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A Note on Russell's Paradox in Locally Cartesian Closed Categories

Abstract. Working in the fragment of Martin-Löf's extensional type theory [12] which has products (but not sums) of dependent types, we consider two additional assumptions: firstly, that there are (strong) equality types; and secondly, that there is a type which is *universal* in the sense that terms of that type name all types, up to isomorphism. For such a type theory, we give a version of Russell's paradox showing that each type possesses a closed term and (hence) that all terms of each type are provably equal. We consider the kind of category theoretic structure which corresponds to this kind of type theory and obtain a categorical version of the paradox. A special case of this result is the degeneracy of a locally cartesian closed category with a morphism which is *generic* in the sense that every other morphism in the category can be obtained from it via pullback.

Introduction

At the heart of the categorical approach to logic lies the ability to identify certain kinds of logical theory with particular kinds of category-theoretic structure. Indeed, the search for such logic-category correspondences has formed a major part of the subject to date. Once achieved, such an identification has its uses in both directions: sometimes algebraic techniques can be applied to the category-theoretic structures to yield results about the logical theoreis; at other times proof-theoretic or model-theoretic results for the logic yield new properties of the categories involved. This note illustrates in a minor way this latter aspect of the subject. Consider the following category-theoretic result:

Proposition. *Let* $\mathbf{C}$ *be a locally cartesian closed category (that is,* $\mathbf{C}$ *has finite limits and for each object* X *in* $\mathbf{C}$*, the slice category* $\mathbf{C}/X$ *is cartesian closed). Suppose further that* $\mathbf{C}$ *contains a generic family, which by definition is a morphism* $t\colon G\to U$ *with the property that any other morphism* $f\colon A\to X$ *is a pullback of* t *along some morphism* $X\to U$*. Then it is the case that* $\mathbf{C}$ *is degenerate: every object in* $\mathbf{C}$ *is isomorphic to the terminal object.*

We will give a proof of this proposition which relies upon the correspondence established by Seely [14] between locally cartesian closed categories and theories over a (strong) version of Martin-Löf's system [12] of dependent types, with sums, products and equality types. Recall that under this correspondence the dependent types $A[x][x\in X]$ of a theory are modelled by morphisms $A\to X$ in the category and substitution is modelled by (chosen) pullback

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operations (which we assume given when specifying the structure of a locally cartesian closed category). Then the assumption that $\boldsymbol{C}$ has a generic family t: $G \to U$ translates into the assumption that the type theory possesses a *universal type* (or "type of all types") in the following sense:

There is a constant type *Type* whose terms are intended to *name* all the types, up to isomorphism; *Type* is modelled by the object U in $\boldsymbol{C}$. There is a dependent type $T[u]$ giving the actual type named by $u \in Type$; this is modelled by t: $G \to U$ in $\boldsymbol{C}$. Whenever A is a type derivable in the theory, possibly dependent on some variables $\bar{x}$ say (and modelled by $A \to X$ in $\boldsymbol{C}$), a constant $n_A(\bar{x}) \in Type$ is introduced along with function constants and axioms to make $A[\bar{x}]$ isomorphic to $T[n_A(\bar{x})]$ (rather than actually equal to it). Since the result of substituting $n_A(\bar{x})$ for u in $T[u]$ is modelled in $\boldsymbol{C}$ by forming the pullback of t: $G \to U$ along the morphism f: $X \to U$ modelling $n_A(\bar{x})$, the provable isomorphism $A[\bar{x}] \cong T[n_A(\bar{x})]$ in the type theory means that in $\boldsymbol{C}$ the morphism $A \to X$ is isomorphic to the (chosen) pullback of t along f (and hence is *a* pullback of t along f).

The proof of the proposition proceeds by giving a type-theoretic version of Russell's paradox. In this version, the roles of equality and universally quantified predicates are played as usual by equality and product types. The role of the powerset operation is played by the function type $(-) \to Type$; and the role of negation is played by the function type $(-) \to A$, where A is a fixed, but arbitrary type. In this form, the paradox produces (via an application of Cantor's familiar diagonal argument) a closed term of type A. Since A was arbitrary, we conclude that all types possess closed terms; since in particular this can be applied to equality types, we conclude further that all terms of any particular type are provably equal. Back in the locally cartesian closed category, this means that all objects possess a global section and that all parallel pairs of morphisms are equal – from which degeneracy follows immediately.

The type-theoretic version of the paradox we present does not use *dependent sums*. (A slightly simpler argument could be given using sums.[2]) As a result, we in fact prove a sharper result (Proposition 1.5) of which the above proposition is a special case.

The use we make of Russell's paradox in this paper should be contrasted with Girard's proof [6] of the inconsistency of Martin-Löf's original theory of types with a universal type, which used a type theory version of the Burali-Forti paradox. Recently there has been renewed interest from the computer science community in type theories with a universal type, but without equality types: see [2], [13] and [3]. Coquand [4] and Howe [7] analyse Girard's version of the paradox and show in particular that it can be carried out in type theory with just dependent products and a universal type. So such a type theory is "inconsistent" in the sense that every type possesses

[2] *Added in proof*: such an argument has been given independently by Boom [1]. We are grateful to A. S. Troelstra for bringing the existence of this preprint to our attention.

a closed term; but since there are no equality types, it does not follow as above that all terms are provably equal. Indeed there are highly non-trivial models of this kind of type theory: see [3], [9] and [15] for example.

Our version of Russell's paradox gives an inconsistency proof considerably less complex than Girard's paradox, but the argument makes use of equality types satisfying strong rules (modelled categorically by *diagonal* morphisms $\Delta = \langle id, id \rangle \colon A \to A \times_X A$). It would be interesting to know whether a version of the argument can be given which does not use these strong equality types, for the following reason. Currently it is not known whether a fixpoint combinator is definable in type theory with just dependent products and a universal type. Howe [7] establishes that the proof of Girard's paradox can be made to yield a term which is a "looping" combinator, but definitely not a fixpoint combinator. Now the paradox we give here produces a closed term of an arbitrary type A by establishing the stronger property that $B \cong (B \to A)$ for some B constructed from A. As we recall below (in 2.1), such an isomorphism implies that A has a fixed point property (so that in particular the fixed point of the identity on A yields a closed term of type A). Thus if such a version of Russell's paradox were possible in the fragment without strong equality types, it would yield a fixpoint combinator for that fragment.

1. Types and categories

We will be working with the fragment of Martin-Löf's extensional theory of types [12] that concerns product and equality types. We refer the reader to Troelstra [16], who provides a useful formalization of Martin-Löf's systems; for convenience we will adopt most of the notation used there. Thus judgements-in-context will be written

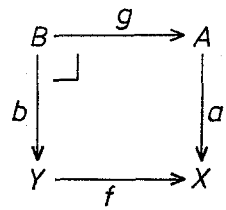
$$\Gamma \Rightarrow \Theta,$$

where Γ is a context and Θ is of the form "A type", "$s \in A$", "$A = B$", or "$s = t \in A$". However, the equality type for $s, s' \in A$ will be denoted by $I_A[s, s']$ (rather than by $I(A, s, t)$); and the application of a term $s \in \prod x \in A.B$ of product type to a term $t \in A$ will be denoted simply by the juxtaposition st (rather than by $\mathrm{Ap}(s, t)$).

The categorical semantics we use for these theories is based upon the approach developed in [15], which in turn is a refinement of the kind of semantics presented in [14] where types $B[x]$ depending on $x \in A$ are interpreted as morphisms $[\![B[x] \mid x \in A]\!] \to [\![A]\!]$ in a category. For the kind of type theory considered by Seely in [14], the categories are locally cartesian closed and *any* morphism can appear as the interpretation of a type. For other (simpler) type theories, the morphisms interpreting types belong to a distinguished class with given properties. We refer the reader to [9, Section 2] for a discussion of this kind of semantics and its relation to similar approaches (such as that of Cartmell [5]).

1.1. Product types. As explained in [9, Section 2], systems of dependent types closed only under the formation of products can be modelled by a category $\boldsymbol{C}$ with finite products together with a collection $\boldsymbol{A}$ of morphisms in $\boldsymbol{C}$ satisfying the following two conditions (where for each $X \in \boldsymbol{C}$, $\boldsymbol{A}(X)$ denotes the full subcategory of the slice category $\boldsymbol{C}/X$ whose objects are the morphisms in $\boldsymbol{A}$ with codomain X):

(1) *If* $a\colon A \to X$ *is in* $\boldsymbol{A}$, $f\colon Y \to X$ *is any morphism in* $\boldsymbol{C}$ *and*



is a pullback square, then b *is also in* $\boldsymbol{A}$. *Moreover, there is a pullback square for each such* a *and* f*: we will denote by* $f^*(a)\colon Y \times_X A \to Y$ *the result of pulling* a *back along* f.

(2) *If* $a\colon A \to X$ *is in* $\boldsymbol{A}$, *then the pullback functor* $a^*\colon \boldsymbol{A}(X) \to \boldsymbol{A}(A)$ *(whose existence is guaranteed by* (1)) *has a right adjoint, denoted by*

$$a_*\colon \boldsymbol{A}(A) \to \boldsymbol{A}(X).$$

Moreover, these right adjoints satisfy the Beck-Chevalley condition for pullback squares in $\boldsymbol{C}$, *namely that for the pullback square* $(*)$, *the canonical natural transformation* $f^* \circ a_* \to b_* \circ g^*$ *is an isomorphism.*

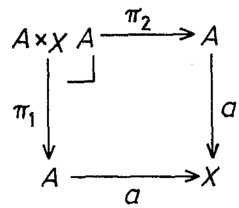
The *constant* types A (those depending on no free variables) are modelled by *objects* $[\![A]\!]$ in $\boldsymbol{C}$ for which the unique morphism from $[\![A]\!]$ to the terminal object 1, is in $\boldsymbol{A}$; types $B[x]$ depending on a variable of a constant type A, are modelled in $\boldsymbol{C}$ by $\boldsymbol{A}$*-morphisms* with codomain $[\![A]\!]$, i.e. by objects of $\boldsymbol{A}([\![A]\!])$; and so on for further type dependencies. If $B(x)$ is modelled by the $\boldsymbol{A}$-morphism $[\![B[x] \mid x \in A]\!] \to [\![A]\!]$, then terms $t \in B[x]$ are modelled by morphisms $[\![A]\!] \to [\![B[x] \mid x \in A]\!]$ in $\boldsymbol{C}$ whose composition with $[\![B[x] \mid x \in A]\!] \to [\![A]\!]$ is the identity on $[\![A]\!]$; the operation of substituting a term for a variable in a type is modelled via the pullbacks of (1). Given $C[x, y]$ dependent on $x \in A$ and $y \in B[x]$, modelled by $\boldsymbol{A}$-morphisms $c\colon [\![C[x, y] \mid x \in A, y \in B[x]]\!] \to [\![B[x] \mid x \in A]\!]$ and $b\colon [\![B[x] \mid x \in A]\!] \to [\![A]\!]$, the product $\prod y \in B[x].C[x, y]$ is modelled by the $\boldsymbol{A}$-morphism $b_*(c)$ given by (2).

If $\boldsymbol{C}$ and $\boldsymbol{A}$ are as above, then there is a corresponding type theory with products which we can use to describe appropriate properties of the category: the theory will have type constants naming the morphisms in $\boldsymbol{A}$ and function constants of various types naming the morphisms in $\boldsymbol{C}$. From the above discussion it is clear that the only objects X of $\boldsymbol{C}$ which can be denoted in this type theory are those for which there is a finite composable chain of A-morphisms starting with X and ending with the terminal object 1. Therefore we impose a further condition:

(3) *For each object X in $\mathbf{C}$, the unique morphism $X \to 1$ belongs to the class of morphisms obtained from $\mathbf{A}$ by closing under composition in $\mathbf{C}$.*

1.2. Equality types. If now we consider extensional type theories with equality types $I_A[x, x']$ [A type, $x \in A$, $x' \in A$] (satisfying the strong rules of [12] or [14]) as well as products, then the corresponding condition on $\mathbf{A}$ is:

(4) *For each morphism a: $A \to X$ in $\mathbf{A}$, forming the pullback of a against itself*



the diagonal morphism Δ: $A \to A \times_X A$ is in $\mathbf{A}$. (Δ is the unique morphism whose composition with both π_1 and π_2 is the identity on A.)

Thus if a: $A \to X$ models a type $A(x)$ dependent on $x \in X$, then
$[\![A[x] \mid x \in X,\ y \in A(x)]\!] \to [\![A[x] \mid x \in X]\!]$ is π_2: $A \times_X A \to A$
and $[\![I_{A[x]}[y, y'] \mid x \in X,\ y \in A[x],\ y' \in A[x]]\!] \to [\![A[x] \mid x \in X,\ y \in A[x]]\!]$ is Δ: $A \to A \times_X A$.

1.3. Remarks. Let us note some properties of the categories $\mathbf{A}(X)$ when $\mathbf{C}$ and $\mathbf{A}$ satisfy conditions (1) to (4):

(i) The main point of condition (1) is that pullbacks in $\mathbf{C}$ of $\mathbf{A}$-morphisms along arbitrary morphisms exist and are again in $\mathbf{A}$. However, the condition is phrased in such a way as to imply also that $\mathbf{A}$ is "replete", i.e. that the composition of an $\mathbf{A}$-morphism on either side with an *isomorphism* again results in an $\mathbf{A}$-morphism.

(ii) Conditions (1) to (4) do not force the existence of anything: the empty collection of morphisms in the trivial category **1** (with one object and one morphism) satisfy the conditions. (Note that in (3), by definition the class of morphisms obtained from $\mathbf{A}$ by closing under composition contains all identity morphisms, since these are given by finite composable chains of $\mathbf{A}$-morphisms of zero length!)

(iii) We are not assuming that $\mathbf{A}$ is closed under composition: such an assumption would correspond to having *dependent sums* satisfying strong rules in the type theory — see Section 2 of [9], which contains a discussion of the categorical semantics of dependent sums for various strengths of rules. Since A is not necessarily closed under composition, condition (1) does not imply that each category $A(X)$ has binary products. Nevertheless, a special case of condition (2) (corresponding to the formation of the exponential $A \to B \equiv \prod x \in A.B$ in the type theory) gives objects which are categorical exponentials for whatever products there are: thus given $a\colon A \to X$ and $b\colon B \to X$ in A, we can form the object $a_*(a^*(b))$ in $A(X)$, which we denote by $(A \to_X B) \to X$. (Whether or not the product functor $(-) \times_X A\colon A(X) \to A(X)$ exists, the morphism $(A \to_X B) \to X$ has a universal property in C by virtue of being the value of the right adjoint a_* at $a^*(b)$.)

1.4. Universal type. As explained in the Introduction, we consider the theory of the kind of universal type which is a type of *names* of types with every type *isomorphic* to a named one. Thus the rules are

$$\frac{}{\mathit{Type}\ \text{type}} \qquad \frac{}{u \in \mathit{Type} \Rightarrow T[u]\,\text{type}} \qquad \frac{\Gamma \Rightarrow A\ \text{type}}{\Gamma \Rightarrow n_A(\bar{x}) \in \mathit{Type}}$$

$$\frac{\Gamma \Rightarrow A\ \text{type}}{\Gamma \Rightarrow i_A(\bar{x}) \in A \to T[n_A(\bar{x})]} \qquad \frac{\Gamma \Rightarrow A\ \text{type}}{\Gamma \Rightarrow j_A(\bar{x}) \in T[n_A(\bar{x})] \to A}$$

$$\frac{\Gamma \Rightarrow A\ \text{type}}{\Gamma \Rightarrow j_A(\bar{x}) \circ i_A(\bar{x}) = id \in A} \qquad \frac{\Gamma \Rightarrow A\ \text{type}}{\Gamma \Rightarrow i_A(\bar{x}) \circ j_A(\bar{x}) = id \in T[n_A(\bar{x})]}$$

where $\bar{x}$ is the list of variables occurring in the context Γ, id abbreviates $\lambda x.x$, $j_A(x) \circ i_A(x)$ abbreviates $\lambda y.j_A(x)(i_A(x)y)$ and similarly for $i_A(\bar{x}) \circ j_A(\bar{x})$. It is convenient to write

$$\Gamma \Rightarrow A \cong T[n_A(\bar{x})]$$

for the combined conclusions of the last four rules.

The condition on C and A corresponding to these rules is:

(5) *There is an object U in $A(1)$ and an object $t\colon G \to U$ in $A(U)$ with the property that for any $X \in C$ and any $a\colon A \to X$ in $A(X)$, there is some $f\colon X \to U$ in C with $f^*(t) \cong a$ in $A(X)$; in other words, t has the property that any other morphism in A can be obtained from it by a pullback.*

Applying condition (5) to the A-morphism $U \to 1$ gives a morphism $u\colon 1 \to U$ such that the pullback of t along u is $U \to 1$. (Thus U contains a name for itself.) Also, applying (4) to $U \to 1$ gives that the diagonal $\Delta\colon U \to U \times U$ is in A. Pulling this diagonal back along $\langle u, u\rangle\colon 1 \to U \times U$ gives that the identity on 1 is in A. Hence by (1), A contains all isomorphisms. In fact A consists *only* of isomorphisms, because we can prove:

1.5. Proposition. *Let C be a category with finite products and A a collection of morphisms of C satisfying conditions* (1) *to* (5) *above. Then C is degenerate: every object in C is isomorphic to the terminal object.*

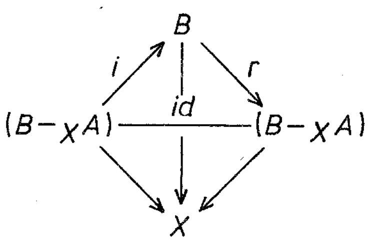
The proposition mentioned in the Introduction is the special case when $\boldsymbol{A}$ consists of all morphisms in $\boldsymbol{C}$. (When $\boldsymbol{A} = mor\boldsymbol{C}$, (1) and (2) are equivalent to $\boldsymbol{C}$ being locally cartesian closed; cf. Remark 1.3(iii).)

2. Proof of the proposition

2.1. Fixed point properties. The degeneracy of $\boldsymbol{C}$ in Proposition 1.5 will be demonstrated by establishing that the objects a: $A \to X$ of the categories $\boldsymbol{A}(X)$ $(X \in \boldsymbol{C})$ have *internal fixpoints* – which is to say that there is a morphism Y: $(A \to_X A) \to A$ in $\boldsymbol{A}(X)$ which is equal to the composition of $\langle id, Y\rangle$: $(A \to_X A) \to (A \to_X A) \times_X A$ with the evaluation morphism ev: $(A \to_X A) \times A \to A$. In other works there is a term $Y(f) \in A$ in the type theory with

$$f \in A \to A \Rightarrow f(Y(f)) = Y(f) \in A.$$

The existence of internal fixed points can be deduced from several stronger properties on an object: see [15, Section 1.5], [8] and [10] for several such properties. The one that concerns us here is that of *reflexivity*. We will say that a: $A \to X$ in $\boldsymbol{A}(X)$ is *reflexive* if there is some other object $B \to X$ in $\boldsymbol{A}(X)$ and a retraction



Given such a retraction, we can take the fixpoint operator Y: $(A \to_X A) \to A$ to be $[\![d(i(\lambda y. f(d(y)))) \in A \mid f \in B \to A]\!]$, where $d(y)$ is the diagonal term $r(y)y$.

The degeneracy of $\boldsymbol{C}$ follows from the combination of the existence of such fixpoint morphisms and conditions (3) and (4) in Section 1. For we can apply Y to the global element of $(A \to_X A) \to X$ determined by the identity on A, to obtain a morphism $X \to A$ whose composition with a is the identity on X (which is to say in the type theory that all types possess closed terms). Applying this observation to the monomorphism Δ: $A \to A \times_X A$ (which we can do by virtue of condition (4)), we conclude that it is actually an isomorphism; consequently a is a monomorphism and hence also an isomorphism (since it has a right inverse). Thus $\boldsymbol{A}$ consists entirely of isomorphisms and therefore by (3) every object is isomorphic to the terminal object.

So to complete the proof of Proposition 1.5 it remains to construct, for each $A \to X$ in $\boldsymbol{A}$ another object $B \to X$ of $\boldsymbol{A}(X)$ with $(B \to_X A)$ a retract of B over X.

In fact we will construct $B \to X$ so that $(B \to_X A)$ is isomorphic to B over X. The construction will be specified using the type theory corresponding to $\boldsymbol{C}$ and $\boldsymbol{A}$ as in Section 1. Thus for each derivable judgement

$$\Gamma \Rightarrow A \text{ type}$$

we will give a type expression B for which one also has

$$\Gamma \Rightarrow B \text{ type} \quad \text{and} \quad \Gamma \Rightarrow B \cong (B \to A).$$

(Recall from 1.4 that the second of these expressions is a short-hand for asserting the existence of term expressions l and m for which one can prove

$$\Gamma \Rightarrow l \in B \to (B \to A), \qquad \Gamma \Rightarrow m \in (B \to A) \to B,$$

$$\Gamma \Rightarrow l \circ m = id \in (B \to A) \to (B \to A) \quad \text{and} \quad \Gamma \Rightarrow m \circ l = id \in B \to B.)$$

2.2. Cantor's diagonal argument. To motivate the type theory argument we are about to give, we first consider the result in (constructive) higher order predicate logic on which it is based. The kind of categories which correspond to theories in this logic are the elementary toposes: see [11, Part II]. If $\boldsymbol{E}$ is a topos (with subobject classifier Ω) in which there is an object Ξ and a monomorphism m: $\Omega^{\Xi} \rightarrowtail \Xi$, then we can carry out Cantor's diagonal argument to see that $\boldsymbol{E}$ is necessarily degenerate. Indeed, we may do this in the internal logic of $\boldsymbol{E}$ by starting with either of the following formulas (i.e. terms of type Ω) involving a free variable x of type Ξ:

$$\pi(x) \equiv \forall X \varepsilon \Omega^{\Xi} \big(x = m(X) \to \neg (x \in X)\big),$$

$$\sigma(x) \equiv \exists X \varepsilon \Omega^{\Xi} \big(x = m(X) \wedge \neg (x \in X)\big).$$

These formulas give corresponding terms $P \equiv \{x \varepsilon \Xi \mid \pi(x)\}$ and $S \equiv \{x \varepsilon \Xi \mid \sigma(x)\}$ of type Ω^{Ξ}, and $\boldsymbol{E}$ satisfies

$$\begin{aligned} m(P) \in P &\leftrightarrow \pi(m(P)) \\ &\leftrightarrow \forall X \varepsilon \Omega^{\Xi} \big(m(P) = m(X) \to \neg (m(P) \in X)\big) \\ &\leftrightarrow \forall X \varepsilon \Omega^{\Xi} \big(P = X \to \neg (m(P) \in X)\big) \quad \text{(since } m \text{ is mono)} \\ &\leftrightarrow \neg \big(m(P) \in P\big) \end{aligned}$$

and similarly $\boldsymbol{E} \models m(S) \in S \leftrightarrow \neg (m(S) \in S)$. From either of these it is easy to deduce that $\boldsymbol{E}$ satisfies $\perp$ (falsity) and hence is degenerate.

2.3. Translation into type theory. In the type-theoretical version of the argument in 2.2 we replace

predicates	by	dependent types,
universal quantification ($\forall$)	by	dependent products ($\prod$),

	implication	by function exponentiation,
	falsity	by an arbitrary type A,
	negation ($\neg$)	by $(-)\to A$,
	the type of propositions (Ω)	by *Type*,
and	the membership predicate ($x \in X$)	by the type $T[Xx]$.

It remains to find a type Ξ and an injection m: $(\Xi \to Type) \rightarrowtail \Xi$. If we allow the use of dependent sums, then we can take

$$\Xi \equiv \textstyle\sum u \in Type.\, T[u] \quad \text{and} \quad m\colon X \in (\Xi \to Type) \mapsto \langle n_\Phi, i_\Phi(X)\rangle \in \Xi,$$

where $\Phi \equiv (\Xi \to Type)$, and n and i are as in 1.4. (As we remarked in 1.3(iii), the assumption that the type theory has dependent sums is equivalent to assuming that $\boldsymbol{A}$ is closed under composition in $\boldsymbol{C}$; in this case the constant type Ξ is modelled in $\boldsymbol{C}$ by the object G in condition (5) of 1.4.) With these definitions, the predicates $\pi(x)$ and $\sigma(x)$ in 2.2 are translated into the following dependent types:

$$P[x] \equiv \textstyle\prod X \in (\Xi \to Type).(I_\Xi[x, \langle n_\Phi, i_\Phi(X)\rangle] \to (T[Xx] \to A))$$
$$S[x] \equiv \textstyle\sum X \in (\Xi \to Type).(I_\Xi[x, \langle n_\Phi, i_\Phi(X)\rangle] \times (T[Xx] \to A)).$$

Starting with either of $P[x]$ or $S[x]$, the analogue of the argument in 2.2 produces a type B and an isomorphism $B \cong (B \to A)$, as required. However, we can do without this use of dependent sums and still get the same result. First note that

$$\Phi \equiv \Xi \to Type \cong \textstyle\prod u \in Type.(T[u] \to Type)$$

which does not involve a sum; so let us redefine Φ to be this product type. We have also to remove the use of the equality type I_Ξ at Ξ. Since each $x \in \Xi$ is provably equal to a pair $\langle u, v\rangle$ where $u \in Type$ and $v \in T[u]$, the use of I_Ξ can be replaced by separate uses of I_{Type} and $I_{T[u]}$. So for each type expression A, we can define a type expression in the product-equality fragment which will play the role of the predicate π in 2.2, namely:

$$R[u, v] \equiv I_{Type}[u, n_\Phi] \to \textstyle\prod X \in \Phi.(I_{T[u]}[v, i_\Phi(X)] \to (T[(Xu)v] \to A))$$

(where we have taken $I_{Type}[u, n_\Phi]$ outside the product because it does not depend on X). Using the rules of the product-equality fragment of the system $\boldsymbol{ML}_0$ in [16] augmented by the rules in 1.4 for the universal type, from

$$\Gamma \Rightarrow A \text{ type}$$

we can prove

$$\Gamma, u \in Type, v \in T[u] \Rightarrow R[u, v] \text{ type}.$$

This is not completely trivial and in particular uses the rules concerning *type equality judgements* (the judgements of the form "$A = B$") to justify the appearance of $i_\Phi(v) \in T[n_\Phi]$ in the equality type for $T[u]$. Consequently we have:

$$\Gamma,\ u \in \mathit{Type},\ v \in T[u] \Rightarrow n_R(u,\ x) \in \mathit{Type}.$$

Hence defining

$$r \equiv \lambda u.\lambda v.n_R(u,\ v).$$

We have

$$\Gamma \Rightarrow r \in \Phi \quad \text{and hence also} \quad \Gamma \Rightarrow i_\Phi(r) \in T[n_\Phi].$$

Using the properties of equality types and the fact that i_Φ has an inverse (namely j_Φ), on substituting n_Φ for u and $i_\Phi(r)$ for v in $R[u,\ v]$ we find that the following isomorphisms are derivable in the context of Γ:

$$\begin{aligned} &R[n_\Phi,\ i_\Phi(r)] \\ &\qquad \equiv I_{Type}[n_\Phi,\ n_\Phi] \to \textstyle\prod X \in \Phi.\big(I_{T[n_\Phi]}[i_\Phi(r),\ i_\Phi(X)] \to (T[(Xn_\Phi)i_\Phi(r)] \to A)\big) \\ &\qquad \cong \textstyle\prod X \in \Phi.\big(I_\Phi[X,\ r] \to (T[(Xn_\Phi)i_\Phi(r)] \to A)\big) \\ &\qquad \cong T[(rn_\Phi)i_\Phi(r)] \to A. \end{aligned}$$

But by definition of r, we have

$$\Gamma \Rightarrow (rn_\Phi)i_\Phi(r) = n_R\big(n_\Phi,\ i_\Phi(r)\big).$$

Therefore

$$\Gamma \Rightarrow T[(rn_\Phi)i_\Phi(r)] = T\big[n_R\big(n_\Phi,\ i_\Phi(r)\big)\big] \cong R[n_\Phi,\ i_\Phi(r)]$$

and hence from above we have that

$$\Gamma \Rightarrow R[n_\Phi,\ i_\Phi(r)] \cong \big(R[n_\Phi,\ i_\Phi(r)] \to A\big).$$

Thus whenever we have $\Gamma \Rightarrow A$ type, we can define $B \equiv R[n_\Phi,\ i_\Phi(r)]$ and derive

$$\Gamma \Rightarrow B \text{ type} \quad \text{and} \quad \Gamma \Rightarrow B \cong (B \to A);$$

and as we indicated in 2.1, this is sufficient to prove Proposition 1.5.

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