INTERPOLATION AND CONCEPTUAL COMPLETENESS FOR PRETOPOSES VIA CATEGORY THEORY

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Absract. An interpolation property of cocomma squares of pretoposes is demonstrated and used to give a constructive version of the Makkai-Reyes Conceptual Completeness Theorem. These results are achieved by using category theory and in particular a certain functorial construction from pretoposes to Grothendieck toposes.

Introduction

One of the many interesting results in Makkai and Reyes' book on first order categorical logic [8] is the conceptual completeness of pretoposes for coherent logic. To explain the meaning of this result, let us first recall the connection that exists between coherent logic and pretoposes. For unexplained terminology and further background, the reader is referred to [8] or to the recent account contained in Makkai [11].

By the term coherent logic is meant the $=,\wedge,\vee,\exists$ fragment of first order logic. The properties a category must possess to enable this fragment to be soundly interpreted in it are precisely the existence of finite limits, pullback stable extremal-epi/mono factorizations and pullback stable finite sups of subobjects; such categories are now commonly called coherent (but were termed "logical" in [8]). Two further categorical concepts, namely that of a finite disjoint coproduct and that of an effective coequalizer of an equivalence relation, are definitely within the realm of coherent logic: that a diagram in a coherent category is either of these things, can be expressed by coherent statements in the internal language of the category. A coherent category which has finite disjoint coproducts and effective coequalizers of equivalence relations is a pretopos. (Although one can take this as a definition of pretopos, the one given at the beginning of Section 1 is more fundamental.) The subcollection of coherent categories which are pretoposes is reflective: that is, letting COH denote the 2-category of coherent categories, morphisms of such and natural transformations, then for every $T \in \mathrm{COH}$ there is $I:T \longrightarrow P(T)$ in COH with $P(T)$ a pretopos and such that for any pretopos P, the induced functor

$$I^{*}:\mathrm{COH}(P(T),P)\longrightarrow \mathrm{COH}(T,P)$$

is an equivalence. In particular, since the category Set of sets is a pretopos, the functor I induces an equivalence I^{*}:Mod(P(T)) ≃ Mod(T) between the categories of (Set-valued)models of T and P(T). (Mod(T) is by definition COH(T,Set).) To summarise:

> P(T) is the completion of T in COH with respect to two particular concepts definable in coherent logic;and in transferring from T to P(T) along I:T ⟶ P(T) we do not change the category of models up to equivalence.

Thus I:T ⟶ P(T) is an example of what Makkai and Reyes call a *strongly conservative* morphism in COH, viz a morphism I:T ⟶ S for which I^{*}:Mod(S) ⟶ Mod(T) is an equivalence of categories.

Following [8], say that a small coherent category T is *conceptually complete* if every strongly conservative extension I:T ⟶ S to another small coherent category is actually an equivalence. (Thus T is conceptually complete if it has no proper small strongly conservative extensions.) Makkai and Reyes prove that the conceptually complete coherent categories are precisely the pretoposes. One part of this is immediate: since I:T ⟶ P(T) is strongly conservative, every conceptually complete coherent category has to be a pretopos. What does require work to prove is the converse assertion that small pretoposes are conceptually complete: this is Theorem 7.1.8 of [8]. Following Kock and Reyes ([7],4.8) and Makkai ([11],Theorem 3.1.1), by "pretopos conceptual completeness" we shall mean an equivalent statement mentioning only pretoposes, viz:

Pretopos Conceptual Completeness Theorem. For a morphism I:S ⟶ T of small pretoposes to be an equivalence, it is sufficient that the induced functor I^{*}:Mod(T) ⟶ Mod(S) between the categories of models be an equivalence.

Since for a small pretopos T, Mod(T) is just the category of pretopos morphisms from T to Set, it is evident that the above theorem is a *purely category-theoretic statement*. But since I:S ⟶ T can also be regarded as an interpretation between coherent theories, the theorem can be couched in more traditional language (from a logician's point of view) as a statement about coherent logic. It is from this latter standpoint that the proof of the theorem in [8] proceeds, using standard techniques of model theory (compactness and the method of diagrams) along the way. Since categorical versions of such techniques have been developed (for example by Freyd and by Barr and Makkai[2]), one could envisage translating the model-theoretic proof of Makkai and Reyes into a categorical proof of a certain sort. This is not our aim here. Whilst a categorical proof of conceptual completeness will be given, it is different in spirit from the original proof. Rather, we shall draw upon the techniques of category theory to prove versions of the theorem which are constructive in the sense that they are valid in category theory over an arbitrary elementary topos with natural number object. Each such topos $\mathfrak{S}$ gives rise to a model of (a constructive version of) category theory in which the usual small/large dichotomy becomes that of internal to $\mathfrak{S}$/fibred over $\mathfrak{S}$.(The details of this for "small" category theory are well documented, but not for the theory "in the large"; cf. the comments in the Introduction of [14] and the references cited there.) The arguments we give in this paper are (except where noted) all valid over an arbirary base topos with natural number object. However, since we are concerned throughout with one fixed such topos (i.e. no "change of base" techniques are needed), we make the following

CONVENTION: The results and arguments in the body of this paper refer to category theory over a fixed elementary topos $\mathfrak{S}$ with natural number object. However we shall generally supress mention of $\mathfrak{S}$ and present the material in the usual informal language of category theory. In particular note that when we assert that a functor $F:C \longrightarrow D$ is an equivalence of categories we shall mean that it is full, faithful and essentially surjective ($\forall Y \in D\ \exists X \in C\ F(X) \cong Y$). (The constructively stronger concept that there exist $G:D \longrightarrow C$ with $FG \cong Id_D$ and $Id_C \cong GF$ is not needed here.)

Letting PT denote the 2-category of large pretoposes in $\mathfrak{S}$ and Pt denote the full sub-2-catgory of small ones, conceptual completeness becomes the statement: "for $I:S \longrightarrow T$ in Pt to be an equivalence, it is sufficient that $I^*:PT(T,\mathfrak{S}) \longrightarrow PT(S,\mathfrak{S})$ be one". We can not expect this to be true for arbitrary $\mathfrak{S}$, since in general the categories $PT(S,\mathfrak{S})$ and $PT(T,\mathfrak{S})$ may contain few objects. When $\mathfrak{S}$ = Set, the Deligne-Joyal Completeness Theorem (which is essentially the same as the usual Completeness Theorem for first order logic) says that $PT(T,\mathfrak{S})$ contains sufficiently many objects to faithfully embed T in a power of $\mathfrak{S}$. For an arbitrary $\mathfrak{S}$, let us say that a collection $\mathcal{V}$ of large pretoposes in $\mathfrak{S}$ is sufficient for the small pretoposes if for each $T \in Pt$, there is a jointly faithful family of morphisms $T \longrightarrow V$ in PT with $V \in \mathcal{V}$. (Thus when $\mathfrak{S}$ = Set, we can take $\mathcal{V} = \{Set\}$.) Isolating the completeness theorem from conceptual completeness by making the existence of such a collection $\mathcal{V}$ a hypothesis, we shall prove (Theorem 2.13):

Pretopos Conceptual Completeness Theorem (Constructive Version). Let $\mathfrak{S}$ be an arbitrary topos with natural number object and let $\mathcal{V} \subseteq PT$ be a collection of large pretoposes which is sufficient for small pretoposes. For a morphism $I:S \longrightarrow T$ of small pretoposes to be an equivalence, it is sufficient that $I^*:PT(T,V) \longrightarrow PT(S,V)$ be an equivalence for each $V \in \mathcal{V}$.

This theorem is an easy consequence of a result about quotient morphisms in Pt (Theorem 2.12), the classical version of which is implicit in [8] and is called by Makkai [11, Theorem 3.1.2] "strong conceptual completeness for pretoposes":

Strong Pretopos Conceptual Completeness Theorem (Constructive Version). With $\mathfrak{S}$ and $\mathcal{V}$ as above, for a morphism $I: S \longrightarrow T$ in Pt to be a quotient, it is sufficient that $I^*: PT(T,V) \longrightarrow PT(S,V)$ be full and faithful for each $V \in \mathcal{V}$.

(Roughly speaking, I is a quotient if, identifying S and T with coherent theories, T is equivalent to a theory obtained from S by adding some new axioms in such a way that I becomes the canonical interpretation of S in the new theory. An equivalent, categorical definition is given in section 2.)

Proceeding as Makkai and Reyes do in the classical case, we derive this strong conceptual completeness result by breaking it into two pieces. Considering separately the hypotheses "I^* is full" and "I^* is faithful", one shows that each implies a condition on I that together are equivalent to I being a quotient.

The case "I^* is full" is Proposition 2.4 below. It should be compared with Theorem 7.1.4 of [8]: both there and here the result is obtained by applying versions of "Beth definability". In our case this takes the form of a special case of a general interpolation theorem for pretoposes (Theorem 1.3):

Pretopos Interpolation Theorem. Let

$$\begin{array}{ccc} S & \xrightarrow{\quad M \quad} & P \\ {\scriptstyle I}\uparrow & \overset{h}{\Longrightarrow} & \uparrow{\scriptstyle N} \\ R & \xrightarrow{\quad J \quad} & T \end{array}$$

be a cocomma square in Pt. Given an object X of R and subobjects $B \rightarrowtail I(X)$ in S and $C \rightarrowtail J(X)$ in T, suppose that the image of the subobject MB along $h_X: MI(X) \longrightarrow NJ(X)$ is contained in the subobject NC of $NJ(X)$. Then there exists a subobject $A \rightarrowtail X$ in R with $B \leq IA$ in $Sub_S(IX)$ and $JA \leq C$ in $Sub_T(JX)$.

Section 1 is devoted to proving this theorem. The method of proof is functorial: the "topos of filters" functor (developed in [13] to prove an interpolation result for Heyting pretoposes) is used to deduce the result from certain properties of Grothendieck toposes, which are developed in [16] as an application of enriched category theory.

Returning to the strong conceptual completeness theorem, the case "I^* is faithful" is Proposition 2.9 below and should be compared with Theorem 7.1.6 of [8]. It is here that our approach, apart from being constructive, is essentially different from that in [8]. For we show that the proposition is in fact a consequence of the previous case (Proposition 2.4) applied to a *codiagonal* morphism; hence it too is a consequence of the interpolation theorem. Although it takes some work to see it, we can thus say that for pretoposes *conceptual completeness is a corollary of the general interpolation theorem*.

This insight is certainly one of the pleasing aspects of the approach developed in this paper. In avoiding non-constructive methods, one of course pays a price. A deep analysis of the classical properties of pretoposes must surely appeal to classical methods: we have in mind here the recent very interesting work involving ultraproducts and pretoposes by Makkai (cf. [9],[10] and [11]). However the methods developed here can be extended very easily to prove a conceptual completeness result for Heyting pretoposes from the interpolation theorem given in [13]: this is in other words a result about first order intuitionistic logic, where one can expect little help from classical model theory. The details will appear in [15].

I am indebted to G.E.Reyes, F.W.Lawvere, A.Joyal and especially to M.Makkai for many helpful conversations on the subject matter of this paper. The research for it was partly carried out whilst a visitor at the Centre Interuniversitaire en Etudes Catégoriques in McGill University and partly whilst a visitor at SUNY at Buffalo; the hospitality of all concerned is gratefully acknowledged.

1. Interpolation

Recall that a *pretopos* is a category with finite limits, finite coproducts that are disjoint and stable under pullback (so in particular, a strict initial object) and effective coequalizers of eqivalence relations that are stable under pullback. (The original definition in SGA4 [1, VI 3.11] also required that all epimorphisms be (stable) effective; but as Makkai and Reyes observe in [8], this is implied by the other conditions.) A *morphism* of pretoposes is a functor preserving finite limits, finite coproducts and coequalizers of equivalence relations.

Notation. We will denote by PT the 2-category of pretoposes, morphisms and natural transformations; Pt will denote the full sub-2-category of small pretoposes.

A *coherent* theory (also called "finitary geometric" or "positive existential") is a first order theory (written in a possibly many-sorted language with equality, relation and function symbols) which can be axiomatized by sentences of the form

$$\forall\bar{x}(\alpha \rightarrow \beta)$$

with α and β *coherent formulas*, i.e. built up from atomic formulas using only finite conjunction, finite disjunction and existential quantification.

We refer the reader to [11] (or of course to [8]) for an account of the relationship between coherent theories, pretoposes and coherent categories (of which pretoposes are the completion with respect to disjoint finite coproducts and effective coequalizers of equivalence relations). A key ingredient of this relationship is the association to each coherent theory $\underline{T}$ of its "classifying pretopos" $P(\underline{T})$: letting $Mod(\underline{T},P)$ denote the category of $\underline{T}$-models and homomorphisms in a pretopos P, then $P(\underline{T})$ is defined up to equivalence by requiring it to be a pretopos in which there is a *generic* $\underline{T}$-model M; thus for any pretopos P, the functor

$$\begin{array}{ccc} PT(P(\underline{T}),P) & \longrightarrow & Mod(\underline{T},P) \\ F & \longmapsto & F(M) \end{array} \qquad (1)$$

is to be an equivalence of categories. This might be called the "semantic" definition of $P(\underline{T})$. But a "syntactic" construction of $P(\underline{T})$ can be given: its objects are quotients by $\underline{T}$-definable partial equivalence relations of finite disjoint unions of finite products of the basic sorts; and its morphisms are $\underline{T}$-provably functional relations (or rather, equivalence classes of such under $\underline{T}$-provability). In this sense $P(\underline{T})$ is is the *category of concepts* of the theory $\underline{T}$. There is some discussion in [11] of the rôle that $P(\underline{T})$ might play in the study of certain parts of first order logic. Here we wish to emphasise that just as in general, category theory is the study of morphisms, so here in particular the association of the pretopos $P(\underline{T})$ to a theory $\underline{T}$ allows us to handle very elegantly an important notion of "morphism of theories":

Definition. Given two coherent theories $\underline{S}$ and $\underline{T}$, an *interpretation* of $\underline{S}$ in $\underline{T}$ will mean an $\underline{S}$-model in the category of concepts $P(\underline{T})$ of $\underline{T}$. Using the equivalence (1), such interpretations can also be regarded as pretopos morphisms from $P(\underline{S})$ to $P(\underline{T})$.

Coupling the first sentence of the above definition with the explicit description of $P(\underline{T})$ mentioned above, one sees both that this notion of interpretation can be given a purely syntactic (albeit rather involved) definition and that as such it coincides with or includes notions of interpretation or translation that various logicians have considered. However it is the second sentence of the definition that allows a very smooth development of the properties of this notion of interpretation. Thus in this paper, whilst the results proven are about morphisms of pretoposes, they are equally results about interpretations between coherent theories.

The passage from coherent theory $\underline{T}$ to pretopos $P(\underline{T})$ is only one half of the connection between such theories and such categories. Each pretopos P is equivalent to $P(\underline{T})$ for some coherent theory $\underline{T}$. For example the underlying graph of P provides a many-sorted language (with no relation symbols) in which we can write down coherent statements that recapture P on taking the category of concepts. This in a precise sense is to present P in terms of generators (the language) and relations (the axioms); and such presentations guarantee the existence of small pseudo-colimits in Pt. The particular pseudo-colimit with which we will be concerned here is the so-called "cocomma" square.

__1.1 Definition.__ Given three categories A, B_1, B_2 and two functors $F_i : B_i \longrightarrow A$ $(i=1,2)$, the __comma category__ (F_1, F_2) has as objects

triples (b_1, f, b_2), where b_i is an object of B_i and $f: F_1(b_1) \longrightarrow F_2(b_2)$ is a morphism in A;

and has as morphisms from (b_1, f, b_1) to (b_1', f', b_2')

pairs (g_1, g_2), where $g_i : b_i \longrightarrow b_i'$ in B_i and

$$\begin{array}{ccc} F_1(b_1) & \xrightarrow{\quad f \quad} & F_2(b_2) \\ {\scriptstyle F_1(g_1)}\downarrow & & \downarrow{\scriptstyle F_2(g_2)} \\ F_1(b_1') & \xrightarrow{\quad f' \quad} & F_2(b_2') \end{array}$$

commutes in A.

Composition and identity morphisms are defined from those in B_1 and B_2 in the obvious way.

For such a comma category, there are projection functors

$$\pi_i : (F_1, F_2) \longrightarrow F_i \qquad (i=1,2)$$

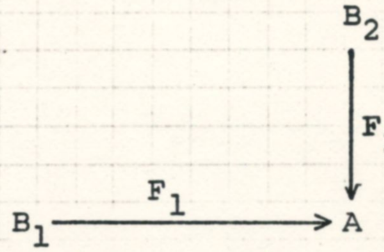
defined by

$$\pi_i\left((b_1, f, b_2) \xrightarrow{(g_1, g_2)} (b_1', f', b_2') \right) = (b_i \xrightarrow{f_i} b_i')$$

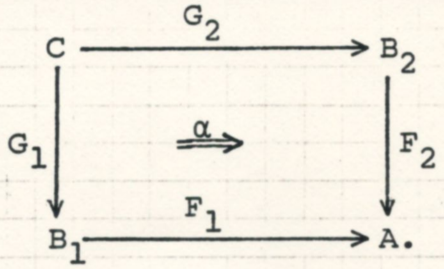
and a natural transformation

$$\alpha : \pi_1 F_1 \longrightarrow \pi_2 F_2$$

whose component at an object (b_1, f, b_2) is f. These provide the universal solution to the problem of completing the diagram



to a square of functors that commutes up to a natural transformation:



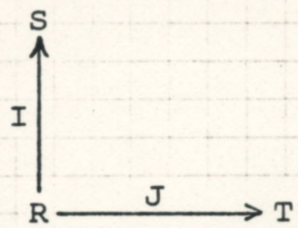
1.2 Definition. A diagram of the above form in a bicategory $\mathcal{B}$ will be called a comma square if for all objects D of $\mathcal{B}$, the canonical functor

$$\mathcal{B}(D,C) \longrightarrow ((F_1)_*, (F_2)_*)$$
$$H \longmapsto (G_1 H, \alpha_H, G_2 H)$$

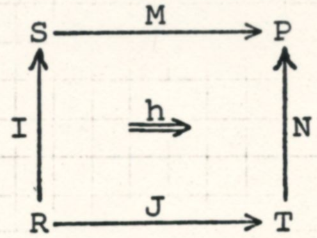
is an equivalence. (As usual, $(F_i)_* : \mathcal{B}(D, B_i) \longrightarrow \mathcal{B}(D, A)$ denotes composition with F_i.) Thus in such a square, G_1, G_2 and α provide the universal solution (in the sense appropriate to bicategories) to the problem of completing the diagram given by F_1 and F_2 to a square commuting up to a 2-cell in a specified direction.

Dually, the above diagram is a cocomma square in $\mathcal{B}$ if it is a comma square in $\mathcal{B}^{op}$.

Starting with



in Pt, the remarks above indicate how one can form a cocomma square containing them: taking T to be a coherent theory in the language of the underlying graph of T with P(T)≃T (and taking a theory S for S similarly), then the cocomma square



is constructed by taking P to be P($\underline{P}$) for $\underline{P}$ the (evidently coherent) theory comprising disjoint copies of $\underline{S}$ and $\underline{T}$ together with function symbols for the components of h, plus axioms saying that it is natural.

Our principal tool for proving the pretopos conceptual completeness results in this paper will be the following interpolation property of cocomma squares in Pt. As usual, for a small pretopos P and object X of P, $Sub_P(X)$ denotes the set of subobjects of X and $\leq$ denotes the partial order on it; similarly, given $f:X \longrightarrow Y$ in P, the operation of pulling back subobjects along f is denoted by f^{-1} and its left adjoint (taking images along f) by $\exists f$.

<u>1.3 Theorem</u> (<u>Pretopos</u> <u>Interpolation</u>). Let

$$\begin{array}{ccc} S & \xrightarrow{M} & P \\ {\scriptstyle I}\uparrow & \xRightarrow{h} & \uparrow{\scriptstyle N} \\ R & \xrightarrow{J} & T \end{array}$$

be a cocomma square in Pt. Given an object X of R and subobjects $B \rightarrowtail I(X)$ in S and $C \rightarrowtail J(X)$ in T, suppose that

$$\exists h_X(MB) \leq NC$$

in $Sub_P(NJX)$. Then there is a subobject $A \rightarrowtail X$ in R such that $B \leq IA$ in $Sub_S(IX)$ and $JA \leq C$ in $Sub_T(JX)$.

In fact we need only a special case of Theorem 1.3, that when I=J:

<u>1.4 Corollary</u>. Given $I:S \longrightarrow T$ in Pt, form the cocomma square

$$\begin{array}{ccc} T & \xrightarrow{M} & P \\ {\scriptstyle I}\uparrow & \xRightarrow{h} & \uparrow{\scriptstyle N} \\ S & \xrightarrow{I} & T. \end{array}$$

For any object X of S and subobject $B \rightarrowtail I(X)$ in T, if

$$\exists h_X(MB) \leq NB$$

in $Sub_P(NIX)$, then $B \cong IA$ in $Sub_T(IX)$ for some $A \rightarrowtail X$ in S.

□

To give Corollary 1.4 a more "model-theoretic" flavour, let us assume that we are given a collection $\mathcal{V}$ of pretoposes that is <u>sufficient</u> for Pt in the sense that for any small pretopos T, the collection of pretopos morphisms $T \longrightarrow V$ with $V \in \mathcal{V}$ is jointly faithful. (Cf. Definition 2.3 below.) (For example, if we are working over the topos of classical sets, where one has the usual completeness theorem for first order logic, then we can take $\mathcal{V}$ to contain just the category of sets.) Let us temporarily call any morphism of a small pretopos T into a member of $\mathcal{V}$, a "model" of T. Then Corollary 1.4 is equivalent to the statement: "given $B \rightarrowtail I(X)$ (i.e. given a coherent formula of $\underline{T}$ whose free variables are of sorts in the image of I), if for any pair M, N of models of $\underline{T}$ and for any $\underline{S}$-model homomorphism h between the restrictions of M and N along I, h preserves B, then B is already in the image of I". Compare this form of the corollary to Theorem 7.1.4' of [8].

Although the proof of Theorem 1.3 which we shall give is somewhat indirect, it is constructive in the sense that it can be carried out in category theory over an arbitrary base topos with natural number object. Moreover it utilises the functorial techniques that are at the heart of category theory. In fact, using a functor $Pt \longrightarrow Pt$ with suitable properties, the proof can be reduced to the special case: I has the property that the operation of applying it to the subobjects of a given object X, has a left adjoint. Now a careful analysis of the cocomma pretopos P enables one to characterise the (finite) sup-preserving maps out of the lattices $Sub_P(NZ)$ (Z an object of T); this characterisation shows that N inherits the property of I just mentioned. But in these circumstances, the interpolation property of a cocomma square is equivalent to a certain equation involving the left adjoints (a kind of Beck-Chevalley condition: cf. the Remark in the introduction of [12]) and this equation is easily verified from the characterisation of the lattices $Sub_P(NZ)$ mentioned above.

The analysis of cocomma pretoposes needed to realize the above proof is just the finitary version of the characterisation, given in [16], of comma squares in GTOP, the bicategory of Grothendieck toposes, geometric morphisms and natural transformations. Rather than give that finitary version, we shall utilise the results for toposes directly (sketching their proof and referring the reader to [16] for complete details). We can do this by using the contravariant pseudofunctor which sends a pretopos P to its associated coherent topos of sheaves on the site consisting of P with the subcanonical topology (cf. [4], section 7.3). Thus the functorial part of our proof of Theorem 1.3 will use the composition of this with the functor $Pt \longrightarrow Pt$ alluded to above. The resulting contravariant pseudofunctor $Pt^{op} \longrightarrow GTOP$ is none other than the "topos of filters" construction of [13]; there it was used to prove an interpolation property of pushout squares of Heyting pretoposes (which are related to intuitionistic first order predicate logic in the same way pretoposes are related to coherent logic).

Let us now give the version of Theorem 1.3 for infinitary geometric logic that the topos of filters construction allows us to use:

<u>1.5 Theorem</u>. Let

$$\begin{array}{ccc} H & \xrightarrow{\;q\;} & G \\ {\scriptstyle p}\downarrow & \overset{\alpha}{\Longrightarrow} & \downarrow{\scriptstyle g} \\ F & \xrightarrow[\;f\;]{} & E \end{array}$$

be a comma square in the bicategory GTOP of Grothendieck toposes, geometric morphisms and natural transformations. Suppose further that f^* <u>preserves all intersections of subobjects</u>. Then given an object X of E and subobjects $B \rightarrowtail f^*(X)$ in F and $C \rightarrowtail g^*(X)$ in G, if

$$\exists\alpha_X(p^*B) \leq q^*C$$

in H, then there is $A \rightarrowtail X$ in E with $B \leq f^*A$ and $g^*A \leq C$.

<u>Proof</u>. Since the operation of applying f^* to subobjects of a given object X of E preserves intersections, this operation has a left adjoint which we shall denote by

$$f_X : Sub_F(f^*X) \longrightarrow Sub_E(X).$$

The theorem follows by showing:

q^* also preserves intersections (so that in particular $q^* : Sub_G(g^*X) \longrightarrow Sub_H(q^*g^*X)$ has a left adjoint, q_{g^*X}; (2)

and:

$$\begin{array}{ccc} Sub_F(f^*X) & \xrightarrow{\;f_X\;} & Sub_E(X) \\ {\scriptstyle p^*}\downarrow & & \\ Sub_H(p^*f^*X) & & \downarrow{\scriptstyle g^*} \\ {\scriptstyle \exists\alpha_X}\downarrow & & \\ Sub_H(q^*g^*X) & \xrightarrow{\;q_{g^*X}\;} & Sub_G(g^*X) \end{array} \quad \text{commutes.} \qquad (3)$$

For then

$$\exists\alpha_X(p^*B) \leq q^*C$$

is equivalent to

$$q_{g^*X}(\exists\alpha_X(p^*B)) \leq C,$$

so that by (3)

$$g^*(f_X B) \leq C.$$

Then since one always has $B \leq f^*(f_X B)$, we can take $A = f_X(B)$.

The proof of (2) and (3) is very much inspired by the work of Joyal and Tierney [6] and follows from a particular way of describing the <u>bicategory of relations</u> of the comma topos H in terms of those of the toposes E,F,G (and the bifunctors induced by f^* and g^*). This description will be given in [16] as a corollary of observations about internal sup-lattices in Grothendieck toposes and sup-lattice enriched category theory. Here we content ourselves with giving some motivation for the description, stating it precisely and applying it to prove (2) and (3).

First some notation. For a topos E, we will denote the associated bicatgory of relations by Rel_E: this has the same objects as E, but a morphism from X to X' in Rel_E is a relation from X to X' in E, i.e. a subobject of $X \times X'$. We will use the notation

$$A:X \mathrel{-\!\!\!+\!\!\!\rightarrow} X'$$

to denote that A is such a relation.

Returning to the comma square in the statement of Theorem 1.5, given objects Y of F and Z of G, we will give a description of $Rel_H(p^*Y, q^*Z)$ as an object in the category Sl of complete posets and sup preserving maps. What relations from p^*Y to q^*Z can there be in H? Certainly given an object X of E and relations $B:Y \mathrel{-\!\!\!+\!\!\!\rightarrow} f^*X$ in F and $C:g^*X \mathrel{-\!\!\!+\!\!\!\rightarrow} Z$ in G, we get by composition a relation $q^*C \circ \alpha_X \circ p^*B : p^*Y \mathrel{-\!\!\!+\!\!\!\rightarrow} q^*Z$ in H. In general an element of $Rel_H(p^*Y, q^*Z)$ is a sup of such relations: M.Makkai (private communication) has pointed out a straightforward way of seeing this in terms of theories. Thinking of E,F and G in terms of the geometric theories they classify, then (just as for the finitary case discussed after Definition 1.2) H is given by the (geometric) theory of an E-model homomorphism from the restriction along f^* of an F-model to the restriction along g^* of a G-model. Then an argument by induction on the complexity of a geometric formula in the language of H of sort $Y \times Z$, shows that it is provably equivalent to one of the form

$$\bigvee_{i \in I} \exists x_i \in X_i [B_i(y, x_i) \wedge C_i(\alpha_{X_i}(x_i), z)],$$

which gives rise to the relation $\bigvee_{i \in I} q^*C_i \circ \alpha_{X_i} \circ p^*B_i$ in H. (In fact, taking the coproduct of the X_i, such a sup is equal to a single relation of the form $q^*C \circ \alpha_X \circ p^*B$, but we do not need this refinement here.)

Thus as an object of the category Sl, $Rel_H(p^*Y, q^*Z)$ is generated by elements of the form $q^*C \circ \alpha_X \circ p^*B$. What relations exist between such elements? Since p^*, q^* and composition preserve sups, we have that

$$q^*(\vee C_i) \circ \alpha_X \circ p^*B = \vee (q^*C_i \circ \alpha_X \circ p^*B)$$

and $$q^*C \circ \alpha_X \circ p^*(\vee B_i) = \vee (q^*C \circ \alpha_X \circ p^*B_i).$$

Also, since α_X is natural for maps in E, it is lax-natural for relations, so that given $A:X \mathrel{-\!\!\!+\!\!\!\rightarrow} X'$ in E, $B:Y \mathrel{-\!\!\!+\!\!\!\rightarrow} f^*X$ in F and $C':g^*X' \mathrel{-\!\!\!+\!\!\!\rightarrow} Z$ in G, we have

$$q^*C'\circ\alpha_{X'}\circ p^*(f^*A\circ B) \leq q^*(C'\circ g^*A)\circ\alpha_X\circ p^*B.$$

It is shown in [16] that the above three kinds of relation are sufficient to generate all the others. Precisely, we mean that given any V in Sl and any collection of elements

$$\Sigma(C,X,B) \in V \qquad (X\in E,\ B\in \mathrm{Rel}_F(Y,f^*X),\ C\in \mathrm{Rel}_G(g^*X,Z)\ )$$

satisfying

$$\Sigma(\vee C_i,X,B) = \vee\Sigma(C_i,X,B), \tag{4}$$

$$\Sigma(C,X,\vee B_i) = \vee\Sigma(C,X,B_i) \tag{5}$$

and
$$\Sigma(C',X',f^*A\circ B) \leq \Sigma(C'\circ g^*A,X,B), \tag{6}$$

then there is a unique morphism $\bar{\Sigma}:\mathrm{Rel}_H(p^*Y,q^*Z)\longrightarrow V$ in Sl with

$$\bar{\Sigma}(q^*C\circ\alpha_X\circ p^*B) = \Sigma(C,X,B).$$

Here we are interested in the case Y=1, the terminal object of F. Identifying $\mathrm{Rel}_F(1,f^*X)$ with $\mathrm{Sub}_F(f^*X)$ and $\mathrm{Rel}_H(p^*1,q^*Z)$ with $\mathrm{Sub}_H(q^*Z)$, the assignment

$$\Sigma(C,X,B) = C\circ g^*(f_X B) \in \mathrm{Sub}_G(Z)$$

satisfies the conditions (4), (5) and (6) and so induces a sup preserving map

$$\bar{\Sigma}:\mathrm{Sub}_H(q^*Z)\longrightarrow \mathrm{Sub}_G(Z).$$

Since $\bar{\Sigma}q^*C = \bar{\Sigma}(q^*C\circ\alpha_1\circ p^*(\top)) = \Sigma(C,1,\top) = C\circ g^*(f_1(\top)) \leq C$

and
$$\begin{aligned} q^*C\circ\alpha_X\circ p^*B &\leq q^*C\circ\alpha_X\circ p^*f^*(f_X B)\\ &\leq q^*C\circ q^*g^*(f_X B)\\ &= q^*\Sigma(C,X,B)\\ &= q^*\bar{\Sigma}(q^*C\circ\alpha_X\circ p^*B), \end{aligned}$$

it follows that $\bar{\Sigma}$ is left adjoint to $q^*:\mathrm{Sub}_G(Z)\longrightarrow \mathrm{Sub}_H(q^*Z)$. This proves (2); and taking $Z = g^*X$, we also have

$$\begin{aligned} q_{g^*X}(\exists\alpha_X(p^*B)) &= \bar{\Sigma}(q^*(id)\circ\alpha_X\circ p^*B)\\ &= \Sigma(id,X,B)\\ &= id\circ g^*(f_X B)\\ &= g^*(f_X B), \end{aligned}$$

which is (3). With these results, Theorem 1.5 follows as indicated above.

□1.5

<u>1.6 Remarks.</u>

(i) Theorem 7.3.5 of [8] is evidently related to the special case of Theorem 1.5 when $f = g$.

(ii) We can say more than (2): it is the case that q is an <u>open</u> geometric morphism. Moreover, if f is surjective then so is q and if g^* reflects 0 (i.e. if $g^*X \cong 0$ implies $X \cong 0$) then so does p^*. For proofs, see [16].

As mentioned above, to deduce Theorem 1.3 from Theorem 1.5 we shall use the "topos of filters" $\Phi(T)$ of a pretopos T, which was defined in [13]. Φ is pseudofunctorial in $T \in Pt$ and there is a pretopos morphism $I_T : T \longrightarrow \Phi(T)$ which is pseudonatural in T. Examining the construction of $\Phi(T)$ and I_T given in section 2 of [13], one can easily deduce the following properties:

<u>1.7</u> $\Phi(T)$ is generated by the collection of those objects which are subobjects of objects in the image of I_T.

<u>1.8</u> Given an object X of T, let $\mathfrak{F}(Sub_T(X))$ denote the (complete, distributive) lattice of filters of $Sub_T(X)$ partially ordered by <u>reverse</u> inclusion; and let $\emptyset(Sub_T(X))$ denote the lattice of ideals of $\mathfrak{F}(Sub_T(X))$ partially ordered by inclusion. (Cf. section 2 of [12].)

Then the map

$$\emptyset(Sub_T(X)) \longrightarrow Sub_{\Phi T}(I_T X)$$
$$\mathcal{A} \longmapsto \bigvee_{\alpha \in \mathcal{A}} \bigwedge_{A \in \alpha} I_T(A)$$

is a lattice isomorphism (and is natural in X).

Thus in particular every subobject of $I_T(X)$ in $\Phi(T)$ is expressible as a join of meets of subobjects in the image of I_T; and it is possible to give an explicit (but rather intricate) formula for when one arbitrary join of meets of subobjects in the image of I_T is less than or equal to another such.

<u>1.9</u> Given $I : S \longrightarrow T$ in Pt and X an object of S, then the map

$$\emptyset(Sub_S(X)) \longrightarrow \emptyset(Sub_T(IX))$$

corresponding under the isomorphisms $\emptyset(Sub_S(X)) \cong Sub_{\Phi S}(I_S X)$ and $\emptyset(Sub_T(IX)) \cong Sub_{\Phi T}(I_T IX) \cong Sub_{\Phi T}((\Phi I)^* I_S X)$, to the map

$$(\Phi I)^* : Sub_{\Phi S}(I_S X) \longrightarrow Sub_{\Phi T}((\Phi I)^* I_S X),$$

is that induced by I: viz it sends $\mathcal{A}$ (an ideal of filters of $Sub_S(X)$) to the ideal $\{\beta \in \mathfrak{F}(Sub_T(IX)) \mid \exists \alpha \in \mathcal{A}\ \forall A \in \alpha\ I(A) \in \beta\}$.

The nature of meets in $\mathfrak{F}(Sub_S(X))$ and $\emptyset(Sub_S(X))$ implies that the map described in 1.9 preserves all meets. Thus

$$(\Phi I)^* : Sub_{\Phi S}(I_S X) \longrightarrow Sub_{\Phi T}((\Phi I)^* I_S X)$$

preserves all intersections. Now for any geometric morphism $f : F \longrightarrow E$ between Grothendieck toposes, the collection of objects X in E for which $f^* : Sub_E(X) \longrightarrow Sub_F(f^* X)$ preserves intersections, is closed under coproducts, subobjects and images. In view of 1.7, we thus have:

<u>1.10</u> For any $I : S \longrightarrow T$ in Pt, $(\Phi I)^* : \Phi S \longrightarrow \Phi T$ preserves all intersections of subobjects of any given object.

We can now give the

<u>Proof of Theorem 1.3</u>. We are given a cocomma square in Pt:

$$\begin{array}{ccc} S & \xrightarrow{\;M\;} & P \\ {\scriptstyle I}\big\uparrow & \overset{h}{\Longrightarrow} & \big\uparrow{\scriptstyle N} \\ R & \xrightarrow{\;J\;} & T. \end{array} \qquad (7)$$

Applying Φ to I and J, let

$$\begin{array}{ccc} H & \xrightarrow{\;q\;} & \Phi(T) \\ {\scriptstyle p}\big\downarrow & \overset{\alpha}{\Longrightarrow} & \big\downarrow{\scriptstyle \Phi J} \\ \Phi(S) & \xrightarrow{\;\Phi I\;} & \Phi(R) \end{array} \qquad (8)$$

be a comma square in GTOP. Using the pseudonaturality of $I_{(-)}:(-)\longrightarrow\Phi(-)$, we get

$$\begin{array}{ccccc} S & \xrightarrow{\;I_S\;} & \Phi(S) & \xrightarrow{\;p^*\;} & H \\ & & & & \big\uparrow{\scriptstyle q^*} \\ {\scriptstyle I}\big\uparrow & & \overset{\beta}{\Longrightarrow} & & \Phi(T) \\ & & & & \big\uparrow{\scriptstyle I_T} \\ R & & \xrightarrow{\;J\;} & & T \end{array}$$

in PT, the bicategory of large toposes; here β is

$$p^*I_SI\cong p^*(\Phi I)^*I_R\xrightarrow{\;\alpha I_R\;}q^*(\Phi J)^*I_R\cong q^*I_TJ.$$

Since (7) is a cocomma square in Pt, evidently it is one in PT as well. So by its universal property, there are $K:P\longrightarrow H$ in PT and isomorphisms

$$KM\cong p^*I_S\ ,\quad KN\cong q^*I_T$$

making

$$\begin{array}{ccc} KMI & \xrightarrow{\;Kh\;} & KNJ \\ \wr\!\wr & & \wr\!\wr \\ p^*I_SI & \xrightarrow{\;\beta\;} & q^*I_TJ \end{array}$$

commute.

In the hypothesis of 1.3 we are given an object X of R and subobjects $B\rightarrowtail I(X)$ and $C\rightarrowtail J(X)$. Write $\overline{X}=I_R(X)$ and $\overline{B}\rightarrowtail(\Phi I)^*(\overline{X})$, $\overline{C}\rightarrowtail(\Phi J)^*(\overline{X})$ for the subobjects

$$I_S(B)\rightarrowtail I_SI(X)\cong(\Phi I)^*(\overline{X})\quad \text{in } I(S)$$

and

$$I_T(C)\rightarrowtail I_TJ(X)\cong(\Phi J)^*(\overline{X})\quad \text{in } I(T).$$

We are given that

$$\exists h_X(MB) \leq NC$$

in $\mathrm{Sub}_P(NJX)$. By definition of β and K, this inequality gives

$$\exists \alpha_{\overline{X}}(p^{\star}\overline{B}) \leq q^{\star}\overline{C}$$

in $\mathrm{Sub}_H(q^{\star}(\Phi J)^{\star}\overline{X})$. In view of 1.10, we can apply Theorem 1.5 to the comma square (8) to conclude that there is a subobject $\overline{A} \rightarrowtail \overline{X}$ in $\Phi(R)$ with $\overline{B} \leq (\Phi I)^{\star}(\overline{A})$ and $(\Phi J)^{\star}(\overline{A}) \leq \overline{C}$. Now under the isomorphism of 1.8 $\overline{A} \in \mathrm{Sub}_{\Phi R}(I_R X)$ corresponds to some ideal of filters $\mathcal{A} \in \phi(\mathrm{Sub}_R(X))$.

Using 1.8 again, plus the definition of $\overline{X}$, we have

$$\mathrm{Sub}_{\Phi S}((\Phi I)^{\star}\overline{X}) \cong \phi(\mathrm{Sub}_S(IX)).$$

By definition of $\overline{B}$ and by 1.8, $\overline{B}$ is identified under this isomorphism with the principal ideal $\{\beta \mid B \in \beta\}$; on the other hand, by definition of $\mathcal{A}$ and by 1.9, $(\Phi I)^{\star}(\overline{A})$ is identified with the ideal $\{\beta \mid \exists \alpha \in \mathcal{A} \forall A \in \alpha\ I(A) \in \beta\}$. So $\overline{B} \leq (\Phi I)^{\star}(\overline{A})$ implies that the principal ideal is contained in this latter ideal and hence that:

$$\text{there is some } \alpha \in \mathcal{A} \text{ such that for all } A \in \alpha \quad B \leq I(A). \qquad (9)$$

Similarly $(\Phi J)^{\star}(\overline{A}) \leq \overline{C}$ implies that the ideal $\{\gamma \mid \exists \alpha \in \mathcal{A} \forall A \in \alpha\ J(A) \in \gamma\}$ is contained in the principal ideal $\{\gamma \mid C \in \gamma\}$. So taking $\gamma = \{C \mid \exists A \in \alpha\ J(A) \leq C\}$ where α is the filter mentioned in (9), we have that $C \in \gamma$, i.e. $J(A) \leq C$ for some $A \in \alpha$; and since $A \in \alpha$, by (9) we also have $B \leq I(A)$.

□1.3

<u>1.11 Remark</u>. The interpolation property of cocomma squares in Pt given in Theorem 1.3 can be shown (using simple properties of quotients in Pt) to be equivalent to the following statement:

If in the cocomma square (7), J reflects the initial object (i.e. if $JX \cong 0$ implies $X \cong 0$), then so does M.

Thus an alternative proof of 1.3 is to deduce this statement from that given in Remark 1.6(ii) and the fact that $I_S : S \longrightarrow \Phi(S)$ reflects the initial object. (I_S is in fact always full and faithful.) An analogous method was used in [13] to prove an interpolation property of pushout squares of Heyting pretoposes.

Let us also remark that Theorem 1.3 can be used to deduce a similar interpolation property for cocomma squares of coherent categories (using the fact that the embedding of such a category into its pretopos completion is <u>powerful</u> in the sense of section 2 below).

2. Conceptual Completeness

Let $I:S \longrightarrow T$ be a morphism of pretoposes.

2.1 Definition.
(i) I is *conservative* if it reflects isomorphisms, i.e. if whenever $f:X \longrightarrow X'$ in S has $I(f)$ an isomorphism, then f is already an isomorphism.
(ii) I is *full on subobjects* if given an object X of S and a subobject $B \rightarrowtail I(X)$ in T, there is $A \rightarrowtail X$ in S with $I(A) \cong B$ in $Sub_T(IX)$.
(iii) Given objects X of S and Y of T, say that X *subcovers* Y *via* I if Y is the codomain of an epimorphism whose domain is a subobject of $I(X)$:

$$\begin{array}{ccc} \cdot & \twoheadrightarrow & Y \\ \downarrow & & \\ I(X) & & \end{array}$$

($\twoheadrightarrow$ will denote an epimorphism in a pretopos and as usual $\rightarrowtail$ will denote a monomorphism.)

Then I is called *subcovering* if each object of T is subcovered via I by some object of S.
(iv) I is a *quotient* morphism if it is both full on subobjects and subcovering.

Note that I is conservative iff it is faithful. Note also that it is conservative iff whenever a subobject of an object X of S is sent by I to the top subobject of $I(X)$, then it was already the top subobject of X. This last observation shows that a conservative morphism of pretoposes exactly captures the usual notion of conservative interpretation of theories.

A similar remark applies to the notion of quotient morphism: it is the case that a pretopos morphism between the categories of concepts of two theories, $P(\underline{S}) \longrightarrow P(\underline{T})$, is a quotient iff $\underline{T}$ is equivalent (in the sense of having equivalent categories of concepts) to a theory obtained from $\underline{S}$ by adding some new axioms. A categorical reformulation of this statement is:

2.2 Remark. $I:S \longrightarrow T$ is a quotient iff it is the universal solution in PT to the problem of inverting some collection Σ of monomorphisms in S, i.e. for all pretoposes P,

$$I^*: PT(T,P) \longrightarrow PT(S,P)$$

is full and faithful with essential image comprising those $F:S \longrightarrow P$ for which $F(\sigma)$ is an isomorphism, all $\sigma \in \Sigma$.

An elegant proof of this characterisation of quotients in PT can be given using "polyadic distributive lattices" as an intermediate step: cf. section 3 of [13]. A detailed discussion of quotients for lex and coherent categories as well as for pretoposes, can be found in section 2.3 of [11].

The collections of conservative and quotient morphisms form a <u>factorization system</u> on the bicategory PT. (See [5] for a precise definition of this concept.) Thus each morphism factors as a quotient followed by a conservative morphism and the two classes are orthogonal. Another factorization system on PT that is perhaps less familiar from the point of view of coherent theories (but is nonetheless very important to the study of global properties of such theories) is that given by the class of subcovering morphisms and the class of morphisms which are both conservative and full on subobjects. Barr and Makkai [2] coin the term <u>powerful embedding</u> for the latter kind of morphism. (Such a morphism is in particular full and faithful.) In fact this factorization is the restriction to pretoposes of the localic-hyperconnected factorization for toposes (cf. [5]): a morphism $I:S \longrightarrow T$ in Pt is subcovering (respectively a powerful embedding) iff the corresponding geometric morphism between coherent toposes is localic (respectively hyperconnected). (Similarly, I is a quotient (repectively conservative) iff the corresponding geometric morphism is an inclusion (respectively surjective).)

<u>2.3 Definition</u>. A collection $\mathcal{V}$ of large pretoposes will be called <u>sufficient</u> for Pt (the bicategory of small pretoposes) if it has the property that for each $T \in \mathrm{Pt}$, the collection of pretopos morphisms $T \longrightarrow V$ with $V \in \mathcal{V}$, is jointly conservative. Thus $f:X \longrightarrow X'$ in T is an isomorphism iff for all $V \in \mathcal{V}$ and all $M \in \mathrm{PT}(T,V)$, $M(f)$ is an isomorphism.

Trivially $\mathcal{V} = \mathrm{Pt}$ is always sufficient for Pt; less trivially, the collection of categories of sheaves on complete Heyting algebras (i.e. localic toposes) is an example of such a $\mathcal{V}$ no matter what base topos we are over (cf. [4], Theorem 7.51). If we are working over the topos of classical sets, then the usual completeness theorem for first order logic implies that we can take $\mathcal{V} = \{\mathrm{Set}\}$ (cf. [11], Theorem 1.2.1).

<u>2.4 Proposition</u>. Suppose that $\mathcal{V} \subseteq \mathrm{PT}$ is sufficient for Pt. Given $I:S \longrightarrow T$ in Pt, if for all $V \in \mathcal{V}$

$$I^*:\mathrm{PT}(T,V) \longrightarrow \mathrm{PT}(S,V)$$

is full, then I is full on subobjects.

__Proof__. Let

$$\begin{array}{ccc} T & \xrightarrow{M} & P \\ {\scriptstyle I}\uparrow & \overset{h}{\Longrightarrow} & \uparrow{\scriptstyle N} \\ S & \xrightarrow[I]{} & T \end{array}$$

be a cocomma square in Pt. Given an object X of S and $B \rightarrowtail I(X)$ in T, we can apply Corollary 1.4 to find $A \rightarrowtail X$ with $I(A) \cong B$ in $\mathrm{Sub}_T(IX)$ provided $\exists h_X(MB) \leq NB$ in $\mathrm{Sub}_P(NIX)$. Since $\mathcal{V}$ is sufficient for Pt, this holds iff for all $V \in \mathcal{V}$ and all $K \in PT(P,V)$, $K(\exists h_X MB) \leq KNB$.

But $Kh: I^*(KM) \longrightarrow I^*(KN)$; so by the hypothesis on I, $Kh = k_I$ for some $k: KM \longrightarrow KN$ in PT(T,V). Then since

$$\begin{array}{ccc} KMB & \rightarrowtail & KMIX \\ {\scriptstyle k_B}\downarrow & & \downarrow{\scriptstyle k_{IX} = Kh_X} \\ KNB & \rightarrowtail & KNIX \end{array}$$

commutes, we have $K(\exists h_X MB) \cong \exists Kh_X(KMB) \leq KNB$, as required.

□

Next we consider the analogue of Proposition 2.4 when the hypothesis becomes: I^* is faithful. It is easy to see that for any pretopos V, a sufficient condition for $I^*: PT(T,V) \longrightarrow PT(S,V)$ to be faithful is that I be subcovering (see Proposition 2.10 below). We shall show that this condition is also necessary.

__2.5 Lemma__. In a pretopos T, an object X subcovers another object Y iff Y is a retract of X in Rel_T, the bicategory of relations of T.

__Proof__. Suppose we have

$$\begin{array}{ccc} Z & \xrightarrow{e} & Y \\ {\scriptstyle m}\downarrow & & \\ X & & \end{array}$$

in T. Letting $R^o: Y \longrightarrow X$ denote the opposite of a relation $R: X \longrightarrow Y$, define

$B = m \cdot e^o : Y \longrightarrow X$

and $C = e \cdot m^o : X \longrightarrow Y.$

Since m is a monomorphism we have $m^o m = id_Z$; and since e is an epimorphism we have $ee^o = id_Y$. Hence

$$CB = em^o me^o = ee^o = id_Y$$

and thus Y is a retract of X in Rel_T.

Conversely, given relations $B: Y \longrightarrow\!\!\!\!\!\!\!+\;\; X$ and $C: X \longrightarrow\!\!\!\!\!\!\!+\;\; Y$ with $CB = id_Y$, then

$$T \models \exists x[B(y_1,x) \wedge C(x,y_2)] \leftrightarrow y_1 = y_2.$$

So letting $A = B^o \wedge C$, in T we have

$$\begin{aligned} A(x,y_1) \wedge A(x,y_2) &\vdash [B(y_1,x) \wedge C(x,y_1)] \wedge [B(y_2,x) \wedge C(x,y_2)] \\ &\vdash B(y_1,x) \wedge C(x,y_2) \\ &\vdash y_1 = y_2, \end{aligned}$$

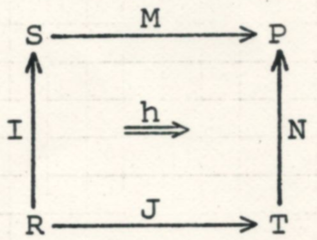
so that $A \rightarrowtail X \times Y \xrightarrow{\pi_1} Y$ is a monomorphism. Similarly

$$\begin{aligned} y = y &\vdash \exists x[B(y,x) \wedge C(x,y)] \\ &\vdash \exists x A(x,y), \end{aligned}$$

so that $A \rightarrowtail X \times Y \xrightarrow{\pi_2} Y$ is an epimorphism. Thus Y is subcovered by X.

□

<u>2.6 Lemma</u>. Suppose that



is a cocomma square in Pt. Given objects Y of S and Z of T, every relation from MY to NZ in P is of the form

$$NC \circ h_X \circ MB$$

for some object X of R and relations $B: Y \longrightarrow\!\!\!\!\!\!\!+\;\; IX$ in S and $C: JX \longrightarrow\!\!\!\!\!\!\!+\;\; Z$ in T.

<u>Proof</u>. This lemma is the finitary version of the remarks after (3) in the proof of Theorem 1.5. We sketch a proof in terms of theories suggested to us by M.Makkai:

Let $\underline{S}$ be a coherent theory in the language of the underlying graph of S whose category of concepts $P(\underline{S})$ is equivalent to S; let $\underline{T}$ be similarly chosen for T. Then as remarked in section 1, $P \simeq P(\underline{P})$ where $\underline{P}$ is the theory comprising disjoint copies $M\underline{S}$ of $\underline{S}$ and $N\underline{T}$ of $\underline{T}$, together with function symbols for the components of h plus axioms saying that it is natural. Thus an element of $Rel_P(MY, NZ)$ is represented by a coherent formula in the language of $\underline{P}$ of sort $MY \times NZ$. An argument by induction on the complexity of such a formula and using the rules of the coherent fragment of first order logic, shows that this formula is $\underline{P}$-provably equivalent to one of the form

$$\bigvee_{i<n} \exists x_i[M\beta_i(y,x_i) \wedge N\gamma_i(h_{X_i}(x_i),z)],$$

where β_i (respectively γ_i) is a coherent formula from $\underline{S}$ (respectively $\underline{T}$) of sort $Y \times IX$ (respectively $JX \times Z$).

Thus every relation $MY \longrightarrow\!\!\!\!\!+\!\!\!\rightarrow NZ$ in P is of the form

$$\bigvee_{i<n} NC_i \circ h_{X_i} \circ MB_i$$

for finitely many objects X_i of R and relations $B_i: Y \longrightarrow\!\!\!\!\!+\!\!\!\rightarrow IX_i$ in S and $C_i: JX_i \longrightarrow\!\!\!\!\!+\!\!\!\rightarrow Z$ in T. But letting

$$X = \coprod_{i<n} X_i$$

be the coproduct of the X_i in R, with coproduct injections $m_i: X_i \rightarrowtail X$ and defining

$$B = \bigvee_{i<n} Im_i \circ B_i \qquad \text{and} \qquad C = \bigvee_{i<n} C_i \circ Im_i^{\circ},$$

since the coproduct is disjoint one has

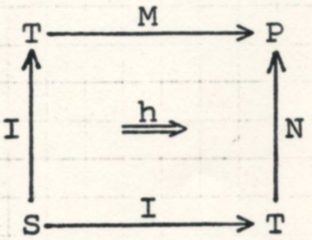
$$\bigvee_{i<n} NC_i \circ h_{X_i} \circ MB_i = NC \circ h_X \circ MB$$

in $Rel_P(MY, NZ)$.

□

Given $I: S \longrightarrow T$ in Pt and an object Y of T, consider the property: "Y is subcovered via I by some object X of S". Evidently it involves quantifying over the objects of S. The next lemma shows that this property is equivalent to a "bounded " one, only involving quantification over the set of subobjects of a particular object in a cocomma pretopos.

<u>2.7. Lemma</u>. Given $I: S \longrightarrow T$ in Pt, form the cocomma square



and let $\nabla: P \longrightarrow T$ be the <u>codiagonal</u> morphism, i.e. a morphism (defined uniquely up to unique isomorphism) for which there are isomorphisms

$$m: \nabla M \cong Id_T \quad \text{and} \quad n: \nabla N \cong Id_T$$

making

$$\begin{array}{ccc} \nabla MI & \xrightarrow{\nabla h} & \nabla NI \\ m_I \wr\| & & \|\wr n_I \\ I & \xrightarrow[id_I]{} & I \end{array} \qquad (10)$$

commute in Pt(S,T).

Then an object Y of T is subcovered via I by some X in S iff there is a relation $D: MY \mathrel{-\!\!\!+\!\!\!\to} NY$ in P making

$$\begin{array}{ccc} \nabla MY & \xrightarrow{\nabla D} & \nabla NY \\ m_Y \wr\| & & \|\wr n_Y \\ Y & \xrightarrow[id_Y]{} & Y \end{array}$$

commute in Rel_T.

<u>Proof</u>. Suppose that I(X) subcovers Y; then by Lemma 2.5 there are relations $B: Y \mathrel{-\!\!\!+\!\!\!\to} IX$ and $C: IX \mathrel{-\!\!\!+\!\!\!\to} Y$ in T with $CB = id_Y$. Put $D = NC \circ h_X \circ MB$. Then by (10) (and using the fact that m and n being natural <u>isomorphisms</u>, they are also natural for relations) we have

$$\begin{aligned} n_Y \nabla D &= n_Y \nabla NC\, \nabla h_X\, \nabla MB \\ &= C\, n_{IX}\, \nabla h_X\, \nabla MB \\ &= C\, m_{IX}\, \nabla MB \\ &= CB\, m_Y \\ &= m_Y \end{aligned}$$

as required.

Conversely suppose we have $D: MY \mathrel{-\!\!\!+\!\!\!\to} NY$ with $n_Y \nabla D = m_Y$. By Lemma 2.6 $D = NC \circ h_X \circ MB$ for some X in S, $B: Y \mathrel{-\!\!\!+\!\!\!\to} IX$ and $C: IX \mathrel{-\!\!\!+\!\!\!\to} Y$. Then as above, we have

$$n_Y \nabla D = n_Y \nabla NC\, \nabla h_X\, \nabla MB = CB\, m_Y,$$

so that $CB = n_Y \nabla D m_Y^{-1}$. But $n_Y \nabla D = m_Y$ and hence $CB = id_Y$. Thus Y is a retract of IX in Rel_T and therefore by Lemma 2.5, is subcovered by IX in T.

□

The following lemma is an elementary consequence of the definition (1.1) of comma categories:

<u>2.8 Lemma</u>. Given a functor $F: B \longrightarrow A$ between categories, consider the diagonal functor $\Delta: B \longrightarrow (F,F)$ defined by

$$\Delta(b \xrightarrow{g} b') = (b, id_{Fb}, b) \xrightarrow{(g,g)} (b', id_{Fb'}, b').$$

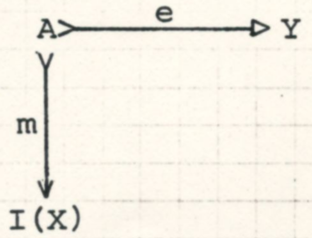
Then F is faithful iff Δ is full.

□

We can now prove:

<u>2.9 Proposition</u>. Suppose $\mathcal{V} \subseteq PT$ is sufficient for Pt. Then $I:S \longrightarrow T$ in Pt is subcovering iff for all $V \in \mathcal{V}$, $I^*:PT(T,V) \longrightarrow PT(S,V)$ is faithful.

<u>Proof</u>. One direction is easy. For suppose I is subcovering and we are given $f,g:J \rightrightarrows K$ in $PT(T,V)$ with $I^*(f) = I^*(g)$, i.e. with $f_I = g_I$. Then for any object Y of T, we can find an object X of S and



in T. Hence

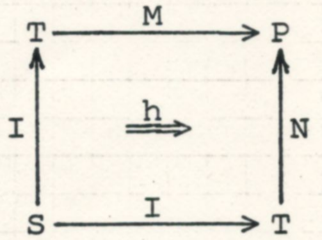
$$Km \cdot f_A = f_{IX} \cdot Jm = g_{IX} \cdot Jm = Km \cdot g_A,$$

so that $f_A = g_A$ (since Km is a monomorphism); and then

$$f_Y \cdot Je = Ke \cdot f_A = Ke \cdot g_A = g_Y \cdot Je,$$

so that $f_Y = g_Y$ (since Je is an epimorphism). Thus $f = g$ as required.

Conversely, suppose that $I^*:PT(T,V) \longrightarrow PT(S,V)$ is faithful for all $V \in \mathcal{V}$. Form the cocomma square



in Pt and let $\nabla:P \longrightarrow T$, $m:\nabla M \cong Id_T$ and $n:\nabla N \cong Id_T$ be as in Lemma 2.7. By Lemma 2.8, the diagonal functor

$$PT(T,V) \longrightarrow (I^*, I^*)$$

is full. But under the equivalence of Definition 1.2, viz

$$PT(P,V) \simeq (I^*, I^*),$$

this diagonal functor is identified with $\nabla^*:PT(T,V) \longrightarrow PT(P,V)$, which is therefore also full. Since this holds for all $V \in \mathcal{V}$, by Proposition 2.4 ∇ is full on subobjects. Hence for any Y in T, the diagonal subobject

$$\langle id, id \rangle : Y \rightarrowtail Y \times Y \cong \nabla(MY \times NY)$$

is in the image of ∇; i.e. there is $D \rightarrowtail MY \times NY$ in P and a commutative square

$$\begin{array}{ccc} Y & \cong & \nabla D \\ \downarrow & & \downarrow \\ Y \times Y & \cong & \nabla(MY \times NY) \end{array}$$

in T. Hence

$$\begin{array}{ccc} \nabla MY & \xrightarrow{\nabla D} & \nabla NY \\ m_Y \wr\| & & \|\wr n_Y \\ Y & \xrightarrow[id_Y]{} & Y \end{array}$$

commutes in Rel_T and so by Lemma 2.7, Y is subcovered via I. Since Y was arbitrary, I is subcovering.

□

2.10 Remark. It is possible to use the pretopos interpolation theorem directly to prove an "object-by-object" version of Proposition 2.9, viz:

> An object Y of T is subcovered via I iff for all $V \in \mathcal{V}$ and all $f,g: J \rightrightarrows K$ in PT(T,V) with $f_I = g_I$, it is also the case that $f_Y = g_Y$.

The "only if" half is as in the first part of the proof of 2.9. For the "if" half, form the cocomma square

$$\begin{array}{ccc} T & \xrightarrow{J} & R \\ \uparrow \nabla & \overset{k}{\Rightarrow} & \uparrow K \\ P & \xrightarrow{\nabla} & T \end{array} \qquad (11)$$

in Pt. Then the pair

$$f: J \cong J\nabla M \xrightarrow{k_M} K\nabla M \cong K,\; g: J \cong J\nabla N \xrightarrow{k_N} K\nabla N \cong N$$

is universal with the property $f_I = g_I$. So the hypothesis implies that $f_Y = g_Y$ (since $\mathcal{V}$ is sufficient for Pt); and from this it follows that in R, the image of $J(Y \rightarrowtail Y \times Y \cong \nabla(MY \times NY))$ along $k_{MY \times NY}$ is contained in $K(Y \rightarrowtail Y \times Y \cong \nabla(MY \times NY))$. Then Corollary 1.4 applied to (11) gives us that $Y \rightarrowtail Y \times Y \cong \nabla(MY \times NY)$ is in the image of ∇, which as in the proof of 2.9 implies that Y is subcovered via I.

2.11 Example. In Proposition 2.9 take $\mathcal{V} = \{Set\}$ and S to be the initial pretopos, viz the category of finite sets. Then the hypothesis "$I^*: PT(T,Set) \longrightarrow PT(S,Set) (\simeq \mathbb{1})$ is faithful" becomes "Mod(T) is preordered". The conclusion that I is subcovering, in this case is equivalent to saying that the classifying topos E(T) of T is localic (cf. the comments before Definition 2.3); and since E(T) has enough points, it is localic iff it is equivalent to the category of sheaves on a topological space. Thus a particular case of Proposition 2.9 is the fact that a coherent topos is spatial iff its category of points is pre-ordered. (M.-F.Coste-Roy and M.Coste [3] derived this result from the Makkai-Reyes conceptual completeness theorem and used it to prove that the real étale spectrum of a ring is spatial.) This property of coherent toposes does not extend to Grothendieck toposes in general, in as much as the analogue of Proposition 2.9 for <u>infinitary</u> geometric logic fails. We produce here a simple example, due to M.Makkai, of <u>a non-localic topos</u> E <u>with the property that for all</u> $F \in$ GTOP, GTOP(F,E) <u>is pre-ordered</u>.

E is the classifying topos of a geometric theory $\underline{T}$ whose models in Set are subsets X of $P(\mathbb{N})$, the powerset of the set of natural numbers. Specifically $\underline{T}$ has one sort X; countably many unary relations $E_n(x)$ ("$n \in x$") and $C_n(x)$ ("$n \notin x$"); and for each $n \in \mathbb{N}$, geometric axioms of the following two kinds:

"C_n is the complement of E_n" (12)

and geometric axioms equivalent to

$$\forall x_1, x_2 [x_1 = x_2 \leftrightarrow \bigwedge_{n\in\mathbb{N}} (E_n(x_1) \leftrightarrow E_n(x_2))]. \qquad (13)$$

(It is possible to achieve (13) in the presence of (12).)

Note that since $\underline{T}$ is countably presented, if E is localic it can have at most $2^{\aleph_0}$ mutually non-isomorphic points. But $\underline{T}$ has $2^{2^{\aleph_0}}$ mutually non-isomorphic models in Set. So E is not localic. But on the other hand, for any $F \in$ GTOP GTOP(F,E) $\cong$ Mod($\underline{T}$,F) is pre-ordered: for given homomorphisms of $\underline{T}$-models in F, $f,g: M \rightrightarrows N$, by (12) ME_n has a complement preserved by f and g, so that

$$F \models NE_n(f_X(x)) \leftrightarrow ME_n(x) \leftrightarrow NE_n(g_X(x))$$

and hence by (13)

$$F \models f_X(x) = g_X(x),$$

i.e. $f = g$.

Combining Propositions 2.4 and 2.9 we obtain a constructive version of what in [11] is termed <u>strong</u> conceptual completeness for pretoposes:

<u>2.12 Theorem</u>. Let $\mathfrak{S}$ be an elementary topos with natural number object and let PT (respectively Pt) denote the 2-category of large (respectively small) pretoposes in $\mathfrak{S}$. Suppose that $\mathcal{V} \subseteq$ PT is sufficient for Pt (in the sense of Definition 2.3). Then for a morphism $I: S \longrightarrow T$ in Pt to be a quotient, it is sufficient that $I^*: PT(T,V) \longrightarrow PT(S,V)$ be full and faithful for each $V \in \mathcal{V}$.

□

An immediate corollary is a constructive version of the usual pre-topos conceptual completeness theorem:

<u>2.13 Theorem</u>. Let $\mathcal{S}$, PT, Pt and $\mathcal{V}$ be as in Theorem 2.12. For $I: S \longrightarrow T$ in Pt to be an equivalence, it is sufficient that $I^*: PT(T,V) \longrightarrow PT(S,V)$ be an equivalence for each $V \in \mathcal{V}$.

<u>Proof</u>. I is an equivalence iff it is both a quotient and conservative. Theorem 2.12 gives the former. For the latter, if $f: X \longrightarrow X'$ is in S and I(f) is an isomorphism, then for each $V \in \mathcal{V}$ and each $M \in PT(S,V)$, since I^* is essentially surjective $M \cong NI$ for some $N \in PT(T,V)$ and hence M(f) is also an isomorphism. Then since $\mathcal{V}$ is sufficient for Pt, f must itself be an isomorphism.

□

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