*J* matrix over *D*"

Unordered systems of linear equations

I, J — finite and non-empty sets

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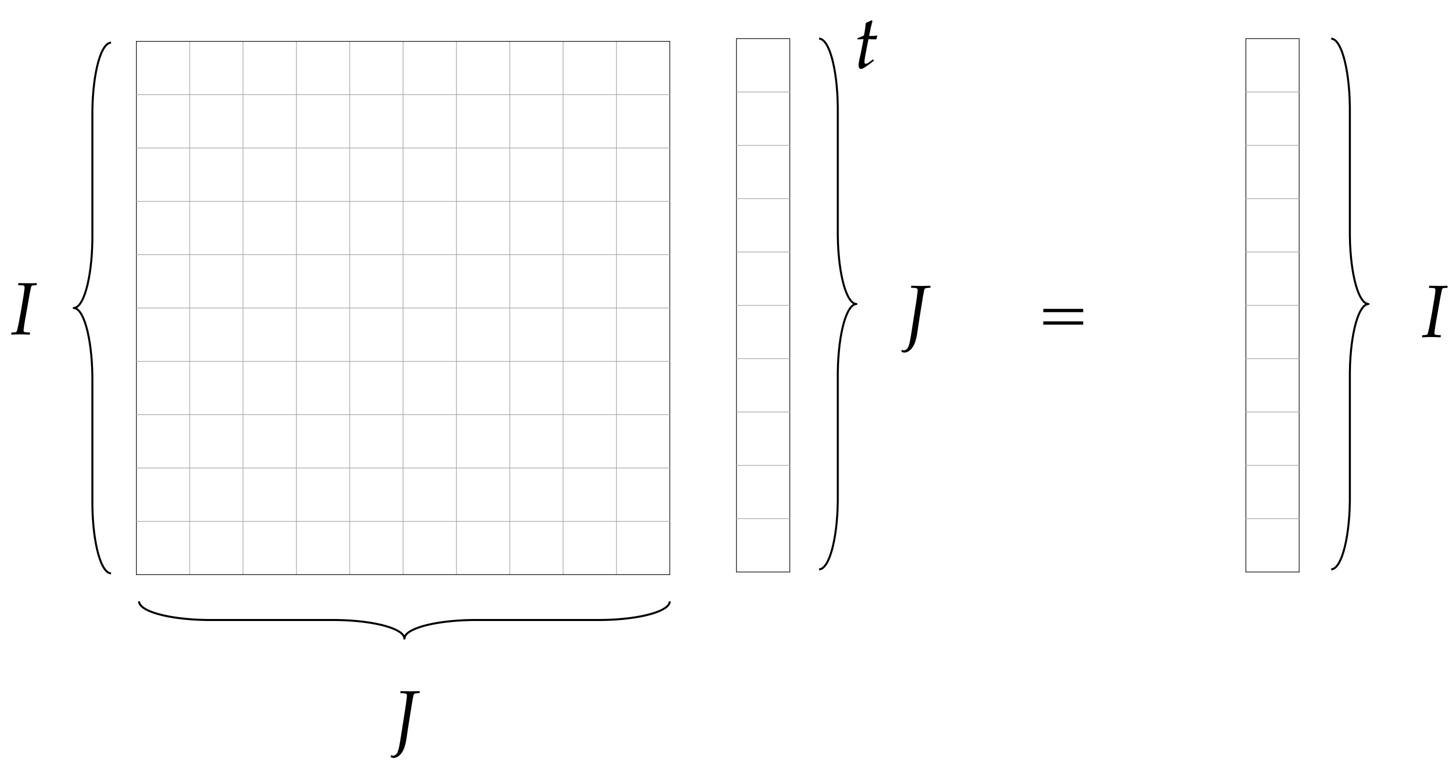

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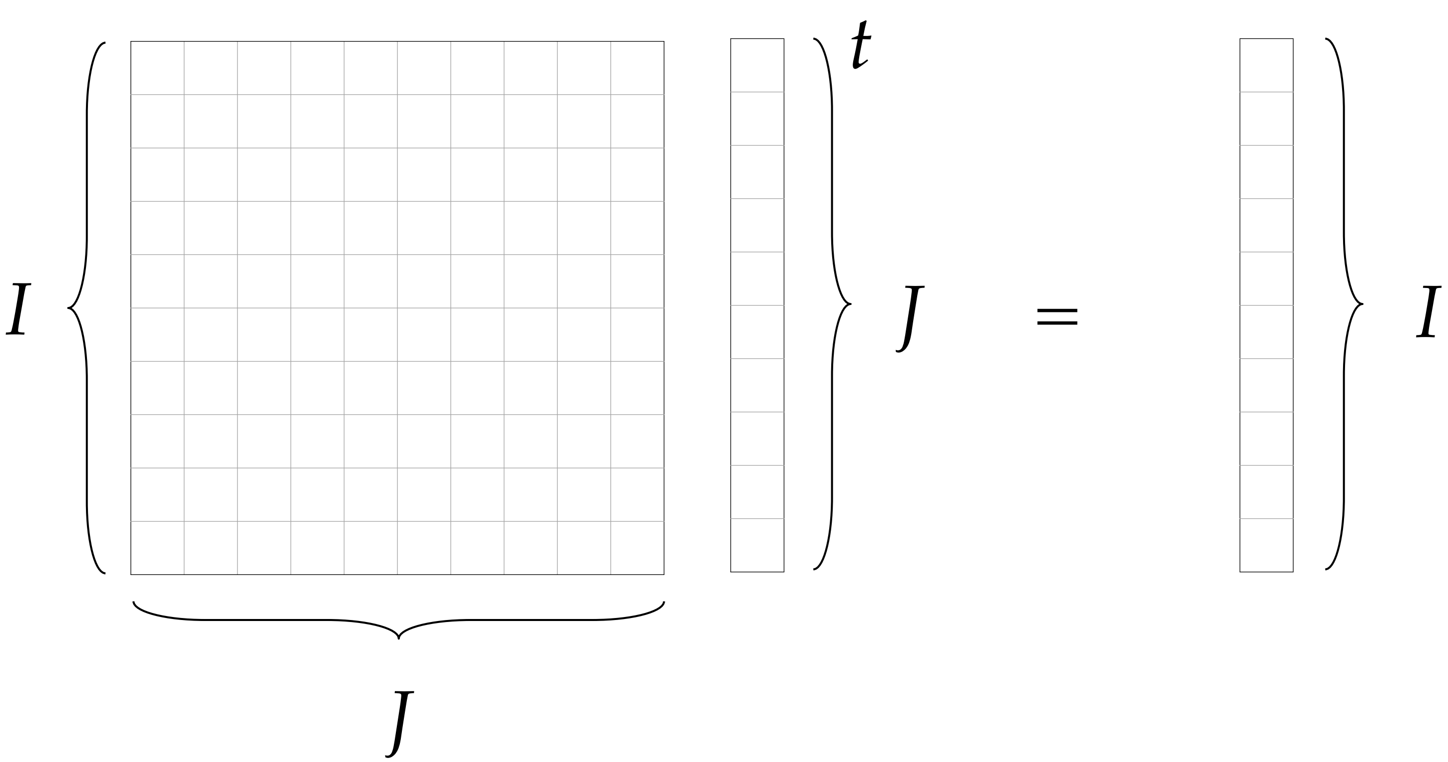
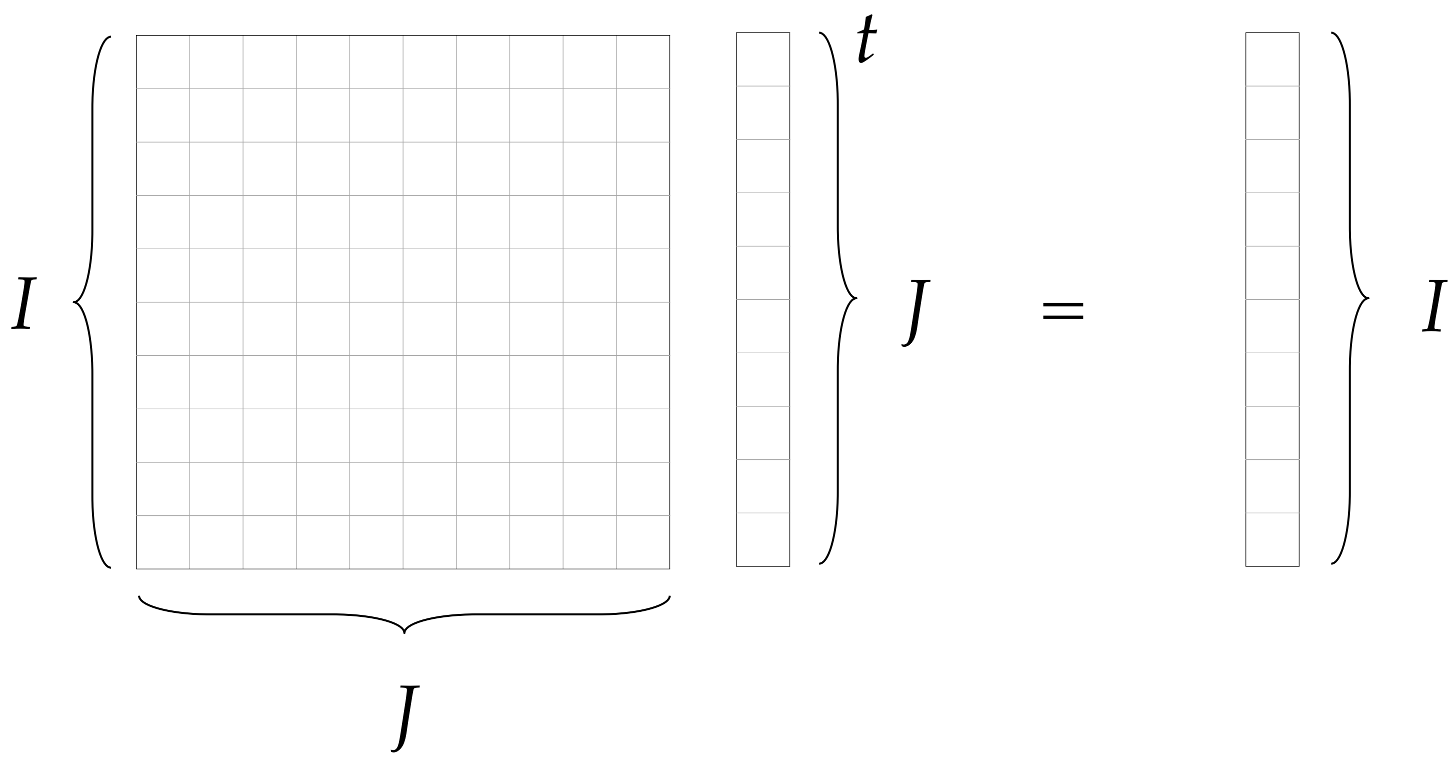


$$A\,\mathbf{x} = \mathbf{b}$$

Unordered systems of linear equations



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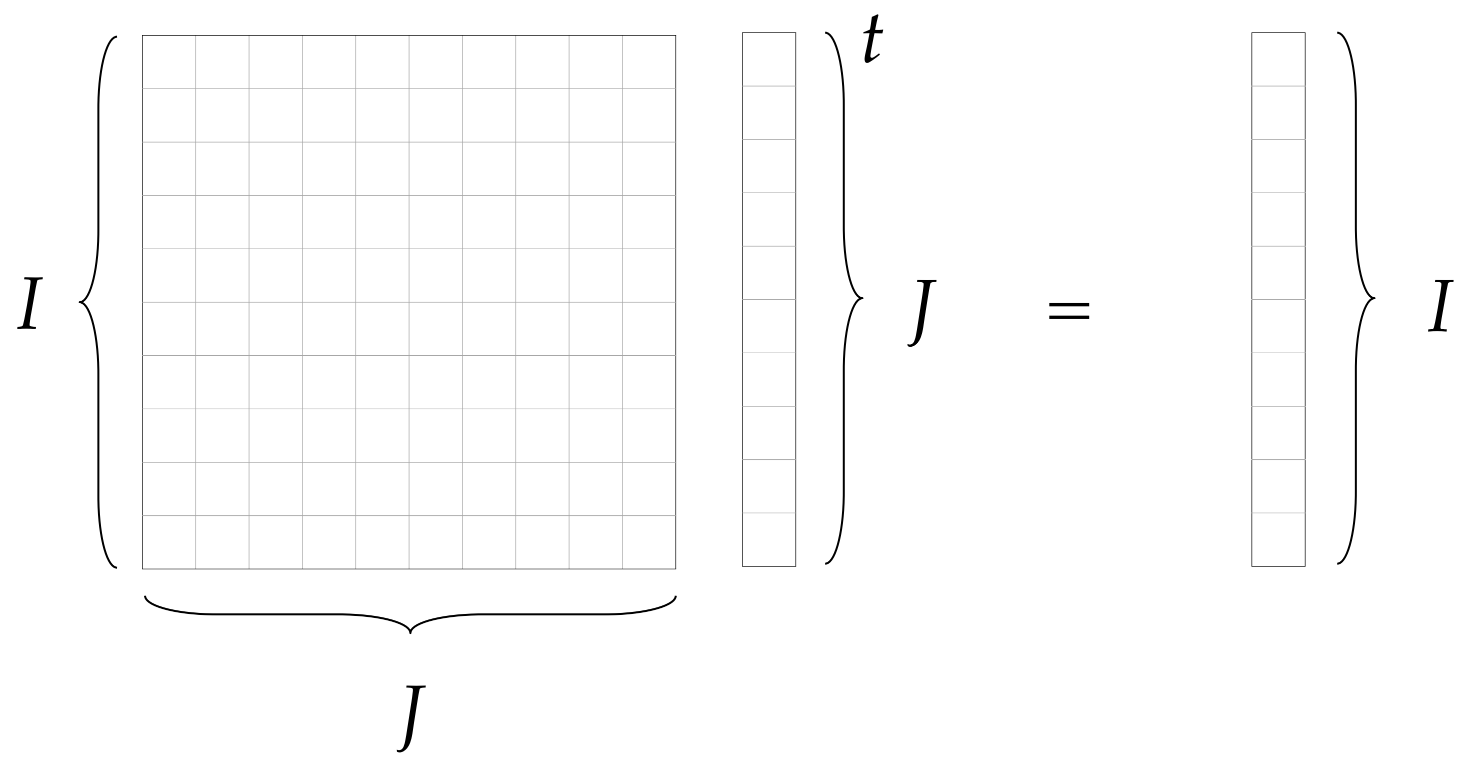

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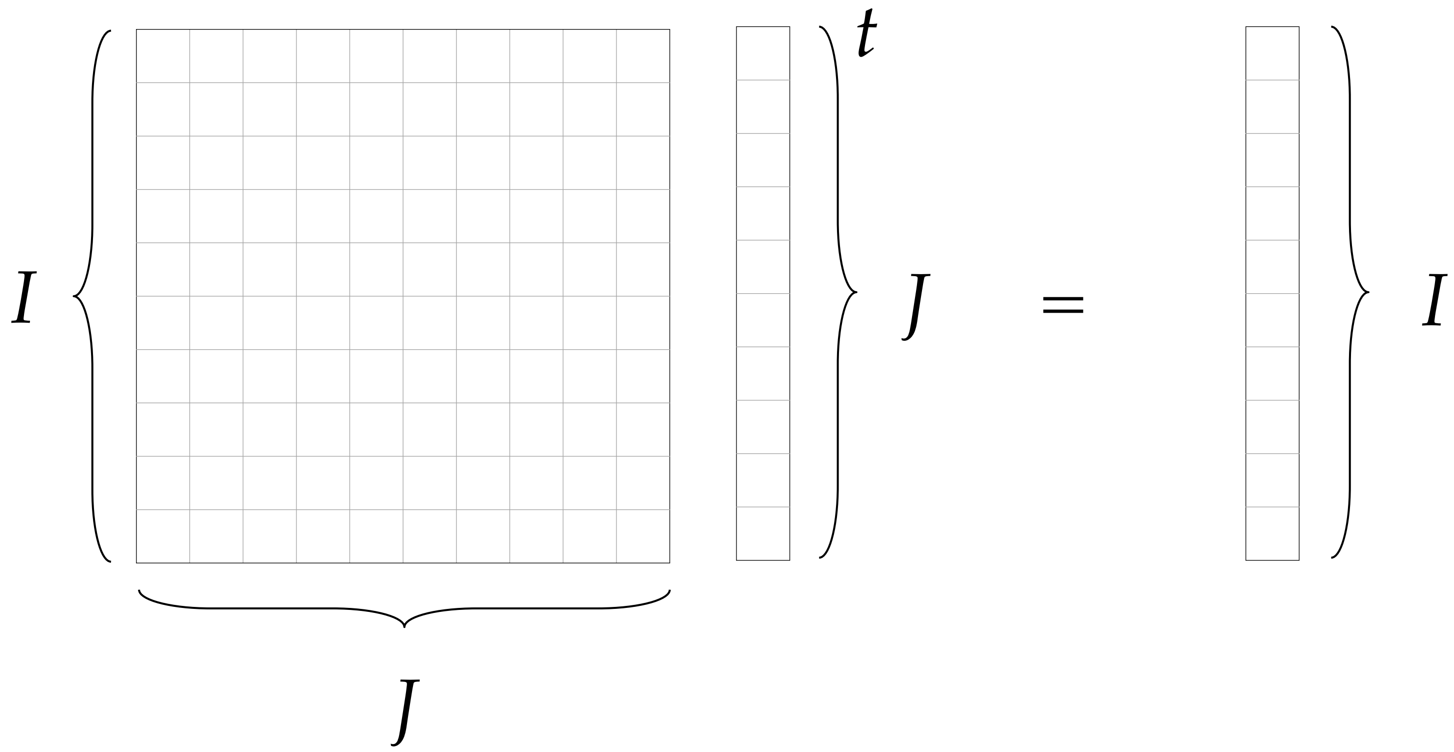

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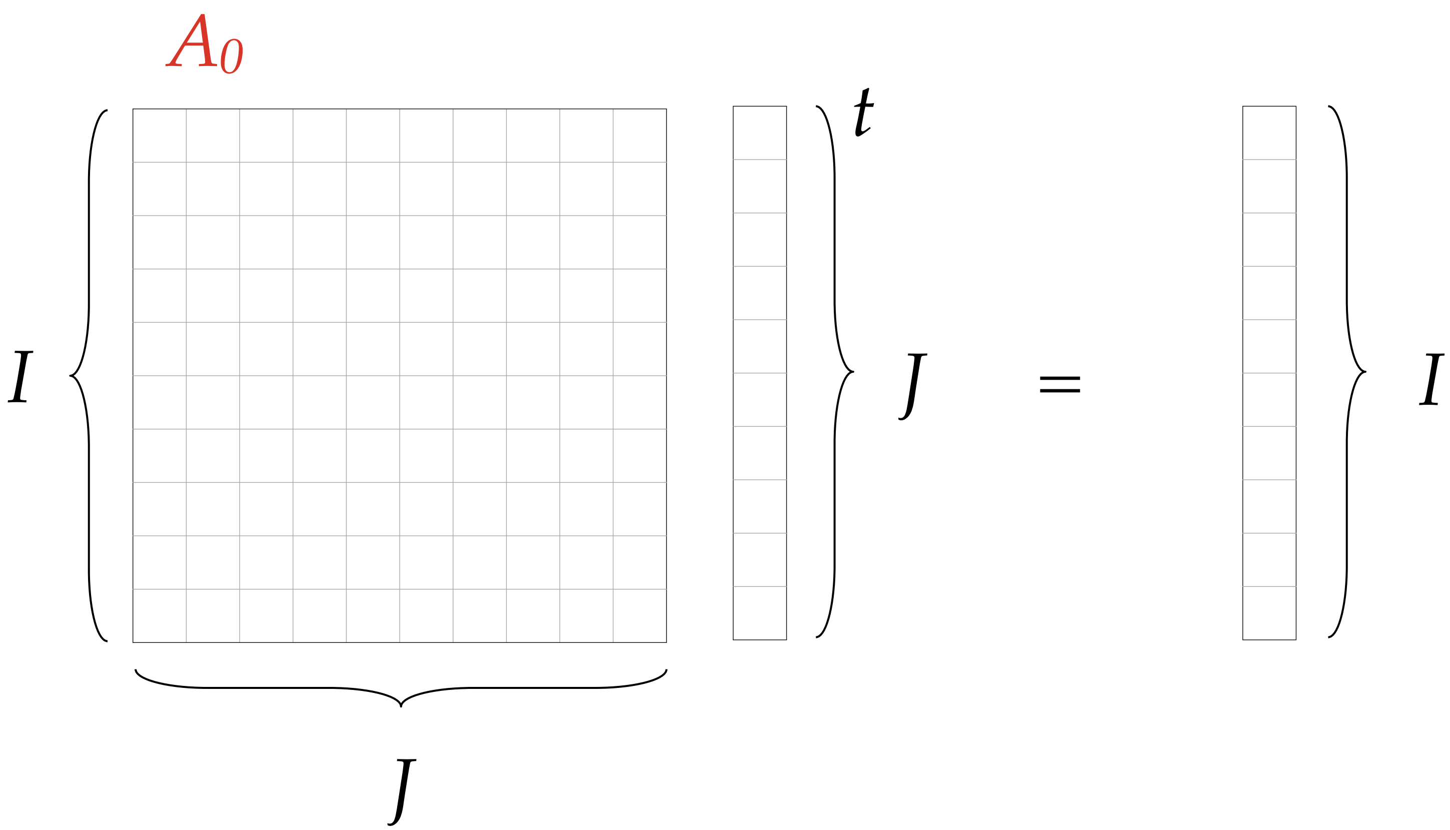

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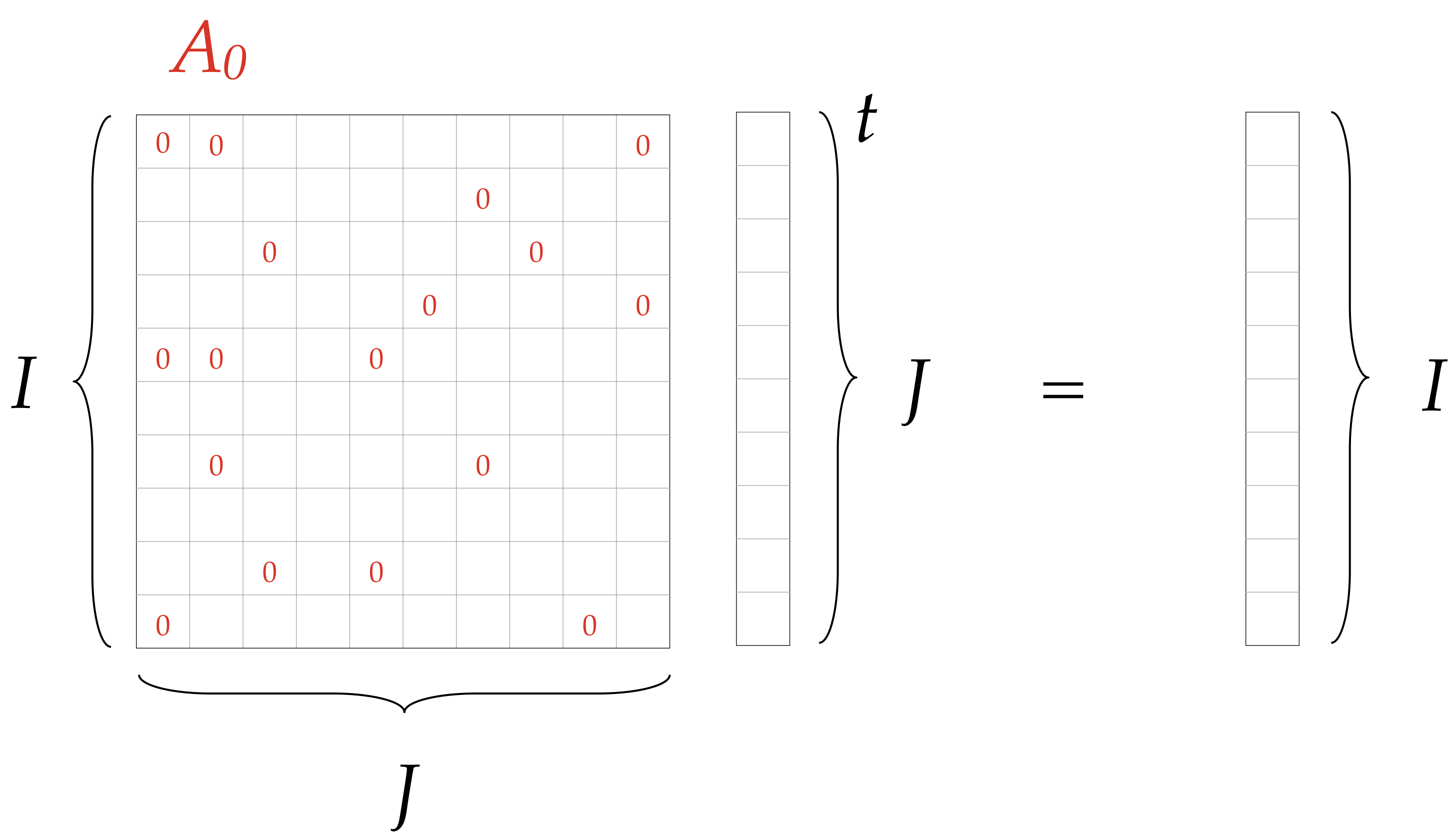

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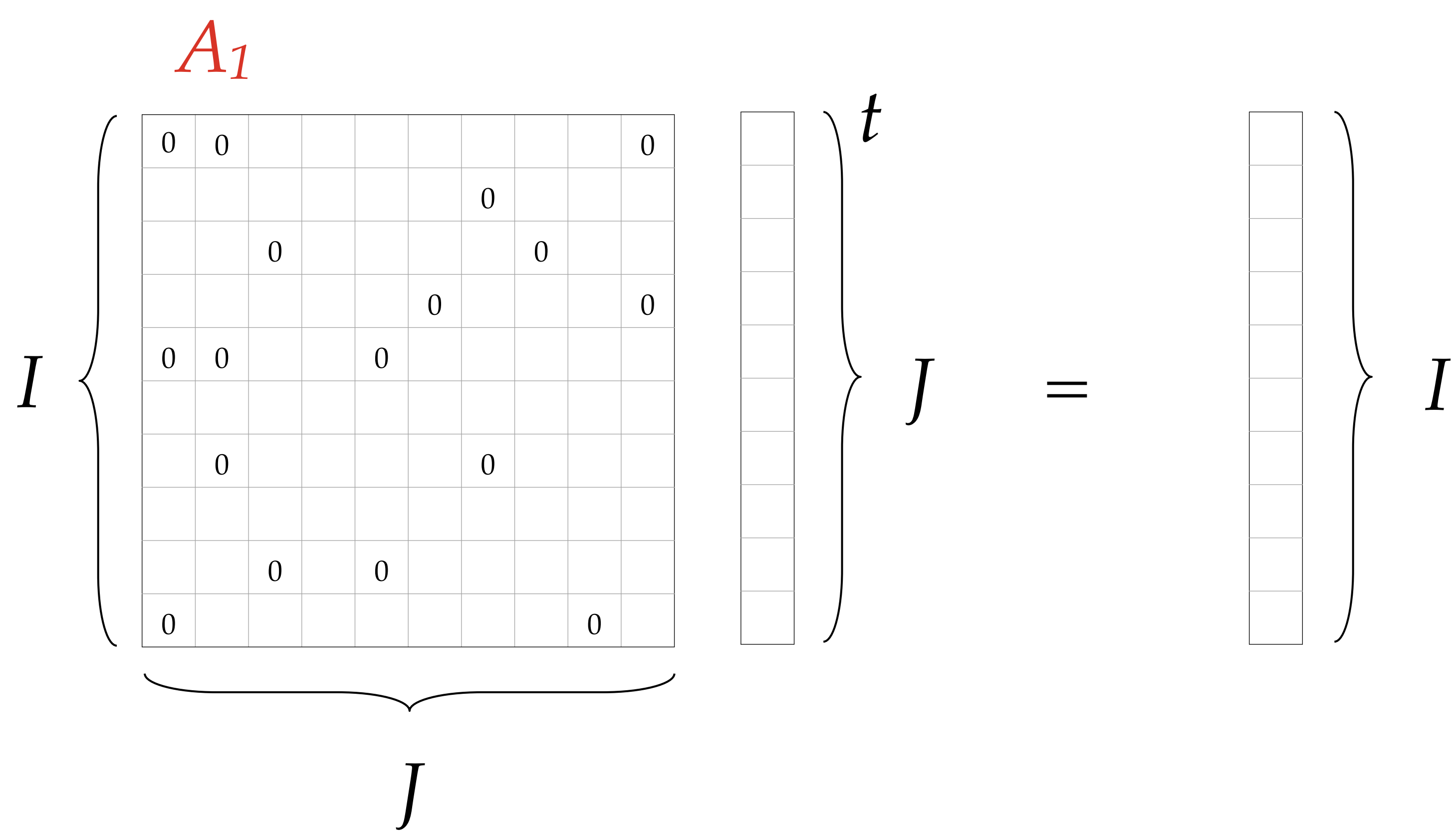

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$$A_1$$

$$I\left\{\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & & 1 & & & & & 0 \\ \hline & & & 1 & & & 0 & & & \\ \hline & & 0 & & 1 & & & 0 & & \\ \hline & & & 1 & 1 & 0 & & & & 0 \\ \hline 0 & 0 & & & 0 & & & & & \\ \hline & & & 1 & & 1 & & & & \\ \hline & 0 & & & & 1 & 0 & & & \\ \hline & & & 1 & & & & 1 & & \\ \hline & & 0 & & 0 & & 1 & 1 & & \\ \hline 0 & & & & & & & & 0 & \\ \hline \end{array}}_{J}\right. \quad \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right\}^{t} J \quad = \quad \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right\} I$$

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In this talk: Focus on $I = J$

FPC — more non-definability results

Solvability of systems of linear equations over any fixed finite Abelian group is not definable in FPC.

Atserias, Bulatov and Dawar (2007)

FPC — more non-definability results

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Matrix rank over finite fields is not definable in FPC.

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(Next talk: solvability problems over groups and rings)

Next step: extend fixed-point logic with ability to define matrix rank

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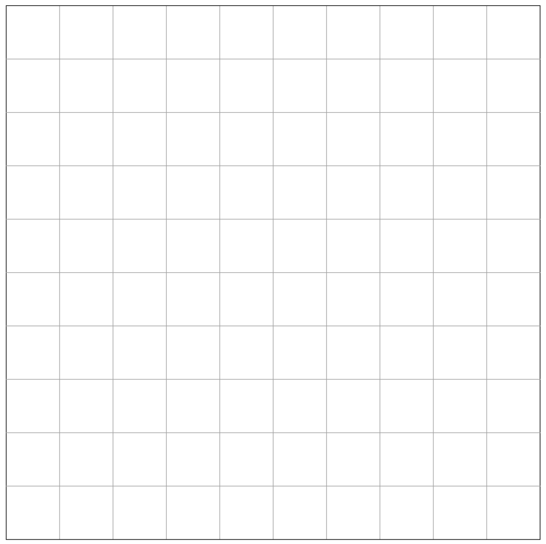

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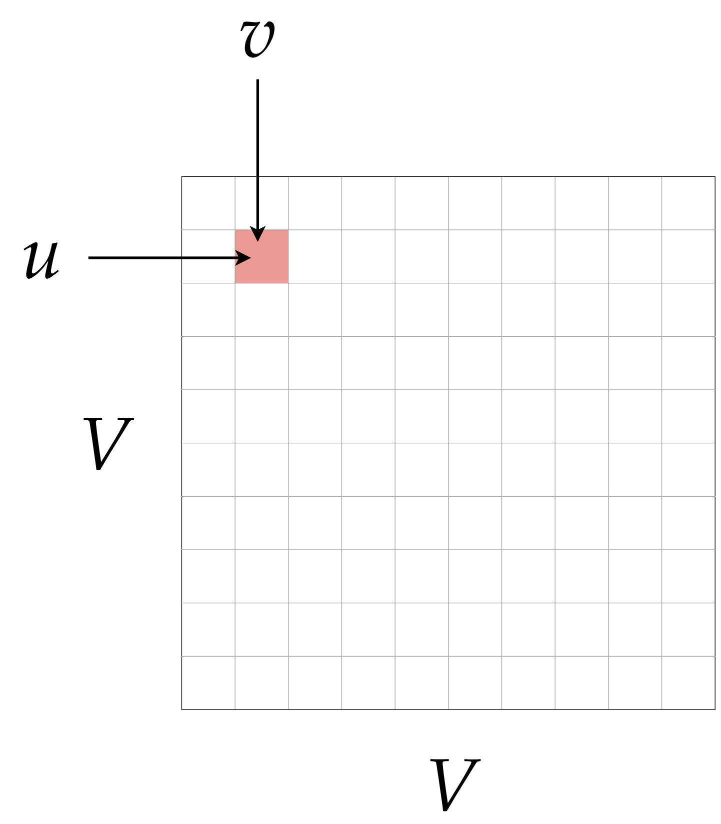

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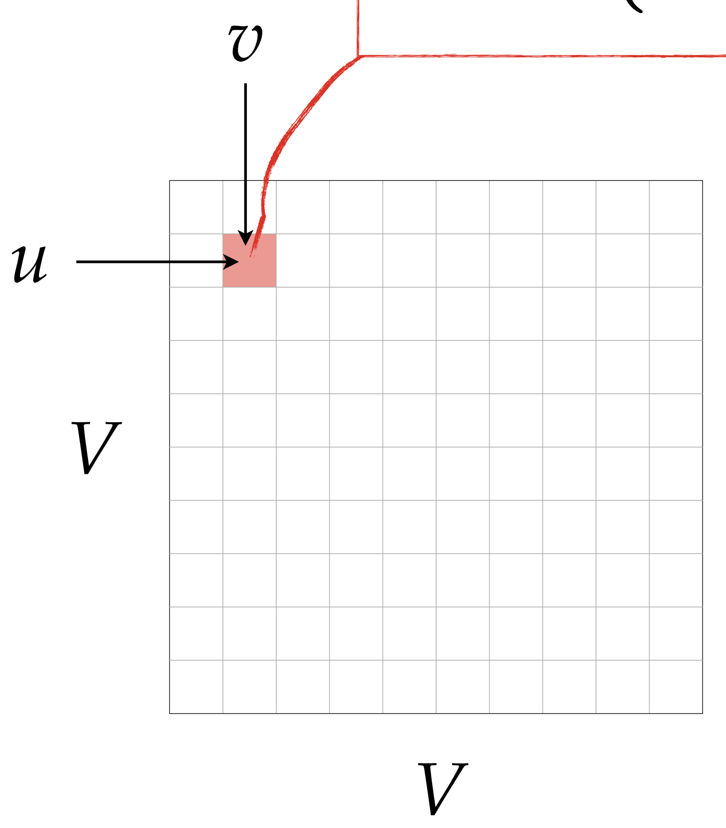

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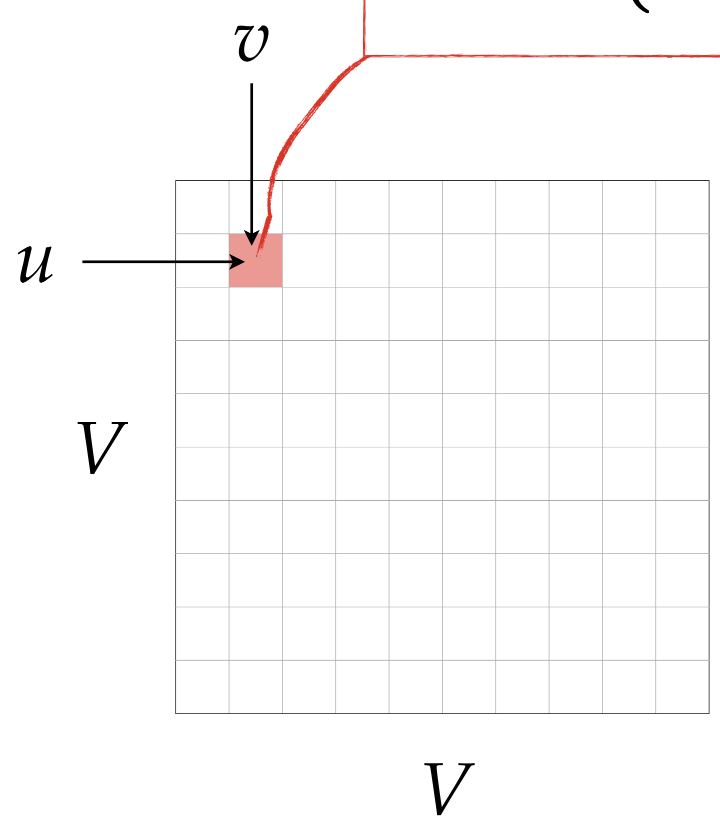

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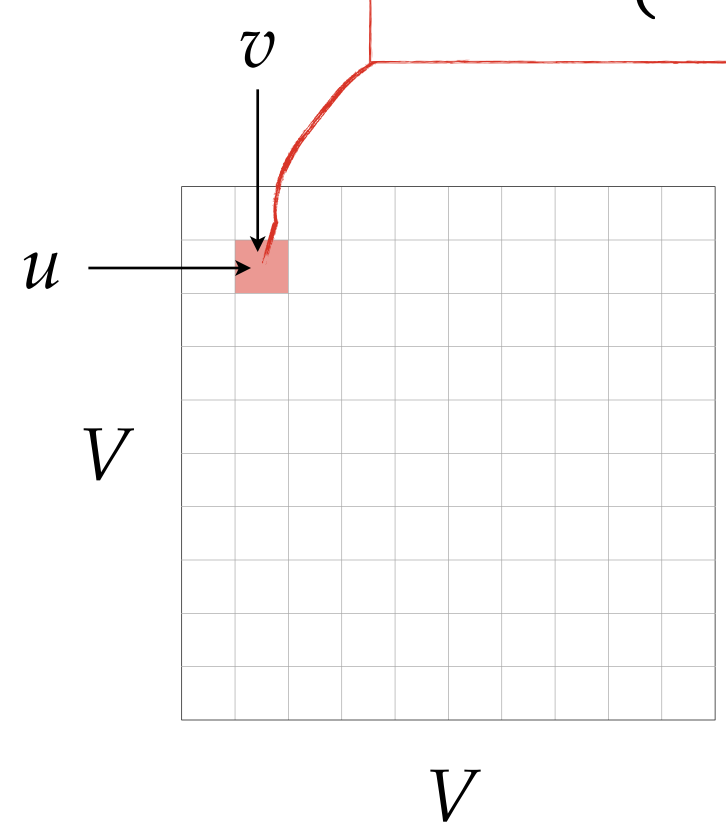

Example: $\varphi(x, y) := E(x, y) \rightsquigarrow M^G_\varphi$ = adjacency matrix of G

More generally: formalise matrices over GF(p), p prime

Fixed-point logic with rank operators

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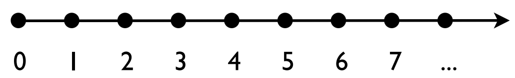

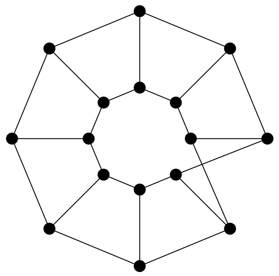

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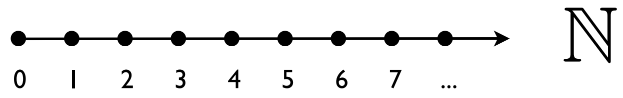

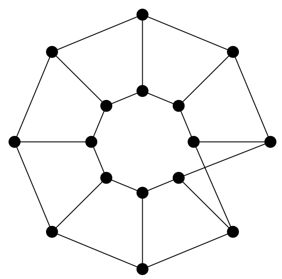

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vertex variables: range over the vertices V

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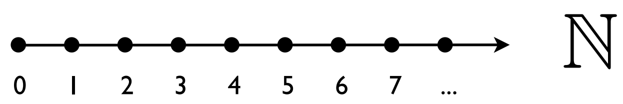

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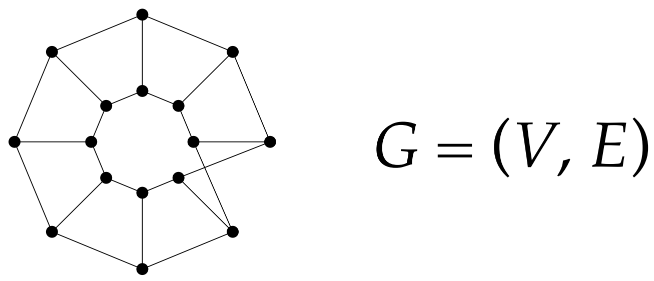

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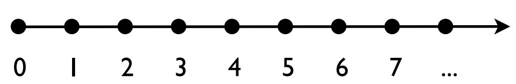

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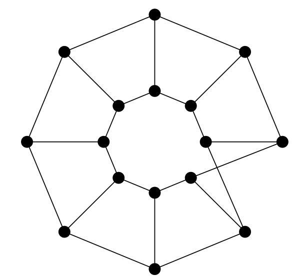

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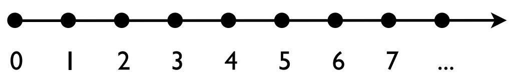

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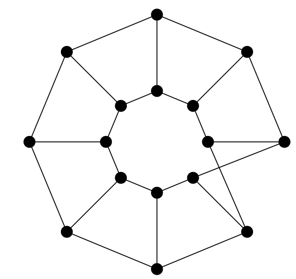

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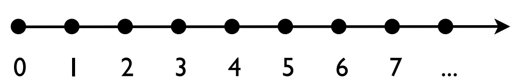

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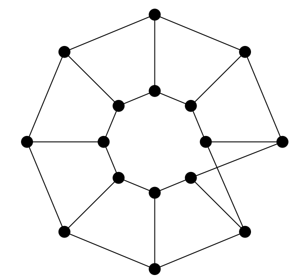

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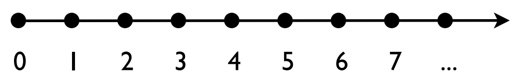

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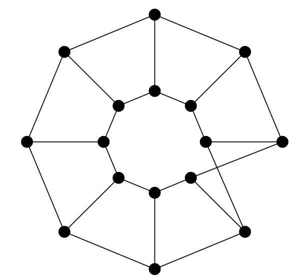

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$\longrightarrow$ Logics FPR$_p$, FPR and similarly FOR$_p$, FOR

Expressive power of rank logics

For any prime p, FPR_p can express solvability of linear equations over $\mathrm{GF}(p)$.

Dawar, Grohe, H., Laubner (2009)

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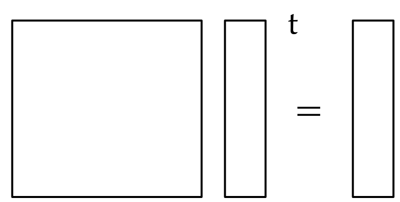

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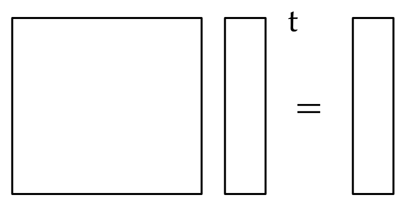

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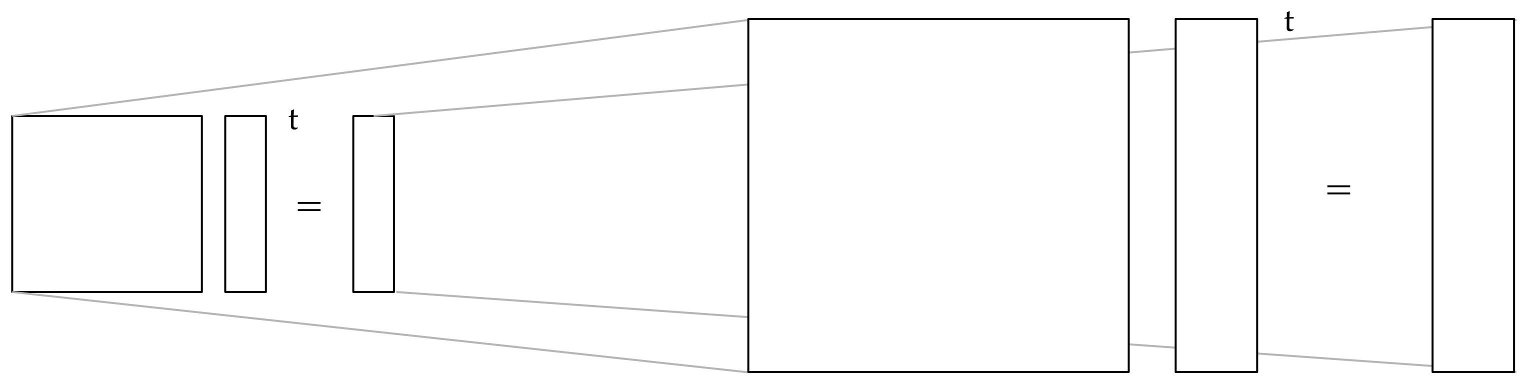

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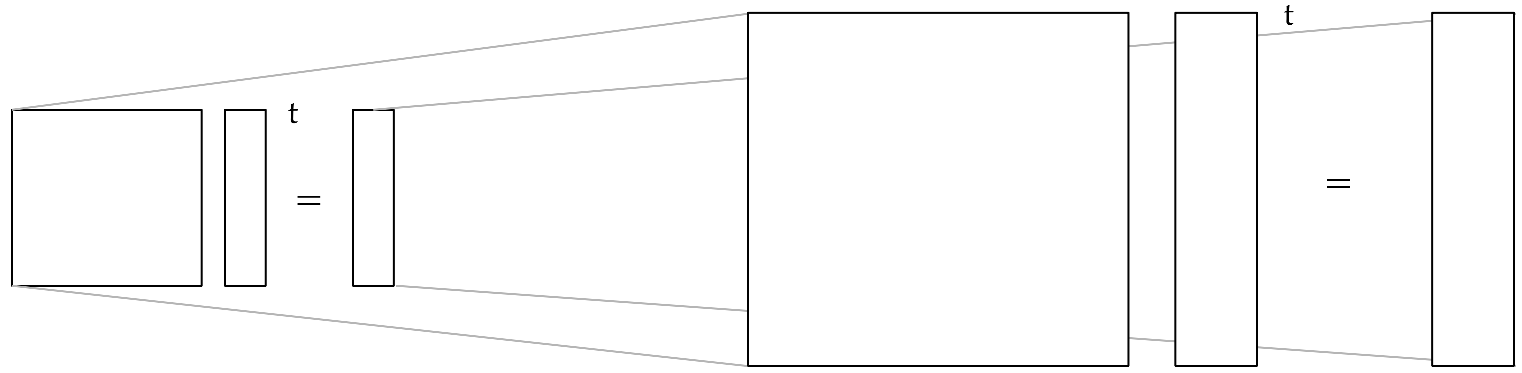

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Corollary

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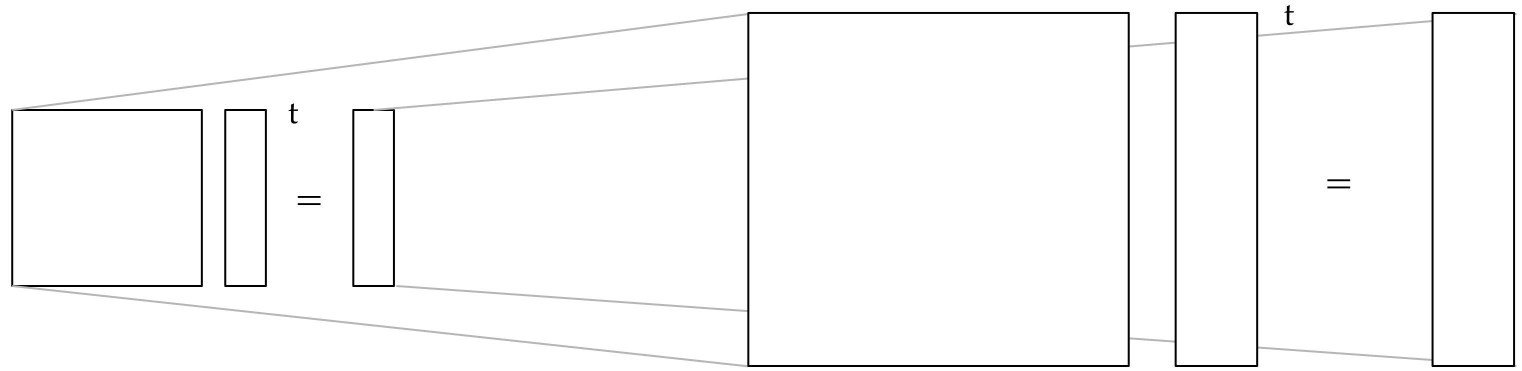

over $\mathrm{GF}(p^m)$ equivalent system over $\mathrm{GF}(p)$

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(we can simulate counting by
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Non-isomorphic CFI graphs can be distinguished by a sentence of FOR_2.

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Isomorphism of graphs of colour class size 4 can be expressed in FOR_2.

Dawar, H. (2011)

Pebble games for rank logics & the Weisfeiler-Lehman method

Proving non-definability in FPR_p

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G and *H* agree on all sentences of *k*-variable rank logic over GF(*p*)	iff	Duplicator has a winning strategy in the *k*-pebble matrix-rank game on *G* and *H*

Matrix-rank game over GF(p)

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Game played on finite graphs G and H

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Strengthening the game rules

Two tuples (A_1, A_2, ..., A_m) and (B_1, B_2, ..., B_m) of n-by-n matrices over a field k are simultaneously similar if there is an invertible S such that $S A_i S^{-1} = B_i$ for all i.

Strengthening the game rules

Two tuples $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_m)$ of n-by-n matrices over a field k are simultaneously similar if there is an invertible S such that $S A_i S^{-1} = B_i$ for all i.

There is a deterministic algorithm that, given two m-tuples $\mathbf{A}$ and $\mathbf{B}$ of n-by-n matrices over a finite field GF(q), determines in time poly(n, m, q) whether $\mathbf{A}$ and $\mathbf{B}$ are simultaneously similar.

Chistov, Karpinsky and Ivanyov (1997)

Game based on invertible linear maps

Invertible-map game on G and H over GF(p):

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Facts:

- We can decide who wins this game in PTIME.
- Refines R_p^k-equivalence: If Duplicator wins the k-pebble invertible-map game on G and H then she also wins the k-pebble matrix rank game on G and H.

Connection with stable colouring

Recall:

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matrix-rank game

invertible-map game

Weisfeiler-Lehman refinement

Input: Graph $G = (V, E)$

Output: Equivalence relation $\approx$ on V.

Weisfeiler-Lehman refinement

"colour refinement"
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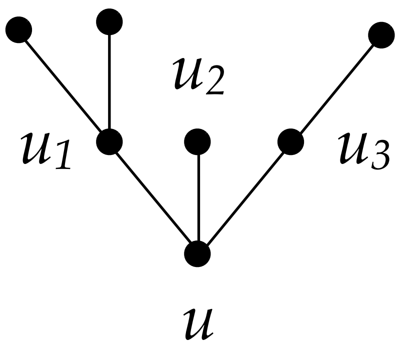
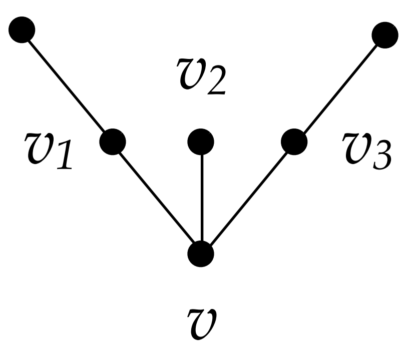
Inductively define: $\sim_0 \supseteq \sim_1 \supseteq \ \dots \ \supseteq \sim_m = \sim_{m+1} =: \approx$





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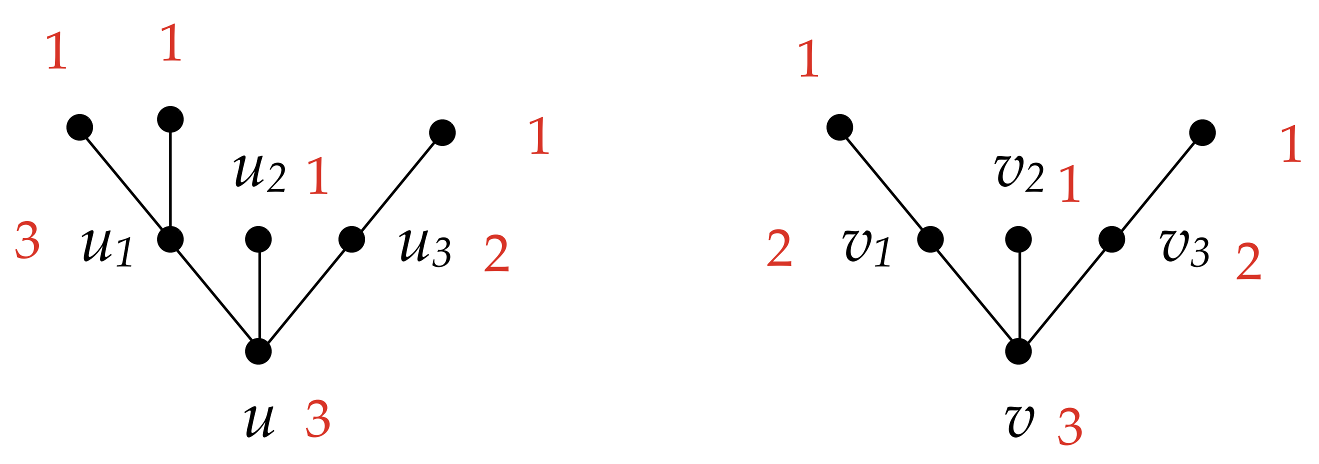

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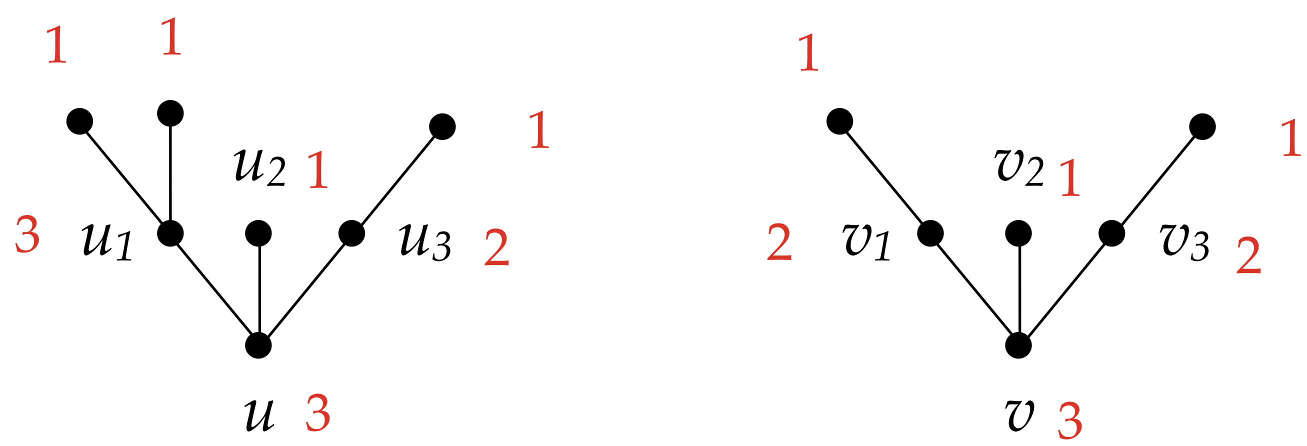

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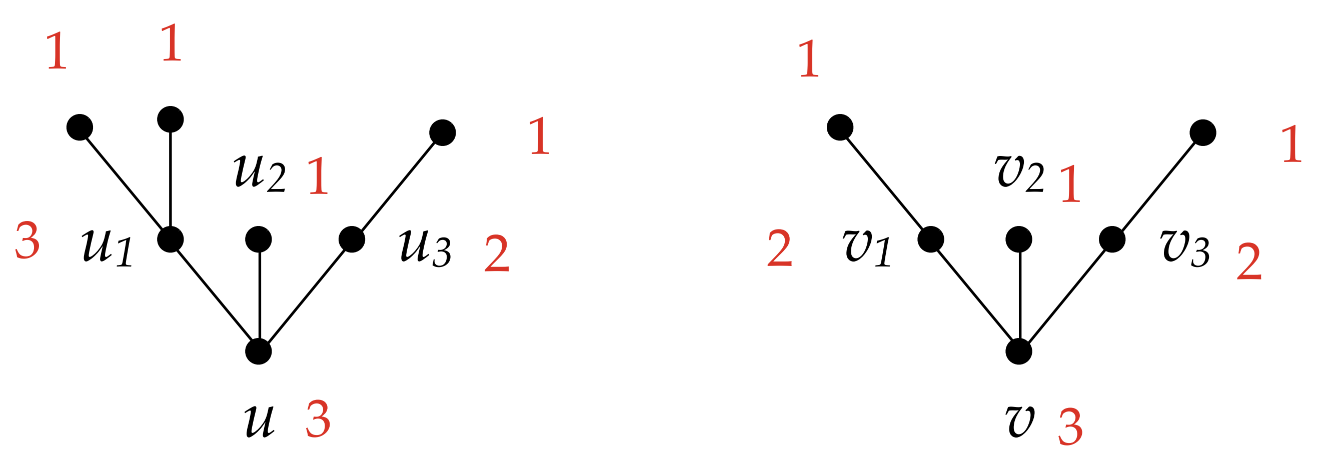

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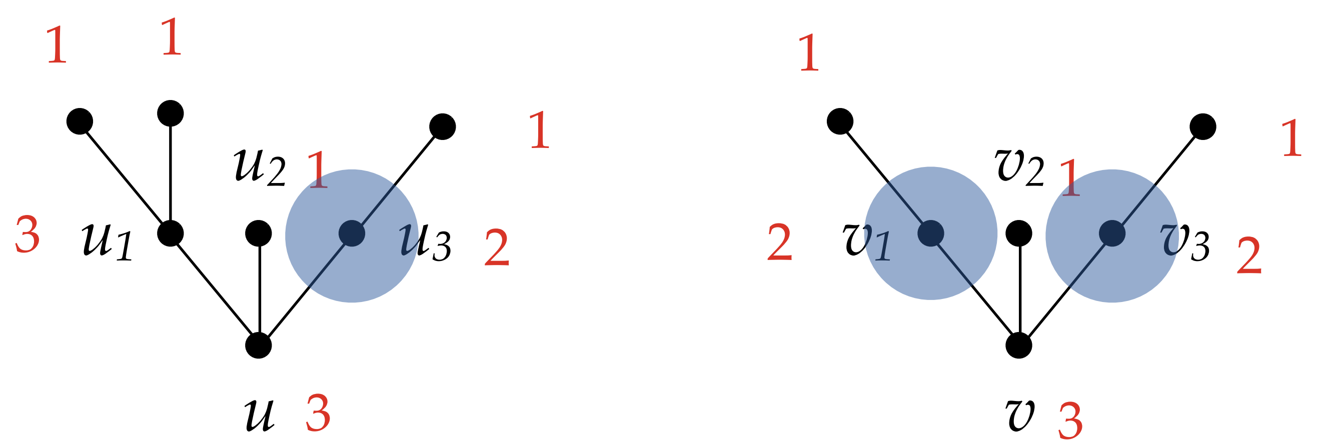

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k-dimensional WL* refinement

One-element extensions in $G = (V, E)$

For $\alpha \subseteq V^k$, a k-tuple $\vec{u} \in V^k$ and $0 \leq i < k$, let:

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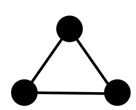
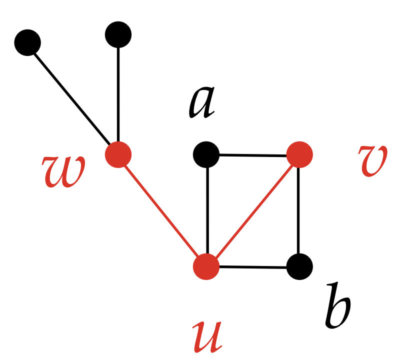
Example: Let $k = 3$ and $\alpha := \{(x, y, z) \in V^3 \mid (x, y, z) =$  $\}$



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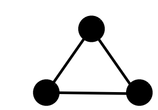
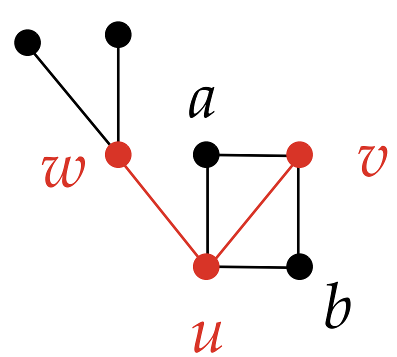
Example: Let $k = 3$ and $\alpha := \{(x, y, z) \in V^3 \mid (x, y, z) =$  $\}$



$$\Gamma_0(uvw, \alpha) = \{a, b\}$$
$$\Gamma_1(uvw, \alpha) = \emptyset$$

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Theorem: $\vec{u} \approx \vec{v}$ iff they agree on all C^k-formulas in G.

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As before: compute k-dimensional WL* refinement and compare across the two graphs.

PTIME for fixed k: k-dim WL* runs in time $O(n^{k+1}\log(n))$.

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There exists a sequence of pairs $\{(G_n, H_n)\}_n$ of non-isomorphic graphs for which it holds that:

- G_n and H_n have O(n) vertices but
- G_n and H_n are not distinguished by the n-dim WL* algorithm.

Cai, Fürer and Immerman (1992)

Refinement by invertible linear maps

Two-element extensions in $G = (V, E)$

For $\alpha \subseteq V^k$, a k-tuple $\vec{u} \in V^k$ and $0 \leq i \neq j < k$, let:

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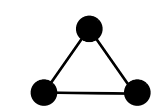
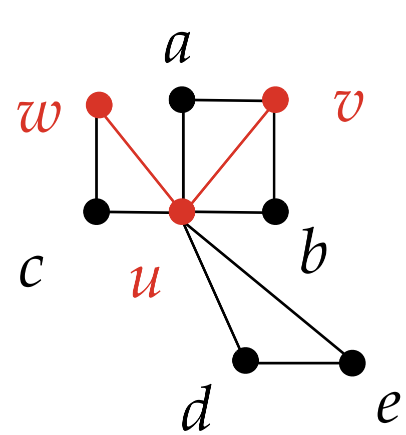
Example: Let $k = 3$ and $\alpha := \{(x,y,z) \in V^3 \mid (x,y,z) =$  $\}$



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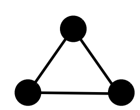
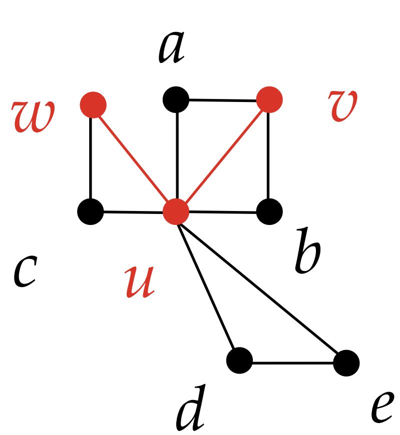
Example: Let $k = 3$ and $\alpha := \{(x, y, z) \in V^3 \mid (x, y, z) =$  $\}$



Γ_{12}:

	u	v	w	a	b	c	d	e
u								
v				1	1			
w						1		
a		1						
b		1						
c			1					
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Dawar and H. (2012)

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For each k and distinct primes p and q, there is a pair of non-isomorphic graphs that can be distinguished by 3-dim IM_p but not by k-dim IM_q.

H. (2010)

k-dimensional IM_p more generally

Consider the invertible-map algorithm for larger matrices (higher arity) and finite sets of primes.

Can we give instances where the general algorithm fails to express graph isomorphism?

Some open problems

Problem 1: Separate FOR_p and FOR_q over empty signatures

For formula $\varphi(x, y)$, integer n and prime p, let $r^p_\varphi(n)$ denote the GF(p)-rank of the matrix defined by $\varphi(x, y)$ over an n-element set.

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Polynomial-rank conjecture

For each $\varphi(x, y)$ and each prime p, there are unary polynomials $f_0, \ldots, f_{p\text{-}1}$ such that $r^p_\varphi(n) = f_i(n)$ for all (sufficiently large) n congruent to i modulo p.

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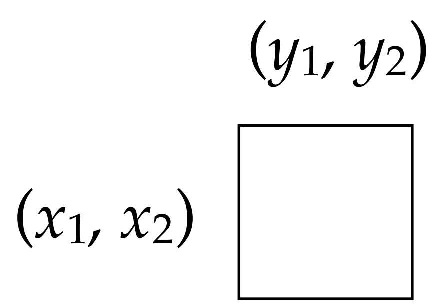

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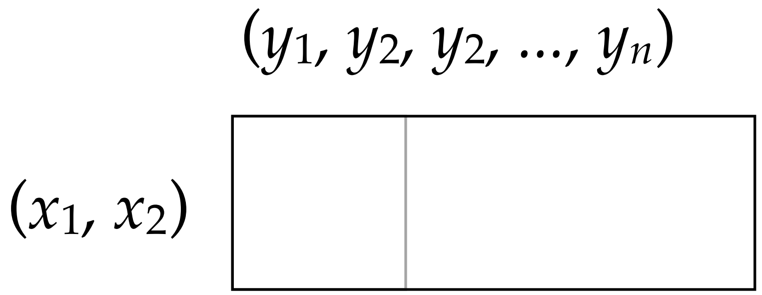

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???

$(y_1, y_2, y_2, \ldots, y_n)$

$(x_1, x_2, \ldots, x_n)$

H. and Laubner (2010)

Kirsten (2012)

Problem 2: Give capturing results for FPR on natural classes of graphs

Consider classes on which we know that FPC does *not* capture PTIME:

- graphs of bounded degree
- graphs of bounded colour-class size

Further questions

- Can FPR express matching in arbitrary graphs?
- Does the “simultaneous similarity game” correspond to a natural logic?

More open problems to come in the next talk!