Journal of Pure and Applied Algebra 29 (1983) 313-326
North-Holland

AN APPLICATION OF OPEN MAPS TO CATEGORICAL LOGIC

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Communicated by P.T. Johnstone
Received 5 January 1983

Introduction

This paper is a sequel to [12]. We are here concerned with properties of theories in full first-order intuitionistic logic; the latter correspond under the identification of theories with categories provided by categorical logic (cf. [8] or [11]), to *Heyting* pretoposes, i.e. pretoposes with universal quantification of subobjects along morphisms. Using the lattice-theoretic machinery developed in [12], we construct a contravariant functor $\Phi:\mathbf{Pt}^{\mathrm{op}}\to\mathbf{Top}$ from the category of pretoposes to the category of Grothendieck toposes, which sends a morphism of Heyting pretoposes to an *open* geometric morphism. This functor allows us to deduce from the fact that open geometric surjections are stable under pullback, that conservative morphisms in the category of Heyting pretoposes are stable under pushout. From this it follows easily that every pushout square in that category has the 'interpolation property'. From the point of view of theories, we thus obtain an essentially very simple, constructive proof of a general form of Craig's Interpolation Theorem. At the end of the paper we make some remarks about the analogues of these properties for the coherent fragment of intuitionistic logic (i.e. for pretoposes).

There are two important ingredients in the construction of the functor $\Phi:\mathbf{Pt}^{\mathrm{op}}\to\mathbf{Top}$. The first is the use of 'indexed lattice theory' as a bridge between propositional and predicate logic: by indexed lattice theory we mean the pre-order part of indexed category theory (cf. [2]). Specifically, we make use of particular kinds of hyperdoctrines which, following Joyal [5], we call *polyadic distributive lattices* and *polyadic Heyting algebras*. The second ingredient is the use of the constructive theory of locales in toposes other than the topos of sets (cf. [7] in particular): in Section 2 we construct locales and (open) continuous maps in various toposes of presheaves. (Indeed, all the arguments given in this paper can be carried out over an arbitrary base topos with natural number object; in particular we never need to resort to the Completeness Theorem or its equivalents.)

1. Polyadic lattice theory

Recall that a category T is a *pretopos* iff it has finite limits, stable images, quotients of equivalence relations and stable disjoint finite sums; a morphism of pretoposes is a functor preserving this structure. Let **Pt** denote the category of (small) pretoposes and morphisms. A *Heyting pretopos* is a pretopos T in which for each $\alpha: X \to Y$, the operation of pulling back subobjects along α, $\alpha^{-1}: \mathrm{Sub}_T(Y) \to \mathrm{Sub}_T(X)$, has a right adjoint $\forall\alpha: \mathrm{Sub}_T(X) \to \mathrm{Sub}_T(Y)$, called 'universal quantification along α'. (Note that this condition implies that each lattice of subobjects $\mathrm{Sub}_T(X)$ is a Heyting algebra.) A morphism of Heyting pretoposes is of course a pretopos morphism which preserves this additional structure. Let **HPt** denote the category of (small) Heyting pretoposes.

In [11] it is shown how we may identify theories in first order intuitionistic logic with Heyting pretoposes and interpretations between theories with morphisms of such; similarly for the coherent fragment $(\wedge, \vee, \exists)$ of intuitionistic logic and the category **Pt**. We can split the passage from theory to category into two stages:

(a) organise the types, terms and formulae into a 'polyadic Lindenbaum algebra' of the theory;

(b) given a polyadic algebra, construct its associated 'syntactic' category.

The kind of structures that arise at stage (a) are the following:

1.1. Definition. Let C be a (small) cartesian category (i.e. C has finite limits) and let **Dl** denote the category of distributive lattices. Then a *polyadic distributive lattice* over C is a functor $A: C^{\mathrm{op}} \to \mathbf{Dl}$ such that for $\alpha: I \to J$ in C, $A\alpha: AJ \to AI$ has a left adjoint $\exists^A\alpha: AI \to AJ$ satisfying

(FR) *Frobenius reciprocity*: if $\phi \in AI$ and $\psi \in AJ$ then

$$\exists^A\alpha(\phi \wedge A\alpha(\psi)) = \exists^A\alpha(\phi) \wedge \psi;$$

(BC) *Beck–Chevalley condition*: if

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & J \arrow[d, "\beta"] \\ I \arrow[r, "\alpha"] & K \end{tikzcd}$$

is a pullback square in C, then $A\beta \circ \exists^A\alpha = \exists^A\delta \circ A\gamma$.

A is a *polyadic Heyting algebra* iff in addition to the above, each AI is a Heyting algebra and each $A\alpha: AJ \to AI$ has a right adjoint $\forall^A\alpha: AI \to AJ$.

If A and B are polyadic distributive lattices over C, a morphism between them is a natural transformation $f: A \to B$ (in the functor category $[C^{\mathrm{op}}, \mathbf{Dl}]$) which 'preserves $\exists$' in the sense that for each $\alpha: I \to J$ in C we have $\exists^B\alpha \circ f_I = f_J \circ \exists^A\alpha$. A morphism of polyadic Heyting algebras should in addition preserve $\to$

and $\forall$. Let **pDl**(C) (respectively **pHa**(C)) denote the category of polyadic distributive lattices (respectively polyadic Heyting algebras) over C.

1.2. Remarks. (i) Since $A\alpha : AJ \to AI$ preserves $\to$ iff $\exists^A\alpha : AI \to AJ$ satisfies (FR), and since $\exists^A$ satisfies (BC) iff $\forall^A$ does, we can define a polyadic Heyting algebra over C to be a contravariant functor $A : C^{\mathrm{op}} \to \mathbf{Ha}$ from C to the category of Heyting algebras, such that each $A\alpha$ has left and right adjoints, the latter satisfying (BC).

(ii) The structure and axioms of such polyadic algebras are due to Lawvere and it is his observation (and no small one) that they embody in a concise, *algebraic* form exactly the language and rules of first order intuitionistic predicate logic. The types and terms become the objects and morphisms of C (which for convenience we have assumed to have equalizers as well as finite products); the formulae (or rather, provable equivalence classes of them) become the elements of the AI, and the propositional connectives $\top, \wedge, \bot, \vee$, become the lattice-theoretic operations; substitution of terms in formulae becomes the maps $A\alpha : AJ \to AI$ and quantification appears as the adjoints to these maps; the distinguished relation of equality at type I is definable as $\exists^A\Delta(\top)$ (where $\Delta : I \to I \times I$ is the diagonal map); and finally (FR) and (BC) ensure that substitution and quantification 'commute' in the correct manner and that equality has the requisite first order properties. See [9] for more details.

Let us now consider stage (b) of the transition from theory to category: given a polyadic algebra A over C, construct a category $C[A]$ in which the abstract predicates of the lattices AI are realized as actual subobjects in $C[A]$. If A is a polyadic distributive lattice then $C[A]$ should be a pretopos, if A a polyadic Heyting algebra then $C[A]$ a Heyting pretopos. Let C-**HPt** denote the 2-category whose objects are cartesian functors $L : C \to T$ from C to a Heyting pretopos T, and whose morphisms are triangles

$$\begin{tikzcd} & T_1 \arrow[dd, "F"] \\ C \arrow[ur, "L_1"] \arrow[dr, "L_2"'] \arrow[r, phantom, "\cong"] & \\ & T_2 \end{tikzcd}$$

commuting up to a specified isomorphism and with $F : T_1 \to T_2$ in **HPt**. Then there is a functor $\mathrm{Sub} : C\text{-}\mathbf{HPt} \to \mathbf{pHa}(C)$ which is defined on objects by sending $L : C \to T$ to $\mathrm{Sub}_T \circ L^{\mathrm{op}} : C^{\mathrm{op}} \to \mathbf{Ha}$. We have the following result (there is a similar proposition about $\mathrm{Sub} : C\text{-}\mathbf{Pt} \to \mathbf{pDl}(C)$):

1.3. Proposition. $\mathrm{Sub} : C\text{-}\mathbf{HPt} \to \mathbf{pHa}(C)$ *has a full and faithful left adjoint* $\Delta : \mathbf{pHa}(C) \to C\text{-}\mathbf{HPt}$ *whose value at the object A we denote by* $\Delta_A : C \to C[A]$. *We* *can identify* **pHa**(C) *via this left adjoint with the full subcategory of* C-**HPt** *whose objects are those* $L: C \to T$ *which are dense, in the sense that any object* X *of* T *is the subquotient of a finite sum of objects from* C:

$$\begin{tikzcd} \cdot \arrow[r, two heads, "e"] \arrow[d, tail, "m"] & X \\ \coprod_{k<n} L(I_k) & \end{tikzcd}$$

(m *is mono and* e *is epi*).

Proof. We give a sketch of the construction of $C[A]$: the details are analogous to those in Section 2 of [1], except that we first have to extend C and A to ensure that the resulting category has (stable, disjoint) finite sums. Accordingly, let C^+ be the result of formally adding finite coproducts to C; we extend A to a polyadic Heyting algebra A^+ over C^+ by defining

$$A^+\left(\coprod_{k<n} I_k\right) = \prod_{k<n} A(I_k)$$

(product of Heyting algebras), and similarly for morphisms of C^+. Now define $C[A]$ to be the category of 'models of equality' in A^+: the objects are pairs (I, E) where I is an object of C^+ and $E \in A^+(I \times I)$ is symmetric and transitive, whilst a morphism from (I_1, E_1) to (I_2, E_2) is given by an element F of $A^+(I_1 \times I_2)$ which is a strict functional relation for the given equalities E_1 and E_2. (From the point of view of Freyd's 'allegories', $C[A]$ is obtained by first splitting the symmetric idempotents in the category of relations associated to A^+ and then taking the corresponding category of maps.)

The value of the functor $\Delta_A : C \to C[A]$ at an object I is defined to be I together with the standard equality relation $\exists^{A^+}\Delta(\top)$ (cf. 3.7 of [1]). Subobjects of $\Delta_A(I)$ in $C[A]$ are in correspondence with the elements of $A(I)$. Specifically, there is an isomorphism $\eta_A : A \cong \mathrm{Sub}(\Delta_A)$, natural in A, and the assignment $F \mapsto \mathrm{Sub}(F) \circ \eta_A$ gives a natural equivalence of categories

$$C\text{-}\mathbf{HPt}(\Delta_A, L) \simeq \mathbf{pHa}(C)(A, \mathrm{Sub}(L)).$$

We thus have a left adjoint to Sub in the sense appropriate to 2-categories, which is full and faithful since the unit η of the adjunction is an isomorphism.

Finally, given $L: C \to T$ in C-**HPt**, the counit of the adjunction at L, $\varepsilon_L : C[\mathrm{Sub}(L)] \to T$, is always full and faithful, and is essentially surjective just in case L is dense in the sense defined. (In fact not only is ε_L full and faithful but also its image is closed under taking subobjects in T: the factorization of L as

$\varepsilon_L \circ \Delta_{\mathrm{Sub}(L)}$ is entirely analogous to the hyperconnected-localic factorization of a geometric morphism; cf. [4].) □

A polyadic Heyting algebra A over C is in particular a Heyting algebra in the topos $[C^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on C. Whilst it is not in general a *complete* Heyting algebra, it does have a certain amount of internal completeness. Specifically $\alpha^*: A^Y \to A^X$ in $[C^{\mathrm{op}}, \mathbf{Set}]$ has left and right adjoints when $\alpha: X \to Y$ is a morphism between *representable* presheaves.

1.4. Proposition. *A locale in* $[C^{\mathrm{op}}, \mathbf{Set}]$ *is a functor* $A: C \to \mathbf{Loc}$ *such that for each* $\alpha: I \to J$ *in* C, $(A\alpha)^*: AJ \to AI$ *has a left adjoint* $(A\alpha)_!: AI \to AJ$ *satisfying the conditions* (FR) *and* (BC) *of* 1.1.

A continuous map between locales A *and* B *in* $[C^{\mathrm{op}}, \mathbf{Set}]$ *is a natural transformation* $f: A \to B$ *in* $[C, \mathbf{Loc}]$ *such that* f^* *preserves* $(-)_!$, *i.e.* $(f_J)^* \circ (B\alpha)_! = (A\alpha)_! \circ (f_I)^*$ *for* $\alpha: I \to J$ *in* C. *In particular this implies that* $(f_I)_*: AI \to BI$ *is natural in* I, *so that* $f^*: B \to A$ *has a* (C-*indexed*) *right adjoint* $f_*: A \to B$. *By definition, the continuous map* $f: A \to B$ *is open iff* f^* *has a left adjoint* $f_!: A \to B$ *satisfying* (FR), *and this is true just in case* f^* *preserves* $\to$, $\bigvee$ *and* $(-)_*$. □

A detailed proof of this proposition may be found in Chapter VI of [7]. Note that a polyadic Heyting algebra A over C, is a locale in $[C^{\mathrm{op}}, \mathbf{Set}]$ just when each AI is a complete lattice; in this case we use the notations $\exists^A\alpha$, $A\alpha$, $\forall^A\alpha$ and $(A\alpha)_!$, $(A\alpha)^*$, $(A\alpha)_*$ interchangeably.

Now recall from [12] the functors $\mathscr{F}$, $\mathscr{I}$ and $\phi = \mathscr{I} \circ \mathscr{F}$ assigning to a distributive lattice its lattice of filters, the locale of ideals and the locale of ideals of filters. We noted in Section 2 of that paper that these functors preserve the relationships of adjointness, (FR) and (BC). It follows that if $A: C^{\mathrm{op}} \to \mathbf{Dl}$ is a polyadic distributive lattice, then $\mathscr{F} \circ A: C^{\mathrm{op}} \to \mathbf{Dl}$ is another such, and $\mathscr{I} \circ A$ and $\phi \circ A$ are locales in $[C^{\mathrm{op}}, \mathbf{Set}]$. Similarly, given $f: A \to B$ in $\mathbf{pDl}(C)$, then $\mathscr{F}f: \mathscr{F} \circ A \to \mathscr{F} \circ B$ is again a morphism in $\mathbf{pDl}(C)$, whilst $\mathscr{I}f: \mathscr{I} \circ B \to \mathscr{I} \circ A$ and $\phi f: \phi \circ B \to \phi \circ A$ are continuous maps of locales in $[C^{\mathrm{op}}, \mathbf{Set}]$. This is because existential quantification is given in ϕA (for example) by $\mathscr{I}\mathscr{F}\exists^A$ and is thus preserved by $(\phi f)_I^* = \mathscr{I}\mathscr{F}(f_I)$, since f_I preserves $\exists^A$. Moreover if f is a morphism of polyadic Heyting algebras over C, then since f_I preserves $\forall^A$, $(\phi f)^*$ preserves universal quantification in $\phi \circ A$: hence by Theorem 2.3 of [12], $\phi f: \phi B \to \phi A$ is an *open* continuous map of locales in $[C^{\mathrm{op}}, \mathbf{Set}]$. We thus get a functor

$$\phi: \mathbf{pHa}(C)^{\mathrm{op}} \to \mathbf{OLoc}[C^{\mathrm{op}}, \mathbf{Set}].$$

Note that just as in the quoted theorem, if $f: A \to B$ is a monomorphism in $\mathbf{pHa}(C)$ (i.e. each f_I is a monomorphism in $\mathbf{Ha}$) then $\phi f: \phi B \to \phi A$ is a continuous surjection. Also the natural monomorphism i, mentioned in that theorem gives for each polyadic Heyting algebra A, a natural monomorphism $i: A \rightarrowtail \phi A$ which preserves $\exists$ and $\forall$ (by definition of the quantifiers in $\phi \circ A$) as well as the lattice operations, and so is a monomorphism in $\mathbf{pHa}(C)$, natural in A.

2. The topos of filters of a pretopos

Suppose that T is a pretopos. Then $\mathrm{Sub}_T: T^{\mathrm{op}} \to \mathbf{Dl}$ is a polyadic distributive lattice over T and so from Section 1, $\mathcal{I} \circ \mathrm{Sub}_T$ is a locale in $[T^{\mathrm{op}}, \mathbf{Set}]$. A simple calculation shows that it is isomorphic to the locale of j-closed sieves for the precanonical topology on T. So taking the topos of sheaves on this locale, $\mathrm{sh}(\mathcal{I} \circ \mathrm{Sub}_T)$, we obtain nothing other than the classifying topos $\mathscr{E}(T)$, of T (cf. [11]). However $\phi \circ \mathrm{Sub}_T$ is also a locale in $[T^{\mathrm{op}}, \mathbf{Set}]$ and we can consider the topos of sheaves on that: call it $\Phi(T)$. Since $\phi \circ \mathrm{Sub}_T$ is $\mathcal{I}$ applied to the polyadic distributive lattice $\mathscr{F} \circ \mathrm{Sub}_T$, if follows that $\phi(T)$ is the classifying topos of the pretopos $T[\mathscr{F} \circ \mathrm{Sub}_T]$ (notation as in 1.3). Alternatively we can describe it as the topos of sheaves on the site consisting of the 'category of filters' of T as defined after the proof of Theorem 1.1 in [10], with the precanonical topology. On applying $\mathcal{I}$ to the monomorphism $\uparrow : \mathrm{Sub}_T \rightarrowtail \mathscr{F} \circ \mathrm{Sub}_T$ in $\mathbf{pDl}(T)$, we obtain a surjective continuous map of locales $\phi \circ \mathrm{Sub}_T \to \mathcal{I} \circ \mathrm{Sub}_T$ and this induces a (localic) surjection $\Phi(T) \to \mathscr{E}(T)$ between toposes.

Restricting attention to Heyting pretoposes, we have:

2.1. Theorem. *The assignment $T \mapsto \Phi(T)$ extends to a contravariant functor $\Phi : \mathbf{HPt}^{\mathrm{op}} \to \mathbf{OTop}$ from the category of Heyting pretoposes to the category of Grothendieck toposes and open geometric morphisms. This functor takes conservative morphisms in* **HPt** *to geometric surjections.*

Moreover, for each object T of **HPt** *there is a conservative morphism $I_T : T \to \Phi(T)$ of Heyting pretoposes which is natural in T.*

Remarks. (i) Recall that a geometric morphism $f : \mathscr{F} \to \mathscr{E}$ between Grothendieck toposes is *open* iff when we take its hyperconnected-localic factorization $\mathscr{F} \to \mathrm{sh}_{\mathscr{E}}(f_*\Omega_{\mathscr{F}}) \to \mathscr{E}$, $f_*\Omega_{\mathscr{F}}$ is an open locale in $\mathscr{E}$, i.e. the unique continuous map $f_*\Omega_{\mathscr{F}} \to \Omega_{\mathscr{E}}$ is open; cf. [7]. Equivalently, f is open iff $f^* : \mathscr{E} \to \mathscr{F}$ preserves universal quantification and hence is in particular a morphism of Heyting pretoposes; cf. [3].

(ii) A morphism $L : S \to T$ in **Pt** is *conservative* iff whenever we have $A, B \in \mathrm{Sub}_S(I)$ with $LA \leq LB$ in $\mathrm{Sub}_T(LI)$, then $A \leq B$ in $\mathrm{Sub}_S(I)$. This accords with the usual notion of conservative extension of theories. Of course L is conservative iff it reflects isomorphisms iff it is faithful. When L is in **HPt**, we only have to check that

$$L(U) = \top \;\Rightarrow\; U = \top, \text{ all } U \in \mathrm{Sub}_S(1)$$

for it to be conservative.

Proof of 2.1. Given $L : S \to T$ in **HPT**, regarding it as a morphism $\mathrm{Id}_S \to L$ in S-**HPt** and applying the functor $\mathrm{Sub} : S\text{-}\mathbf{HPt} \to \mathbf{pHa}(S)$, we obtain a morphism $\lambda : \mathrm{Sub}_S \to \mathrm{Sub}(L) = \mathrm{Sub}_T \circ L^{\mathrm{op}}$ whose component at an object I of S sends a subobject $A \rightarrowtail I$ to $LA \rightarrowtail LI$. Then $\phi\lambda : \phi\mathrm{Sub}(L) \to \phi\mathrm{Sub}_S$ is an open continuous map of locales in

$[S^{op}, \mathbf{Set}]$. Now L also induces a geometric morphism $l : [T^{op}, \mathbf{Set}] \to [S^{op}, \mathbf{Set}]$ where l_* is precomposition with L^{op} and l^* is left Kan extension along L^{op}. Then since

$$l_*(\phi \mathrm{Sub}_T) = \phi \circ \mathrm{Sub}_T \circ L^{op} = \phi \circ \mathrm{Sub}(L),$$

it follows that the hyperconnected-localic factorization of

$$\Phi(T) \to [T^{op}, \mathbf{Set}] \xrightarrow{l} [S^{op}, \mathbf{Set}]$$

is of the form

$$\Phi(T) \xrightarrow{h} \mathrm{sh}(\phi \circ \mathrm{Sub}(L)) \to [S^{op}, \mathbf{Set}].$$

We define $\Phi(L) : \Phi(T) \to \Phi(S)$ to be the composite

$$\Phi(T) \xrightarrow{h} \mathrm{sh}(\phi \circ \mathrm{Sub}(L)) \xrightarrow{\phi\lambda} \mathrm{sh}(\phi \circ \mathrm{Sub}_S) = \Phi(S).$$

Since h is hyperconnected and $\phi\lambda$ open, $\Phi(L)$ is an open geometric morphism. This definition does indeed make Φ into a (pseudo)functor $\mathbf{HPt}^{op} \to \mathbf{OTop}$. If L is conservative, then by definition λ is a monomorphism, so $\phi\lambda$ is a surjection of locales and hence $\Phi(L)$ is a geometric surjection.

Define $I_T : T \to \Phi(T)$ to be the composition of the Yoneda embedding with the constant sheaf functor:

$$I_T : T \xrightarrow{H} [T^{op}, \mathbf{Set}] \xrightarrow{\Delta} \mathrm{sh}(\phi \circ \mathrm{Sub}_T) = \Phi(T).$$

Since I_T is cartesian we can form the polyadic Heyting algebra $A = \mathrm{Sub}(I_T)$ over T. For each object X of T we have

$$A(X) \cong \Phi(T)(\Delta H_X, \Omega) \cong [T^{op}, \mathbf{Set}](H_X, \phi \circ \mathrm{Sub}_T) \cong \phi \circ \mathrm{Sub}_T(X),$$

giving an isomorphism $\phi \circ \mathrm{Sub}_T \cong A$ in $\mathbf{pHa}(T)$. Under this isomorphism the natural transformation $\mathrm{Sub}_T \to A$ sending $U \in \mathrm{Sub}_T(X)$ to $I_T(U) \in A(X)$ is (necessarily) identified with the monomorphism $i : \mathrm{Sub}_T \rightarrowtail \phi \circ \mathrm{Sub}_T$ in $\mathbf{pHa}(T)$ defined at the end of Section 1. It follows that I_T is a conservative morphism of Heyting pretoposes. The naturality of I_T is a simple calculation which we omit. □

2.2. Corollary. *Conservative morphisms are stable under pushout in* **HPt**.

Proof. The proof in [12] that monomorphisms in **Ha** are stable under pushout hinged on the fact that the pullback in **Loc** of an open (surjective) continuous map is again open (surjective). Here we use the corresponding property of geometric morphisms between Grothendieck toposes, for a proof of which see [7] or [3].

Suppose we have morphisms $K: R\to S$ and $L: R\to T$ in **HPt**. Applying the functor $\Phi: \mathbf{HPt}^{\mathrm{op}}\to\mathbf{OTop}$, let

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "q"] \arrow[d, "p"] & \Phi T \arrow[d, "\Phi(L)"] \\
\Phi S \arrow[r, "\Phi(K)"] & \Phi R
\end{tikzcd}$$

be a pullback square in **Top**. Since $\Phi(K)$ and $\Phi(L)$ are open, so are p and q, and we therefore obtain a square of Heyting pretopos morphisms

$$\begin{tikzcd}
S \arrow[r, "I_S"] & \Phi S \arrow[r, "p^*"] & \mathcal{E} \\
 & & \Phi T \arrow[u, "q^*"] \\
R \arrow[uu, "K"] \arrow[rr, "L"] & & T \arrow[u, "I_T"]
\end{tikzcd}$$

which commutes up to isomorphism by the naturality of I. Then if K is conservative, by 2.1 $\Phi(K)$ is an open surjection and hence so is q; therefore $q^*\circ I_T$ is conservative. But since the pushout of K along L in **HPt** factors through $q^*\circ I_T$, that pushout is conservative. ($\mathcal{E}$ is a 'large' Heyting pretopos, but this creates no difficulty.) $\square$

3. The interpolation property

The Interpolation Theorem for the intuitionistic predicate calculus (IPC) states that if ϕ and ψ are sentences in some (many-sorted) language $\mathcal{L}$ such that $\mathrm{IPC}\vdash\phi\to\psi$, then there is a sentence θ of $\mathcal{L}$ involving only the sorts, relation and function symbols common to both ϕ and ψ with $\mathrm{IPC}\vdash\phi\to\theta$ and $\mathrm{IPC}\vdash\theta\to\psi$. More generally we make the following definition:

3.1. Definition. Suppose that

$$\begin{tikzcd}
S \arrow[r, "M"] & P \\
R \arrow[u, "K"] \arrow[r, "L"] \arrow[ur, phantom, "\cong"] & T \arrow[u, "N"]
\end{tikzcd}$$

is a square of morphisms in **HPt** commuting up to isomorphism. We say that it has the *interpolation property at an object X of R* iff given $V \in \mathrm{Sub}_S(KX)$ and $W \in \mathrm{Sub}_T(LX)$ with $MV \leq NW$ as subobjects of $MKX \cong NLX$, there is $U \in \mathrm{Sub}_R(X)$ with $V \leq KU$ in $\mathrm{Sub}_S(KX)$ and $LU \leq W$ in $\mathrm{Sub}_T(LX)$. The square has the *interpolation property* iff it has it at each object X of R.

Denoting the free Heyting pretopos on a language $\mathscr{L}$ by $F(\mathscr{L})$, we can interpret the Interpolation Theorem as saying that a pushout square of the form

$$\begin{tikzcd} F(\mathscr{L}_1) \arrow[r] & F(\mathscr{L}_1 \cup \mathscr{L}_2) \\ F(\mathscr{L}_1 \cap \mathscr{L}_2) \arrow[u] \arrow[r] & F(\mathscr{L}_2) \arrow[u] \end{tikzcd}$$

has the interpolation property (at 1). We shall show below that in fact every pushout square in **HPt** has the interpolation property. To do this we need some facts about quotients of Heyting pretoposes. At the level of theories, quotienting corresponds to adding some new axioms without changing the language; at the level of categories, it corresponds to forcing a collection of monomorphisms to be isomorphisms. Which monomorphisms in S are sent by a morphism $L : S \to T$ to isomorphisms in T, is completely determined by the filter of subobjects of 1

$$\ker(L) = \{U \in \mathrm{Sub}_S(1) \mid L(U) \rightarrowtail 1 \text{ iso}\}$$

(since S has and L preserves $\forall$). Conversely, if σ is a filter of subobjects of 1 in S, there is a morphism $Q : S \to S/\sigma$ in **HPt** with the property that

$$Q^* : \mathbf{HPt}(S/\sigma, T) \to \mathbf{HPt}(S, T)$$

is full and faithful and has essential image the full subcategory whose objects are those functors $L : S \to T$ with $\sigma \subseteq \ker(L)$. We can construct Q as $\Delta_A : S \to S[A]$ (cf. 1.3), where A is the polyadic Heyting algebra over S with

$$A(X) = \mathrm{Sub}_S(X)/\sigma_X \quad \text{(quotient of Heyting algebras)}$$

and

$$\sigma_X = \{U \in \mathrm{Sub}_S(X) \mid \forall_X(U) \in \sigma\}.$$

Alternatively we can think of S/σ as the filtered colimit of slice categories: $S/\sigma \cong \varinjlim_{U \in \sigma}(S/U)$. Call morphisms of the form $Q : S \to S/\sigma$ *quotient* morphisms: they are characterised by the two properties

(a) Q is full on subobjects, i.e. given $V \rightarrowtail Q(X)$, there is $U \rightarrowtail X$ with $QU \cong V$;

(b) every object Y of the codomain is covered by one in the domain via Q, i.e. there is an epimorphism $Q(X) \twoheadrightarrow Y$.

Since a morphism $L : S \to T$ in **HPt** is conservative iff $\ker(L)$ is trivial, any L factors as a quotient followed by a conservative morphism, viz. $L : S \to S/\ker(L) \to T$. Moreover the class of quotient morphisms is orthogonal to the class of conservative morphisms; cf. [4]. (Taking (a) and (b) above as the definition of quotient morphism, this is also true of the category of pretoposes, although quotients there correspond not to filters but to more complicated sets of monomorphisms: see [6].) The behaviour of quotients in **HPt** under factorization and pushout is precisely analogous to that for Heyting algebra quotients noted in 3.1 of [12].

3.2. Theorem. *Every pushout square in* **HPt** *has the interpolation property.*

Proof. First note that a pushout square

$$\begin{tikzcd}
S \arrow[r, "M"] & P \\
R \arrow[u, "K"] \arrow[r, "L"] & T \arrow[u, "N"']
\end{tikzcd}$$

has the interpolation property at an object X of R iff the pushout square obtained by slicing

$$\begin{tikzcd}
S/KX \arrow[r] & P/MKX \\
R/X \arrow[u] \arrow[r] & T/LX \arrow[u]
\end{tikzcd}$$

has it at 1. It therefore suffices to prove that every pushout square has the interpolation property at 1.

Suppose then that we have $V \rightarrowtail 1$ in S in S and $W \rightarrowtail 1$ in T with $MV \leq NW$ in P. Define filters

$$\begin{aligned}
&\sigma = \uparrow(V) = \{V' \in \mathrm{Sub}_S(1) \mid V \leq V'\}, \\
&\varrho = K^{-1}(\sigma) = \{U \in \mathrm{Sub}_R(1) \mid KU \in \sigma\}, \\
&\tau = \mathscr{F}L(\varrho) = \{W' \in \mathrm{Sub}_T(1) \mid \exists U \in \varrho \ LU \leq W'\}, \\
&\pi = \uparrow(MV).
\end{aligned}$$

Just as in Theorem B of [12], quotienting by these filters we obtain a pushout square

$$\begin{tikzcd}
S/\sigma \arrow[r, "\hat{M}"] & P/\pi \\
R/\varrho \arrow[u, "\hat{K}"] \arrow[r, "\hat{L}"] & T/\tau \arrow[u, "\hat{N}"']
\end{tikzcd}$$

with $\bar{R}$ conservative. So by 2.2, $\bar{N}$ is conservative. Then since $NW \in \pi$, we have $W \in \tau$, i.e. there is $U \rightarrowtail 1$ in R with $V \leq KU$ and $LU \leq W$, as required. $\square$

Corollary. **The Beth Definability Theorem.** *Given an interpretation of one theory in another, that is a morphism $I: S \to T$ in* **HPt**, *the generic pair of models M_1, M_2 of T which are isomorphic when restricted along I, $I^*M_1 \cong I^*M_2$, is given by the pushout of I along itself:*

$$\begin{tikzcd} T \arrow[r, "M_1"] & P \\ S \arrow[u, "I"] \arrow[r, "I"'] \arrow[ur, phantom, "\cong"] & T \arrow[u, "M_2"'] \end{tikzcd}$$

Then Beth's theorem says that this pushout square has the property: given $V \rightarrowtail I(X)$ in T, if $M_1(V) = M_2(V)$ as subobjects of $M_1IX \cong M_2IX$, then there is $U \rightarrowtail X$ in S with $I(U) = V$ in $\mathrm{Sub}_T(IX)$.

This is of course a direct corollary of 3.2.

We conclude with some examples to show that the analogues of 2.2 and 3.2 fail for the category of pretoposes, **Pt**.

3.3. Example. Let P be a pretopos with an uncomplemented subobject of 1, $U \rightarrowtail 1$ say. Let $P \to P[\vdash U]$ and $P \to P[U \vdash\,]$ be the quotients forcing U to be true and false respectively. Then the pushout square

$$\begin{tikzcd} P[U\vdash\,] \arrow[r] & \mathbf{1} \\ P \arrow[u] \arrow[r] & P[\vdash U] \arrow[u] \end{tikzcd}$$

(where $\mathbf{1}$ is the trivial pretopos) fails to have the interpolation property. The argument is just as in Section 4, (a) of [12].

3.4. Example. Let R be the coherent theory with two sorts X and Y and axioms

$$\exists x \in X(x=x) \dashv\vdash \exists y \in Y(y=y).$$

Let S be the coherent theory with two sorts and one function symbol $f: X \to Y$ together with the axiom

$$\vdash_y \exists x(fx=y).$$

There is an obvious interpretation $K: R \to S$ and it is conservative since the induced geometric morphism between the classifying toposes, which are both toposes of presheaves on the opposites of the categories of finite models, is (essential and) a surjection. Now let $L:R\to T$ be the quotient in which the sort X is forced to be terminal. Form a pushout square in **Pt**.

$$\begin{tikzcd} S \arrow[r, "M"] & P \\ R \arrow[u, "K"] \arrow[r, "L"] & T \arrow[u, "N"'] \end{tikzcd}$$

In the theory P, X is terminal and covers Y; therefore Y is also terminal. Hence P is the initial theory and N is not conservative.

This shows that *conservative morphisms are not stable under pushout in* **Pt**. However we can do better than this. The final example (which generalises one suggested to the author by G.E. Reyes) shows that the pushout of a conservative morphism in **Pt** along another such can fail to be conservative, i.e. **Pt** *fails to have the 'amalgamation property' for conservative morphisms.*

3.5. Example. Let **BPt** denote the full subcategory of **Pt** whose objects are *Boolean* pretoposes, i.e. those in which each subobject lattice is a Boolean algebra. let $(-)_c:\mathbf{Pt}\to\mathbf{BPt}$ denote the left adjoint to the inclusion $\mathbf{BPt}\hookrightarrow\mathbf{Pt}$. Thus T_c is the classical theory generated by the coherent theory T. The unit of the adjunction at T gives a morphism $(\hat{-}):T\to T_c$ in **Pt** which is conservative (since for example, every topos is covered by a Boolean topos).

Now suppose we have a pretopos T such that

(a) T is *well-pointed*, i.e. it is non-trivial and its terminal object is a generator;

(b) T_c contains a *proper* subobject of 1, i.e. $U\rightarrowtail 1$ such that $U\neq\bot,\top$.

(In Reyes' example T was the pretopos whose objects are the recursively enumerable subsets of $\mathbb{N}$ and whose morphisms are (restrictions of) partial recursive functions.) Given subobjects $A\rightarrowtail X$, $B\rightarrowtail X$ in T, if $A\not\leq B$ then by (a) we can find $x:1\to X$ with $x\in A$ and $x\notin B$. So if we had

$$\hat{X}^{-1}(U)\wedge\hat{A}\leq\hat{B}\quad\text{in }\mathrm{Sub}_{T_c}(\hat{X}),$$

then pulling back along $x:1\to X$ we would get

$$U\wedge\top\leq\bot,$$

contradicting (b). Thus distinct subobjects are sent to distinct subobjects by the morphism $T \to T_c \to T_c/U$, which is therefore conservative. Similarly $T \to T_c \to T_c/\neg U$ is conservative. However, if

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & T_c \arrow[r] & T_c/U \arrow[dd] \\
T_c \arrow[d] & & \\
T_c/\neg U \arrow[rr] & & P
\end{tikzcd}$$

is a pushout square in **Pt**, then

$$\begin{tikzcd}
T_c \arrow[rr] \arrow[dd] & & T_c/U \arrow[d] \\
& & P \arrow[d] \\
T_c/\neg U \arrow[r] & P \arrow[r] & P_c
\end{tikzcd}$$

commutes up to isomorphism in **BPt** and therefore $P_c \simeq \mathbf{1}$, and hence P is also trivial (since $P \to P_c$ is conservative). Thus $T_c/U \to P$ and $T_c/\neg U \to P$ are not conservative.

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