Advanced Topics in
Types and Programming Languages

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7 *Typed Operational Reasoning*

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The aim of this chapter is to explain, by example, some methods for reasoning about equivalence of programs based directly upon a type system and an operational semantics for the programming language in question. We will concentrate on methods for reasoning about equivalence of representations of abstract data types. This provides an excellent example: it is easy to appreciate why such methods are useful and at the same time non-trivial problems have to be solved to get a sound reasoning principle in the presence of nontermination and recursion. Rather than just treat abstract data types, we will cover full existential types, using a programming language combining a pure fragment of ML (including records and recursive functions) with System F.

7.1 Introduction

As explained in *TAPL*, Chapter 24, type systems involving existentially quantified type variables provide a useful foundation for explaining and relating various features of programming languages to do with information hiding. To establish the properties of such type-theoretic interpretations of information hiding requires a theory of semantic equivalence for expressions of existential type. Methods involving type-indexed families of *relations* between between expressions have proved very useful in this respect. Study of relational properties of typed calculi goes back to the *logical relations* for simply typed lambda calculus in Plotkin (1973) and Statman (1985) and discussed in Chapter 6, and the notion of *relational parametricity* for polymorphic types in Reynolds (1983). More relevant to the kind of example considered in this chapter is Mitchell's principle for establishing the denotational equivalence of programs involving higher-order functions and different implementations of an abstract datatype in terms of the existence of a *simulation* relation be-

tween the implementations (Mitchell, 1991a). This principle was extended by
Plotkin and Abadi (1993) to encompass all the (possibly impredicative) exis-
tential types of the Girard-Reynolds polymorphic lambda calculus.

One feature of these works is that they develop proof principles for *deno-
tational* models of programming languages. The relevance of such principles
to the operational behavior of programs relies upon 'goodness of fit' results
(some published, some not) connecting operational and denotational seman-
tics. Another feature of the above works is that they do not treat the use of
general recursive definitions; and so the languages considered are not Tur-
ing powerful. It is folklore that a proof principle for denotational equality at
existential type, phrased in terms of the existence of certain simulation rela-
tions, is still valid in the presence of recursively defined functions of higher
type, provided one imposes some *admissibility* conditions on the notion of
relation. In fact using techniques for defining operationally based logical re-
lations developed in Pitts (2000), we will see in this chapter that suitable
admissibility conditions for relations and an associated proof principle for
operational equivalence at existential type can be phrased directly, and quite
simply, in terms of the syntax and operational semantics of a programming
language combining existential types with recursively defined, higher-order
functions. The programming language we work with combines a pure frag-
ment of ML (including records and recursive functions) with the polymorphic
lambda calculus of Girard (1972) and Reynolds (1974).

7.2   Overview

In order to get the most out of this chapter you should have some familiarity
with *TAPL*, Chapters 23 and 24. The material in this chapter is technically
quite intricate (especially the definition and properties of the logical relation
in §7.6) and it is easy to lose sight of the wood for the trees. So here is an
overview of the chapter.

Equivalence of programs One application of formal semantics of program-
ming languages is to give a mathematically precise definition of what it
means for one program to be semantically equal to another. In this chapter
we use operational semantics and discuss a notion of program equivalence
called *contextual equivalence* (§7.5).

Extensionality principles In order to reason about program equivalence, it
is useful to establish the validity of proof methods for it. The most basic
method uses the congruence property—reasoning by "replacing equals by
equals"—which holds of contextual equivalence by construction. In §7.1

we discuss informally some methods for proving contextual equivalence of implementations of abstract datatypes. The discussion culminates with the Extensionality Principle 7.3.6. One goal of this chapter is to give a mathematically precise formulation of this principle and to establish its validity.

Logical relations The Extensionality Principle is phrased in terms of type-respecting *relations* between the terms of our example language. In order to formulate this principle precisely and then prove it we develop an alternative characterisation of contextual equivalence in terms of a certain "logical relation" (§7.6). The combination of features in our language—higher-order recursive functions and fully impredicative polymorphic types—force us to use a form of logical relation with quite a difficult definition. Chapter 6 presents another use of logical relations with a simpler definition; as such, that chapter provides a useful warm-up for this one.

7.3 Motivating Examples

In this section we motivate the use of logical relations for reasoning about existential types by giving some examples.

To begin, let us recall the syntax for expressions involving existentially quantified type variables from *TAPL*, Chapter 24. If T is a type expression and X is a type variable, then we write $\{\exists X, T\}$ for the corresponding existentially quantified type. Free occurrences of X in T become bound in this type expression. We write $[X \mapsto S]T$ for the result of substituting a type S for all free occurrences of X in T, renaming bound type variables as necessary to avoid capture.[1] It t is a term of type $[X \mapsto S]T$, then we can "pack" the type S and the term t together to get a term

$$\{*S, t\} \text{ as } \{\exists X, T\} \tag{7.1}$$

of the indicated existential type. To eliminate such terms we use the form

$$\text{let } \{*X, x\} = t_1 \text{ in } t_2 \tag{7.2}$$

This is a binding construct: free occurrences of the type variable X and the value variable x in t_2 become bound in the term. The typing of such terms goes as follows:

if t_1 has type $\{\exists X, T\}$ and t_2 has type T_2 when we assume the variable x has type T, then *provided* X *does not occur free in* T_2, we can conclude that the term in (7.2) has type T_2.

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1. Throughout this chapter we will always identify expressions, be they types or terms, up to renaming of bound variables.

(Such rules are better presented symbolically, but we postpone doing that until we give a formal definition of the language we will be using, in the next section.) The italicized restriction on free occurrences of X in T_2 in the above rule is what distinguishes an existential type from a type-indexed dependent sum, where there is free access both to the type component as well as the term component of a "packed" term: see Mitchell and Plotkin (1988), p. 474 *et seq*, for a discussion of this point.

Since we wish to consider existential types in the context of an ML-like language, we adopt an eager strategy for evaluating expressions like (7.1) and (7.2). Thus to evaluate the first, one evaluates t to canonical form, v say, and returns the canonical form {*S,v} as {∃X,T}; to evaluate the second, one evaluates t_1 to canonical form, {*S,v} as {∃X,T} say, and then evaluates $[X \mapsto S][x \mapsto v]t_2$.

7.3.1   EXAMPLE: Consider the existentially quantified record type

```
type Counter = {∃X, {mk:X, inc:X→X, get:X→Int}}
```

where Int is a type of integers. Values of type Counter consist of some type together with values of the appropriate types implementing mk, inc, and get. For example

```
val counter₁ = {*Int, {mk  = 0,
                        inc = λx:Int.x+1,
                        get = λx:Int.x   } as Counter
```

and

```
val counter₂ = {*Int, {mk  = 0,
                        inc = λx:Int.x-1,
                        get = λx:Int.0-x } as Counter
```

are both values of type Counter. The terms

```
let {*X,x} = counter₁  in x.get(x.inc(x.mk))
let {*X,x} = counter₂ in x.get(x.inc(x.mk))
```

(where we use the syntax r.f for selecting field f of record r) are both terms of type Int which evaluate to 1. By contrast, of the terms

```
let {*X,x} = counter₁  in x.get(x.inc(1))
let {*X,x} = counter₂ in x.get(x.inc(1))
```

the first evaluates to 2, whereas the second evaluates to 0; but in this case neither term is well-typed. Indeed, it is the case that *any* well-typed closed term involving occurrences of the term $counter_1$ will exhibit precisely the same evaluation behavior if we replace those occurrences by $counter_2$. In other words, $counter_1$ and $counter_2$ are equivalent in the following sense. □

7.3.2   DEFINITION [CONTEXTUAL EQUIVALENCE, INFORMALLY]: We write $t_1 =_{ctx} t_2 : T$
to indicate that two terms t_1 and t_2 of the same type T are *contextually equiv-
alent*. By definition, this means that for all well-typed terms $t[t_1]$ containing
instances of t_1, if $t[t_2]$ is the term obtained by replacing those instances
by t_2, then $t[t_1]$ and $t[t_2]$ give exactly the same observable results when
evaluated.   □

This notion of program equivalence assumes we have already fixed upon a
definition of the "observable results" of evaluating terms. It also presupposes
that the meaning of a well-typed term should only depend upon the final
result (if any) of evaluating it. This is reasonable for deterministic and non-
interactive programming even in the presence of computational effects like
side-effecting state or raising exceptions, provided we include those effects as
part of the observable results of evaluation. Certainly, contextual equivalence
is a widely used notion of program equivalence in the literature and it is the
one we adopt here.

For the terms in Example 7.3.1, it is the case that

$$counter_1 =_{ctx} counter_2 : Counter \tag{7.3}$$

but the quantification over all possible contexts $t[-]$ in the definition of $=_{ctx}$
makes a direct proof of this and similar facts rather difficult. Thus one is
led to ask whether there are proof principles for contextual equivalence that
make proving such equivalences at existential types more tractable. Since
values {*S,v} as {∃X,T} of a given existential type {∃X,T} are specified by
pairs of data S and v, as a first stab at such a proof principle one might
try componentwise equivalence. Equivalence in the second component will of
course mean contextual equivalence; but in the first component, where the
expressions involved are types, what should equivalence mean? If we take
it to mean syntactic identity, =, (which for us includes renaming of bound
variables) we obtain the following proof principle.[2]

7.3.3   PRINCIPLE [EXTENSIONALITY FOR ∃-TYPES, VERSION I]: For an existential type
$E \stackrel{\text{def}}{=} \{∃X,T\}$, types T_1, T_2, and values v_1, v_2, if $T_1 = T_2$ and $v_1 =_{ctx} v_2 : [X \mapsto T_2]T$, then ({*$T_1$,$v_1$} as E) $=_{ctx}$ ({*T_2,v_2} as E):{∃X,T}.   □

The hypotheses of Principle 7.3.3 are far too strong for it to be very useful.
For example, it cannot be used to prove (7.3), since in this case $T_1 = Int = T_2$,
but

---

2. This and subsequent proof principles for {∃X,T} are called *extensionality* principles by
analogy with the familiar extensionality principle for functions; it is a convenient terminology,
but perhaps the analogy is a little stretched.

```
val v₁ = {mk=0, inc=λx:Int.x+1, get=λx:Int.x}
```

and

```
val v₂ = {mk=0, inc=λx:Int.x-1, get=λx:Int.0-x}
```

are clearly not contextually equivalent values of the record type

```
{mk:Int,inc:Int→Int,get:Int→Int}
```

(for example, we get different integers when evaluating $t[v_1]$ and $t[v_2]$ when $t[-]$ is $(-.inc)0$). However, they do become contextually equivalent if in the second term we use a variant of integers in which the roles of positive and negative are reversed. Such "integers" are of course in bijection with the usual ones and this leads us to our second version of an extensionality principle for existential types—in which the use of syntactic identity as the notion of type equivalence is replaced by the more flexible one of *bijection*. A bijection $i : T_1 \cong T_2$ means a closed term $i : T_1 \to T_2$ for which there is a closed term $i^{-1} : T_2 \to T_1$ which is a two-sided inverse up to contextual equivalence: $i^{-1}(i\ x_1) =_{ctx} x_1 : T_1$ and $i(i^{-1}\ x_2) =_{ctx} x_2 : T_2$.

7.3.4   PRINCIPLE [EXTENSIONALITY FOR ∃-TYPES, VERSION II]: For each existential type $E \overset{\text{def}}{=} \{\exists X, T\}$, types T_1, T_2, and values v_1, v_2, if there is a *bijection* $i : T_1 \cong T_2$ such that $T(i)\ v_1 =_{ctx} v_2 : [X \mapsto T_2]T$, then

$$(\{*T_1, v_1\}\ \text{as}\ E) =_{ctx} (\{*T_2, v_2\}\ \text{as}\ E) : \{\exists X, T\}.$$

In stating this principle we have used the notation $T(i)$ for the "action" of types T on bijections i: given a type T, possibly containing free occurrences of a type variable X, one can define an induced bijection $T(i) : [X \mapsto T_1]T \cong [X \mapsto T_2]T$ (with inverse $T(i^{-1})$). For example, if T is the type

```
{mk:X, inc:X→X, get:X→Int}
```

then $T(i)$ is

```
λx:{ mk:T₁, inc:T₁→T₁, get:T₁→Int}.
    { mk = i(x.mk),
      inc = λx₂:T₂.i(x.inc(i⁻¹ x₂)),
      get = λx₂:T₂.x.get(i⁻¹ x₂)) }
```

and $T(i^{-1})$ is

```
λx:{ mk:T₂, inc:T₂→T₂, get:T₂→Int}.
    { mk = i⁻¹(x.mk),
      inc = λx₁:T₁.i⁻¹(x.inc(i x₁)),
      get = λx₁:T₁.x.get(i x₁)) }.
```

(In general, if T is a simple type then the definition of $T(i)$ and $T(i^{-1})$ can be done by induction on the structure of T; for recursively defined types, the definition of the induced bijection is not so straightforward.) □

We can use this second version of the extensionality principle for existential types to prove the contextual equivalence in (7.3), using the bijection

$$i \stackrel{\text{def}}{=} (\lambda\text{x:Int.0-x}) : \text{Int} \cong \text{Int}.$$

This does indeed satisfy $T(i)$ v_1 $=_{\text{ctx}}$ v_2 : Int when v_1, v_2, and T are defined as above. (Of course these contextual equivalences, and indeed the fact that this particular term i is a bijection, all require proof; but the methods developed in this chapter render this straightforward.) However, the use of bijections between types is still too restrictive for proving many common examples of contextual equivalence of abstract datatype implementations, such as the following.

7.3.5 EXAMPLE: Consider the following existentially quantified record type, where Bool is a type of booleans.

```
type Semaphore = {∃X, {bit:X, flip:X→X, read:X→Bool}}
```

The following terms have type Semaphore:

```
val semaphore₁ =
    {*Bool, {bit  = true
             flip = λx:Bool.not x,
             read = λx:Bool.x      } as Semaphore;
val semaphore₂ =
    {*Int,  {bit  = 1,
             flip = λx:Int.0-2*x,
             read = λx:Int.x >= 0} as Semaphore
```

There is no bijection Bool $\cong$ Int, so one cannot use Principle 7.3.4 to prove

$$\text{semaphore}_1 =_{\text{ctx}} \text{semaphore}_2 : \text{Semaphore}. \tag{7.4}$$

Nevertheless, this contextual equivalence does hold. An informal argument for this makes use of the following relation $r : \text{Bool} \leftrightarrow \text{Int}$ between values of type Bool and of type Int.

$$r \stackrel{\text{def}}{=} \quad \{(\text{true},m) \mid m = (-2)^n \text{ for some even } n \geq 0\}$$
$$\cup \quad \{(\text{false},m) \mid m = (-2)^n \text{ for some odd } n \geq 0\}.$$

Write s_i for the second component of semaphore_i ($i = 1, 2$). Then

- s_1.bit evaluates to true; s_2.bit evaluates to 1; and $(\text{true}, 1) \in r$;

- if $(t_1, t_2) \in r$, then $(s_1.\text{flip})t_1$ and $(s_2.\text{flip})t_2$ evaluate to a pair of values which are again *r*-related;

- if $(t_1, t_2) \in r$, then $(s_1.\text{read})t_1$ and $(s_2.\text{read})t_2$ evaluate to the same boolean value.

The informal argument for the contextual equivalence (7.4) goes as follows: "any context t[−] which is well-typed whenever its hole '−' is filled with a term of type Semaphore can only make use of a term placed in its hole by opening it as an abstract pair {X,x} and applying the methods bit, flip, and read in some combination; therefore the above observations about *r* are enough to show that t[semaphore$_1$] and t[semaphore$_2$] always have the same evaluation behavior." □

The validity of this informal argument and in particular the assumptions it makes about the way a context can "use" its hole are far from immediate and need formal justification. Leaving that for later, at least we can state the relational principle a bit more precisely.

7.3.6   PRINCIPLE [EXTENSIONALITY FOR ∃-TYPES, FINAL VERSION]: For each existential type $E \overset{\text{def}}{=} \{\exists X, T\}$, types T_1, T_2, and values v_1, v_2, if there is a *relation* $r : T_1 \leftrightarrow T_2$ between terms of type T_1 and of type T_2, such that $(v_1, v_2) \in T[r]$, then $(\{*T_1, v_1\} \text{ as } E) =_{\text{ctx}} (\{*T_2, v_2\} \text{ as } E) : \{\exists X, T\}$. □

Evidently this principle presupposes the existence of an "action" of types on term-relations that sends relations $r : T_1 \leftrightarrow T_2$ to relations $T[r] : [X \mapsto T_1]T \leftrightarrow [X \mapsto T_2]T$ and with certain other properties. It is the definition of this action that is at the heart of the matter. It has to be phrased with some care in order for the above extensionality principle to be valid for languages involving non-termination of evaluation (through the presence of fixpoint recursion for example). We will give a precise definition in §7.6 (Definition 7.6.9) for a language combining impredicative polymorphism with fixpoint recursion at the level of terms. How best to define such relational actions in the presence of recursion at the level of types is still a matter for research (see Exercise 7.8.1).

7.3.7   NOTE: Principle 7.3.4 generalizes Principle 7.3.3, because if $T_1 = T_2$, then the identity function $i \overset{\text{def}}{=} \lambda x : T_1 . x$ is a bijection $T_1 \cong T_2$ satisfying

$(T(i) \; v) =_{\text{ctx}} v$ (for any v)

so that $v_1 =_{ctx} v_2$ implies $(T(i)\ v_1) =_{ctx} v_2$. Principle 7.3.6 generalizes Principle 7.3.4, because each bijection $i : T_1 \cong T_2$ can be replaced by its *graph*

$$r_i \stackrel{\text{def}}{=} \{(u_1, u_2) \mid i\ u_1 =_{ctx} u_2\}$$

which in fact has the property that $(v_1, v_2) \in T[r_i]$ if and only if $(T(i)\ v_1)$ is contextually equivalent to v_2. □

As mentioned in the Introduction, Principle 7.3.6 is an operational generalization of similar principles for the denotational semantics of abstract datatypes over the simply typed lambda calculus (Mitchell, 1991a) and relationally parametric models of the polymorphic lambda calculus (Plotkin and Abadi, 1993). It permits many examples of contextual equivalence at existential types to be proved rather easily. Nevertheless, we will see in §7.7 that it is incomplete for the particular ML-like language we consider here, in the sense that $(\{*T_1, v_1\}\ \text{as}\ E) =_{ctx} (\{*T_2, v_2\}\ \text{as}\ E) : \{\exists X, T\}$ can hold even though there is no relation r for which $(v_1, v_2) \in T[r]$ holds (see Example 7.7.4).

7.4 The Language

In this section we define a small, ML-like programming language that we will use in the rest of the chapter. It combines Girard's *System F* (1972) (in other words, the *polymorphic lambda calculus* of Reynolds [1974]) with recursively defined functions, record types and ground types; in common with ML (Milner, Tofte, Harper, and MacQueen, 1997), evaluation order is strict (i.e., left-to-right, call-by-value). We will call the language F_{ML}. Its syntax and type system are specified in Figure 7-1 and its operational semantics in Figure 7-2.

Syntax

In Figure 7-1, X and x respectively range over disjoint countably infinite sets of *type variables* and *value variables*; l ranges over a countably infinite set of *field labels*; c ranges over the constants `true`, `false` and n (for $n \in \mathbb{Z}$); Gnd is either the type of booleans `Bool` or the type of integers `Int`; and op ranges over a fixed collection of arithmetic and boolean operations (such as `+`, `=`, `not`, etc.).

To simplify the definition of the language's operational semantics we employ the now quite common device of using a syntax for terms that is in a "reduced" (or "A-normal") form, with all sequential evaluation expressed via `let`-expressions. For example, the general form of (left-to-right, call-by-value) function application is coded by

$$t_1\ t_2 \stackrel{\text{def}}{=} \text{let}\ x_1 = t_1\ \text{in}\ (\text{let}\ x_2 = t_2\ \text{in}\ x_1\ x_2). \tag{7.5}$$

Syntax

$$\Gamma \vdash c : \textit{Typeof}(c) \qquad \text{(T-CONST)}$$

t	::=		*terms:*
		v	*value*
		if v then t else t	*conditional*
		op($v_i{}^{i \in 1..n}$)	*operation*
		v v	*application*
		v.l	*projection*
		v T	*type application*
		let {*X,x}=v in t	*unpacking*
		let x=t in t	*sequencing*
v	::=		*values:*
		x	*value variable*
		c	*constant*
		fun x(x:T)=t:T	*recursive function*
		{l_i=$v_i{}^{i \in 1..n}$}	*record value*
		λX.v	*type abstraction*
		{*T,v} as {∃X,T}	*package value*
T	::=		*types:*
		X	*type variable*
		Gnd	*ground type*
		T→T	*function type*
		{l_i:$T_i{}^{i \in 1..n}$}	*record type*
		∀X.T	*universally quantified type*
		{∃X,T}	*existentially quantified type*
Γ	::=		*typing contexts:*
		∅	*empty context*
		Γ,x:T	*non-empty context*
		Γ,X	*non-empty context*

Typing terms $\boxed{\Gamma \vdash t : T}$

$$\frac{x:T \in \Gamma}{\Gamma \vdash x : T} \qquad \text{(T-VAR)}$$

$$\frac{\Gamma, f:T, x:T_1 \vdash t : T_2 \quad T = T_1 \rightarrow T_2}{\Gamma \vdash \text{fun } f(x:T_1)=t:T_2 : T} \qquad \text{(T-FUN)}$$

$$\frac{(\Gamma \vdash v_i : T_i)^{\ i \in 1..n}}{\Gamma \vdash \{l_i=v_i{}^{\ i \in 1..n}\} : \{l_i:T_i{}^{\ i \in 1..n}\}} \qquad \text{(T-RCD)}$$

$$\frac{\Gamma, X \vdash v : T \quad X \notin \textit{ftv}(\Gamma)}{\Gamma \vdash \lambda X.v : \forall X.T} \qquad \text{(T-TABS)}$$

$$\frac{\Gamma \vdash v_1 : [X \mapsto T_1]T \quad T' = \{\exists X, T\}}{\Gamma \vdash \{*T_1, v_1\} \text{ as } T' : T'} \qquad \text{(T-PACK)}$$

$$\frac{\Gamma \vdash v : \text{Bool} \quad \Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : T}{\Gamma \vdash \text{if } v \text{ then } t_1 \text{ else } t_2 : T} \qquad \text{(T-IF)}$$

$$\frac{op:\text{Gnd}_1, \ldots, \text{Gnd}_n \rightarrow \text{Gnd} \quad (\Gamma \vdash v_i : \text{Gnd}_i)^{\ i \in 1..n}}{\Gamma \vdash op(v_i{}^{\ i \in 1..n}) : \text{Gnd}} \qquad \text{(T-OP)}$$

$$\frac{\Gamma \vdash v_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash v_2 : T_1}{\Gamma \vdash v_1 \ v_2 : T_2} \qquad \text{(T-APP)}$$

$$\frac{\Gamma \vdash v : \{l_i:T_i{}^{\ i \in 1..n}\}}{\Gamma \vdash v.l_j : T_j} \qquad \text{(T-PROJ)}$$

$$\frac{\Gamma \vdash v : \forall X.T}{\Gamma \vdash v \ T_1 : [X \mapsto T_1]T} \qquad \text{(T-TAPP)}$$

$$\frac{\Gamma, X, x:T \vdash t : T_1 \quad X \notin \textit{ftv}(\Gamma, T_1) \quad \Gamma \vdash v : \{\exists X, T\}}{\Gamma \vdash \text{let } \{*X,x\}=v \text{ in } t : T_1} \qquad \text{(T-UNPACK)}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma, x:T_1 \vdash t_2 : T_2}{\Gamma \vdash \text{let } x=t_1 \text{ in } t_2 : T_2} \qquad \text{(T-SEQ)}$$

Figure 7-1: F_{ML} **syntax and typing**

As a further simplification, function abstraction and recursive function declaration have been rolled into the one form fun f(x:T_1) = t:T_2, which corresponds to the expressions

$$\text{let fun } f\ (x{:}T_1) = t{:}T_2 \text{ in } f \text{ end} \quad \text{in Standard ML}$$
$$\text{or } \quad \text{let rec } f\ (x{:}T_1) = t{:}T_2 \text{ in } f \quad\quad \text{in Ocaml.}$$

Ordinary function abstraction can be coded as

$$\lambda x{:}T_1.t \stackrel{\text{def}}{=} \text{fun } f(x{:}T_1) = t{:}T_2 \tag{7.6}$$

where f does not occur freely in t (and T_2 is the type of t, given f has type $T_1{\rightarrow}T_2$ and x has type T_1). In what follows we shall use the abbreviations (7.5) and (7.6) without further comment. We shall also use infix notation for application of constant arithmetic and boolean operators such as +, =, etc.

7.4.1   REMARK [VALUE-RESTRICTION]: Note that the operation $\lambda X.(-)$ of polymorphic generalization is restricted to apply only to values. This is a real restriction since for a non-value term t, one cannot define $\lambda X.t$ to be the term let x=t in $\lambda X.x$, since the latter will in general be an ill-typed term. In an ML-like language $\lambda X.t$ is not yet fully evaluated if t is a non-value; and thus evaluation must go under type abstraction $\lambda X.(-)$ and work on terms at types with free type variables. By imposing the restriction that $\lambda X.t$ is only well-formed when t is a value we can restrict attention to the evaluation of closed terms of *closed* type, simplifying the technical development. The restriction does not seem to affect the expressiveness of F_{ML} in practice and is comparable to the "value restriction" on let-bound polymorphism used in the 1997 revision of Standard ML (Milner et al., 1997) and in Objective Caml (Leroy, 2000). However, this restriction does have an effect on the properties of F_{ML}. For example, with the restriction the type $\forall X.X$ contains no closed values (see Exercise 7.7.6); whereas without the restriction there are closed values of that type, such as $\lambda X.$ (fun f(x:Bool) = f x : X) true. The "emptiness" of $\forall X.X$ plays a role in the properties explored in Example 7.7.4 and Remark 7.7.7. □

Operational Semantics

Although we do not do so, the operational semantics of F_{ML} could be specified in the style of the *Definition of Standard ML* (Milner, Tofte, Harper, and MacQueen, 1997) as a syntax-directed, inductively defined relation between terms and values.[3] Here we are interested primarily in the notion of contex-

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3. That Definition uses environments assigning values to value variables. For reasons of technical convenience we eliminate the use of environments by substituting them into the term and only considering the evaluation relation between *closed* terms and values.

Frame stack syntax

S ::=   *frame stacks:*

$\quad\quad Id$ *nil stack*

$\quad\quad S \circ (\text{x.t})$ *stack cons*

Typing frame stacks $\boxed{\Gamma \vdash S : T_1 \multimap T_2}$

$$\Gamma \vdash Id : T \multimap T \quad\quad\quad\quad (\text{S-Nil})$$

$$\frac{\Gamma, \text{x}{:}T_1 \vdash t : T_2 \quad\quad \Gamma \vdash S : T_2 \multimap T_3}{\Gamma \vdash S \circ (\text{x.t}) : T_1 \multimap T_3} \quad (\text{S-Cons})$$

Primitive reductions $\boxed{t_1 \rightsquigarrow t_2}$

$$\text{if true then } t_1 \text{ else } t_2$$
$$\rightsquigarrow t_1 \quad\quad\quad (\text{R-IfTrue})$$

$$\text{if false then } t_1 \text{ else } t_2$$
$$\rightsquigarrow t_2 \quad\quad\quad (\text{R-IfFalse})$$

$$\frac{\text{the value of } op(c_i^{\ i\in 1..n}) \text{ is } c}{op(c_i^{\ i\in 1..n}) \rightsquigarrow c} \quad (\text{R-Op})$$

$$\frac{v_1 \text{ is fun } f(\text{x}{:}T_1)=t{:}T_2}{v_1\ v_2 \rightsquigarrow [f \mapsto v_1][\text{x} \mapsto v_2]t} \quad (\text{R-AppAbs})$$

$$\{l_i{=}v_i^{\ i\in 1..n}\}.j \rightsquigarrow v_j \quad\quad (\text{R-ProjRcd})$$

$$(\lambda X.v)T \rightsquigarrow [X \mapsto T]v \quad\quad (\text{R-TappTabs})$$

$$\frac{v \text{ is } \{*T_1,v_1\} \text{ as } \{\exists X,T\}}{\text{let } \{*X,x\}=v \text{ in } t} \quad (\text{R-UnpackPack})$$
$$\rightsquigarrow [X \mapsto T_1][x \mapsto v_1]t$$

Termination $\boxed{\langle S,t\rangle\downarrow \quad and \quad t\downarrow}$

$$\langle Id,v\rangle \downarrow \quad\quad\quad\quad (\text{S-NilVal})$$

$$\frac{\langle S,[x \mapsto v]t\rangle \downarrow}{\langle S \circ (\text{x.t}),v\rangle \downarrow} \quad\quad (\text{S-ConsVal})$$

$$\frac{\langle S \circ (\text{x.}t_2),t_1\rangle \downarrow}{\langle S, \text{let } \text{x}{=}t_1 \text{ in } t_2\rangle \downarrow} \quad\quad (\text{S-Seq})$$

$$\frac{t_1 \rightsquigarrow t_2 \quad\quad \langle S,t_2\rangle \downarrow}{\langle S,t_1\rangle \downarrow} \quad\quad (\text{S-Red})$$

$$\frac{\langle Id,\text{t}\rangle \downarrow}{\text{t} \downarrow} \quad\quad\quad\quad (\text{Term})$$

Figure 7-2: F_{ML} **operational semantics**

tual equivalence (Definition 7.3.2) that this evaluation relation determines by observing the results of evaluating terms in context. Because evaluation in F_{ML} is strict and the language has a sufficiently expressive collection of constructs for deconstructing values, it turns out that the notion of contextual equivalence is not affected much by the choice of what to observe of evaluation. Most reasonable choices give rise to the same equivalence as the one we adopt (see Exercise 7.5.10 below), which is based upon observing *termination*: whether or not a term evaluates to some value, we care not which. So instead of defining the relation of evaluation between terms and values, we proceed directly to a definition of the termination relation, $t\downarrow$, for F_{ML}. This is given in Figure 7-2, using an auxiliary notion of *frame stack*. (The conventions and notations used in Figure 7-2 in connection with binding, free variables and substitution are summarized in Figure 7-3.)

Frame stacks are finite lists of individual "evaluation frames." They provide a convenient syntax for the notion of *evaluation context* $E[-]$ (Felleisen and Hieb, 1992; Wright and Felleisen, 1994). Every closed term can be decomposed

Binding constructs

> let {*X,x}=v in (−)
> let x=t in (−)
> fun f(x:T_1)=(−:T_2)
> λX.(−)
> ∀X.(−)
> {∃X,(−)}
> S ∘ (x.(−))

We identify expressions up to renaming of bound value and type variables.

Notation for free variable sets

$\boxed{ftv(E)}$ is the finite set of free type variables of the expression E (a type, a term, or a frame stack);

$\boxed{fv(E)}$ is the finite set of free value variables of an expression E (a term, or a frame stack, but not a type, since types do not contain occurrences of value variables).

Closed types, terms and frame stacks

> A type T is *closed* if $ftv(T) = \varnothing$.
>
> A term or frame stack E is *closed* if $fv(E) = \varnothing$ (even if $ftv(E) \neq \varnothing$).

Notation for substitution

$\boxed{[X \mapsto T]E}$ denotes the result of capture-avoiding substitution of a type T for all free occurrences of a type variable X in E (a type, a term, or a frame stack);

$\boxed{[x \mapsto v]E}$ denotes the result of capture-avoiding substitution of a value v for all free occurrences of the value variable x in a term or frame stack E.

(Note that as their name suggests, value variables stand for unknown *values*—the substitution of a non-value term for a variable makes no sense syntactically, in that it may result in an ill-formed expression.)

Figure 7-3: Binding, free variables and substitution

uniquely as E[t] where the evaluation context E[−] is a context with a unique hole (−) occurring in the place where the next step of evaluation (called a *primitive reduction* in Figure 7-2), if any, will take place. With F_{ML}'s reduced syntax, such evaluation contexts turn out to be just nested sequences of the let-construct

$$E[-] = \text{let } x_1 = (\ldots(\text{let } x_n = (-) \text{ in } t_n)\ldots) \text{ in } t_1.$$

The corresponding frame stack

$$S = Id \circ (x_1.t_1) \circ \cdots \circ (x_n.t_n)$$

records this sequence as a list of *evaluation frames*, $x_i.t_i$ (with free occurrences of x_i in t_i being bound in $x_i.t_i$). Under this correspondence it can be shown that E[t] evaluates to some value in the standard evaluation-style (or "big-step") structural operational semantics if and only if $\langle S, t \rangle \downarrow$ holds, for the relation $\langle -, - \rangle \downarrow$ defined in Figure 7-2. Not only does the use of frame

stacks enable a conveniently syntax-directed inductive definition of termination, but also frame stacks play a big role in §7.6 when defining the logical relation that we use to establish properties of F_{ML} contextual equivalence.

7.4.2   EXERCISE [RECOMMENDED, ★★]: Consider a relation $\langle S_1, t_1 \rangle \longrightarrow \langle S_2, t_2 \rangle$ defined by cases according to the structure of the term t_1 and the frame stack S_1, as follows:

- $\langle S \circ (x.t), v \rangle \longrightarrow \langle S, [x \mapsto v]t \rangle$

- $\langle S, \text{let } x=t_1 \text{ in } t_2 \rangle \longrightarrow \langle S \circ (x.t_2), t_1 \rangle$

- $\langle S, t_1 \rangle \longrightarrow \langle S, t_2 \rangle$, if $t_1 \rightsquigarrow t_2$.

Show that

$$\langle S'@S, t \rangle \downarrow \quad \Leftrightarrow \quad (\exists v) \langle S, t \rangle \longrightarrow^* \langle Id, v \rangle \;\&\; \langle S', v \rangle \downarrow \tag{7.7}$$

where $\longrightarrow^*$ denotes the reflexive-transitive closure of the $\longrightarrow$ relation, and $S'@S$ is the frame stack obtained by appending the two lists of evaluation frames S' and S. Deduce that $t \downarrow$ holds if and only if there is some value v with $\langle Id, t \rangle \longrightarrow^* \langle Id, v \rangle$.   □

Typing

We will consider the termination relation only for frame stacks and terms that are *well-typed*. A term t is well-typed with respect to a particular typing context Γ if a typing judgment

$$\Gamma \vdash t : T \tag{7.8}$$

can be derived for some type T using the rules in Figure 7-1. We identify typing contexts Γ up to rearranging their constituent hypotheses ("X" or "$x : X$") and eliminating duplicates. Thus a typical typing context looks like

$$\Gamma = X_1, \ldots, X_m, x_1 : T_1, \ldots, x_n : T_n$$

where the type variables X_i and the value variables x_j are all distinct (and $m = 0$ or $n = 0$ is allowed). The typing judgments that are derivable from the rules all have the property that the free type variables of T and each T_j occur in the set $\{X_1, \ldots, X_m\}$, and the free value variables of t occur in the set $\{x_1, \ldots, x_n\}$. This is ensured by including some explicit side-conditions about free variable occurrences in the typing rules (T-ABS) and (T-UNPACK). In *TAPL*, Chapters 23 and 24, such side-conditions are implicit, being subsumed by

extra well-formedness conditions for typing judgments. Also, we have chosen to include sufficient explicit type information in terms to ensure that for any given Γ and t, there is at most one T for which (7.8) holds. Apart from such minor differences, the rules in Figure 7-1 for inductively generating the valid F_{ML} typing judgments are all quite standard.

The judgment for typing frame stacks takes the form

$$\Gamma \vdash S : T_1 \multimap T_2 \tag{7.9}$$

where, in terms of the evaluation context corresponding to S, T_2 is the overall type of the context, given that T_1 is the type of the hole. The rules for generating this judgment are given in Figure 7-2. Unlike for terms, we have not included explicit type information in the syntax of frame stacks; for example, *Id* is not tagged with a type. However, it is not hard to see that, given Γ, S, and T_1, there is at most one T_2 for which (7.9) holds. This property is enough for our purposes, since the argument type of a frame stack will always be supplied in any particular situation in which we use it.

7.4.3   EXERCISE [★, ↦]: Write $\Gamma \vdash \langle S, t \rangle : T$ to mean that $\Gamma \vdash S : T' \multimap T$ and $\Gamma \vdash t : T'$ hold for some type T'. Using the relation $\longrightarrow$ from Exercise 7.4.2, show that if $\varnothing \vdash \langle S_1, t_1 \rangle : T$ and $\langle S_1, t_1 \rangle \longrightarrow \langle S_2, t_2 \rangle$, then $\varnothing \vdash \langle S_2, t_2 \rangle : T$.   □

Unwinding Recursive Functions

In what follows we will need a finiteness property of recursively defined functions with respect to the termination relation. This *unwinding property*, as it is called, is a syntactic analog of the fact that the denotation of a recursively defined function is constructed as the least upper bound (*lub*) of finite approximations obtained by successively unfolding its definition starting with the *bottom* denotation, i.e., the totally undefined partial function. This gives rise to the useful principle of *Scott induction* in denotational semantics: given an *admissible* property of denotations, i.e., one closed under the formation of lubs of increasing chains, to show that it holds of the denotation of recursively defined data it suffices to show that it holds of bottom and is closed under application of the function that defines the data as a fixed point. Here we use a syntactic analog of Scott induction for recursively defined functions, $\text{fun } f(x{:}T_1) = u{:}T_2$, in order to prove the "fundamental property" (Lemma 7.6.17) of the logical relation constructed in §7.6.

The proof of the unwinding property that we give here is made easier by our syntax-directed definition of termination using frame stacks. For statements and proofs of similar properties see for example: Mason, Smith, and Talcott (1996), Section 4.3, Pitts and Stark (1998), Theorem 3.2, Birkedal and Harper (1999), Section 3.1, and Lassen (1998), Section 4.5.

7.4.4   THEOREM [UNWINDING]: Given any closed recursive function value F of the form fun f(x:T$_1$)=u:T$_2$, define the followings abbreviations[4] :

$$F_0 \overset{\text{def}}{=} \text{fun } f(x:T_1) = (f\ x) : T_2$$

$$F_{n+1} \overset{\text{def}}{=} \text{fun } f(x:T_1) = [f \mapsto F_n]u : T_2$$

Thus F$_0$ is a closed function value describing a function of type T$_1$→T$_2$ that diverges when applied to any argument, and the F$_n$ are obtained from this by repeatedly substituting for the the value variable f in the body u of the original function value F. Then for all terms t containing at most f free we have $[f \mapsto F]t \downarrow$ if and only if $(\exists n)\ [f \mapsto F_n]t \downarrow$.   □

Proof: By definition of the relation t↓ in terms of the relation $\langle S, t\rangle \downarrow$ (via rule (TERM) in Figure 7-2), it suffices to prove the more general property that for all terms t and frame stacks S (containing at most f free) we have

$$\langle [f \mapsto F]S, [f \mapsto F]t\rangle \downarrow \ \Leftrightarrow\ (\exists n)\ \langle [f \mapsto F_n]S, [f \mapsto F_n]t\rangle \downarrow \tag{7.10}$$

The proof of (7.10) is via a series of straightforward, if somewhat tedious, inductions that we leave as an exercise.   □

7.4.5   EXERCISE [★★★, ↛]: This exercise leads you through a proof of (7.10). First prove that

$$\langle [f \mapsto F_n]S, [f \mapsto F_n]t\rangle \downarrow\ \Rightarrow\ \langle [f \mapsto F]S, [f \mapsto F]t\rangle \downarrow \tag{7.11}$$

holds for all *n*, S and t by induction on the derivation of $\langle [f \mapsto F_n]S, [f \mapsto F_n]t\rangle \downarrow$ from the rules in Figure 7-2. Conversely show that

$$\langle [f \mapsto F]S, [f \mapsto F]t\rangle \downarrow\ \Rightarrow\ (\exists n)\ \langle [f \mapsto F_n]S, [f \mapsto F_n]t\rangle \downarrow \tag{7.12}$$

holds for all S and t, by induction on the derivation of $\langle [f \mapsto F]S, [f \mapsto F]t\rangle \downarrow$ from the rules. To do this, you will first need to prove by induction on *n* that

$$\langle [f \mapsto F_n]S, [f \mapsto F_n]t\rangle \downarrow\ \Rightarrow\ \langle [f \mapsto F_{n+1}]S, [f \mapsto F_{n+1}]t\rangle \downarrow \tag{7.13}$$

holds for all *n*, S and t; the base case *n* = 0 involves yet another induction, this time over the derivation of $\langle [f \mapsto F_0]S, [f \mapsto F_0]t\rangle \downarrow$ from the rules.   □
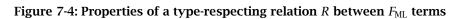
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4. Note that in the definition of F$_{n+1}$, the outer binding instance of f is a dummy, since f does not occur free in $[f \mapsto F_n]u$.

7.5   Contextual Equivalence

Definition 7.3.2 gave an informal definition of the notion of *contextual equivalence* that applies to any (typed) programming language. In giving a precise definition of this notion for the F_{ML} language we will take the more abstract, relational approach of Gordon (1998) and Lassen (1998) that avoids the explicit use of program contexts t[−] in favor of congruence relations. For one thing, program contexts are an inconveniently concrete notion, because substitution of terms t′ for the hole "−" in a context t[−] to produce a term t[t′] may involve the capture of free variables in t′ by binders in t[−]. For example, when we replace the hole "−" in the context fun f(x:T) = f [−] by the term f x, its free value variables are captured by the fun-binder. Consequently, contexts have to be treated more concretely than terms since renaming their bound variables may not preserve their meaning. For example, if we identified fun f(x:T) = f [−] with fun g(x:T) = g [−] (where f and g are distinct value variables), then we should have to identify the results of filling the hole with f x, that is, we should have to identify the syntactically unequal terms fun f(x:T) = f(f x) and fun g(x:T) = g(f x). But more than this, the abstract treatment of contextual equivalence that we use focuses attention upon the key features of this kind of program equality, namely that it is a congruence and is "adequate" for observing termination. In a nutshell, we will define contextual equivalence to be the largest type-respecting congruence relation between F_{ML} terms that is adequate for observing termination.

7.5.1   DEFINITION: A *type-respecting binary relation* between F_{ML} terms is a set R of quadruples $(\Gamma, t, t′, T)$, each consisting of a typing context, two terms and a type satisfying $\Gamma \vdash t : T$ and $\Gamma \vdash t′ : T$. Figure 7-4 defines the properties of *reflexivity*, *symmetry*, *transitivity*, *substitutivity*, and *compatibility* for such relations; R has one of these properties if it is closed under the axioms and rules under the corresponding heading in the figure. In these figures, and elsewhere, we write $\Gamma \vdash t \, R \, t′ : T$ instead of $(\Gamma, t, t′, T) \in R$. We say that R is

- an *equivalence relation* if it has the reflexivity, symmetry and transitivity properties;

- a *congruence relation* if it is an equivalence relation with the *substitutivity* and *compatibility* properties;

- *adequate* (for the termination relation ⇓ defined in Figure 7-2) if whenever $\varnothing \vdash t \, R \, t′ : T$ holds, then t⇓ holds if and only if t′⇓ does.   □

7.5.2   DEFINITION: We will need to use the following constructions on type-respecting binary relations.

Reflexivity
$$\frac{\Gamma \vdash t : T}{\Gamma \vdash t \; R \; t : T}$$

Symmetry
$$\frac{\Gamma \vdash t \; R \; t' : T}{\Gamma \vdash t' \; R \; t : T}$$

Transitivity
$$\frac{\Gamma \vdash t \; R \; t' : T \qquad \Gamma \vdash t' \; R \; t'' : T}{\Gamma \vdash t \; R \; t'' : T}$$

Substitutivity
$$\frac{\Gamma \vdash v \; R \; v' : T_1 \qquad \Gamma, x : T_1 \vdash t \; R \; t' : T_2}{\Gamma \vdash [x \mapsto v]t \; R \; [x \mapsto v']t' : T_2}$$

$$\frac{\Gamma, X \vdash t \; R \; t' : T}{\Gamma \vdash [X \mapsto T_1]t \; R \; [X \mapsto T_1]t' : [X \mapsto T_1]T}$$

Compatibility
$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x \; R \; x : T}$$

$$\Gamma \vdash c \; R \; c : \textit{Typeof}(c)$$

$$\frac{\Gamma, f : T_1 \to T_2, x : T_1 \vdash t \; R \; t' : T_2}{\begin{array}{c}\Gamma \vdash \mathsf{fun}\; f(x : T_1) = t : T_2 \; R \\ \mathsf{fun}\; f(x : T_1) = t' : T_2 : T_1 \to T_2\end{array}}$$

$$\frac{(\Gamma \vdash v_i \; R \; v_i' : T_i)^{\; i \in 1..n}}{\begin{array}{c}\Gamma \vdash \{l_i = v_i^{\; i \in 1..n}\} \; R \; \{l_i = v_i'^{\; i \in 1..n}\} \\ : \{l_i : T_i^{\; i \in 1..n}\}\end{array}}$$

$$\frac{\Gamma, X \vdash v \; R \; v' : T \qquad X \notin \textit{ftv}(\Gamma)}{\Gamma \vdash \lambda X.v \; R \; \lambda X.v' : \forall X.T}$$

$$\frac{\Gamma \vdash v_1 \; R \; v_1' : [X \mapsto T_1]T}{\begin{array}{c}\Gamma \vdash \{*T_1, v_1\} \;\mathsf{as}\; \{\exists X, T\} \; R \\ \{*T_1, v_1'\} \;\mathsf{as}\; \{\exists X, T\} : \{\exists X, T\}\end{array}}$$

$$\frac{\begin{array}{c}\Gamma \vdash v \; R \; v' : \mathsf{Bool} \\ \Gamma \vdash t_1 \; R \; t_1' : T \qquad \Gamma \vdash t_2 \; R \; t_2' : T\end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{if}\; v \;\mathsf{then}\; t_1 \;\mathsf{else}\; t_2 \; R \\ \mathsf{if}\; v' \;\mathsf{then}\; t_1' \;\mathsf{else}\; t_2' : T\end{array}}$$

$$\frac{\begin{array}{c}\mathsf{op} : \mathsf{Gnd}_1, \ldots, \mathsf{Gnd}_n \to \mathsf{Gnd} \\ (\Gamma \vdash v_i \; R \; v_i' : \mathsf{Gnd}_i)^{\; i \in 1..n}\end{array}}{\Gamma \vdash \mathsf{op}(v_i^{\; i \in 1..n}) \; R \; \mathsf{op}(v_i'^{\; i \in 1..n}) : \mathsf{Gnd}}$$

$$\frac{\Gamma \vdash v_1 \; R \; v_1' : T_1 \to T_2 \qquad \Gamma \vdash v_2 \; R \; v_2' : T_1}{\Gamma \vdash v_1 \; v_2 \; R \; v_1' \; v_2' : T_2}$$

$$\frac{\Gamma \vdash v \; R \; v' : \{l_i : T_i^{\; i \in 1..n}\}}{\Gamma \vdash v.l_j \; R \; v'.l_j : T_j}$$

$$\frac{\Gamma \vdash v \; R \; v' : \forall X.T}{\Gamma \vdash v \; T_1 \; R \; v' \; T_1 : [X \mapsto T_1]T}$$

$$\frac{\begin{array}{c}\Gamma, X, x : T \vdash t \; R \; t' : T_1 \\ X \notin \textit{ftv}(\Gamma, T_1) \qquad \Gamma \vdash v \; R \; v' : \{\exists X, T\}\end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{let}\; \{*X, x\} = v \;\mathsf{in}\; t \; R \\ \mathsf{let}\; \{*X, x\} = v' \;\mathsf{in}\; t' : T_1\end{array}}$$

$$\frac{\Gamma \vdash t_1 \; R \; t_1' : T_1 \qquad \Gamma, x : T_1 \vdash t_2 \; R \; t_2' : T_2}{\Gamma \vdash \mathsf{let}\; x = t_1 \;\mathsf{in}\; t_2 \; R \; \mathsf{let}\; x = t_1' \;\mathsf{in}\; t_2' : T_2}$$

Figure 7-4: Properties of a type-respecting relation R between F_{ML} terms

(i) The *identity* relation is $Id \stackrel{\mathrm{def}}{=} \{(\Gamma, t, t, T) \mid \Gamma \vdash t : T\}$.

(ii) The *reciprocal* of the relation R is $R^{op} \stackrel{\mathrm{def}}{=} \{(\Gamma, t', t, T) \mid \Gamma \vdash t \; R \; t' : T\}$.

(iii) The *composition* of relations R_1 and R_2 is
$R_1 \circ R_2 \stackrel{\mathrm{def}}{=} \{(\Gamma, t, t'', T) \mid \exists t'. \, \Gamma \vdash t \; R_1 \; t' : T \; \& \; \Gamma \vdash t' \; R_2 \; t'' : T\}$.

(iv) The *transitive closure* of the relation R is the countable union $R^+ \overset{\text{def}}{=} \bigcup_{i \in \mathbb{N}} R_i$, where $R_0 = R$ and $R_{i+1} = R \circ R_i$.

(v) The *open extension* of the relation R is denoted R° and consists of all quadruples $(\Gamma, \mathtt{t}, \mathtt{t'}, \mathsf{T})$ such that $\varnothing \vdash \sigma(\mathtt{t}) \ R \ \sigma(\mathtt{t'}) : \sigma(\mathsf{T})$ holds for all Γ-closing substitutions σ. If $\Gamma = \mathsf{X}_1, \ldots, \mathsf{X}_m, \mathtt{x}_1 : \mathsf{T}_1, \ldots, \mathtt{x}_n : \mathsf{T}_n$, then a Γ-*closing substitution* is given by a function $[\mathsf{X}_i \mapsto \mathsf{T}_i \mid i = 1..m]$ mapping the type variables X_i to closed types T_i and by a function $[\mathtt{x}_j \mapsto \mathtt{v}_j \mid j = 1..n]$ mapping the value variables $\mathtt{x}_j$ to closed values $\mathtt{v}_j$ of appropriate type, namely satisfying $\varnothing \vdash \mathtt{v}_j : [\mathsf{X}_i \mapsto \mathsf{T}_i \mid i = 1..m]\mathsf{T}_j$.

(Note that R° only depends on the quadruples of the form $(\varnothing, \mathtt{t}, \mathtt{t'}, \mathsf{T})$ in R.)   □

We wish to define contextual equivalence to be the largest adequate congruence relation, but it is not immediately clear why a largest such relation exists. Therefore we give a theorem rather than a definition.

7.5.3   THEOREM [F_{ML} CONTEXTUAL EQUIVALENCE, $=_{\text{ctx}}$]: There exists a largest type-respecting binary relation between F_{ML} terms that is a congruence and adequate. We call it *contextual equivalence* and write it $=_{\text{ctx}}$.   □

Proof: The proof makes use of the following series of facts, only the last of which is not entirely straightforward to prove (see Exercise 7.5.4).

(i)   The identity relation *Id* is an adequate congruence relation.

(ii)   The collection of adequate relations is closed under taking unions.

(iii)   Every compatible relation is reflexive, i.e., contains *Id*.

(iv)   The set of all of compatible relations is closed under the operations of composition and reciprocation; similarly for the set of all substitutive relations and the set of all adequate relations.

(v)   If the union of a non-empty family of compatible relations is transitive, it is also compatible; similarly, if the union of a non-empty family of reflexive and substitutive relations is transitive, it is also (reflexive and) substitutive.

Let $=_{\text{ctx}}$ be the union of the family of relations that are adequate, compatible and substitutive. Note that this family is non-empty by (i). By (ii), $=_{\text{ctx}}$ is adequate. So it suffices to show that it is a congruence relation. It is certainly reflexive by (i); and (iv) implies that it is also symmetric and transitive. So it just remains to show that it is compatible and substitutive, and this follows from (v), whose proof needs (iii).   □

7.5.4   EXERCISE [⋆⋆]: Prove properties (iii) and (v) stated in the above proof.   □

It is not easy to use either the formulation in terms of contexts in Definition 7.3.2 or the more abstract characterisation of Theorem 7.5.3 to prove that a particular pair of terms are contextually equivalent. For example, it is not easy to see from these characterisations that terms in the primitive reduction relation of Figure 7-2 are contextually equivalent (Corollary 7.5.8). That this is so follows from the coincidence of $=_{\mathrm{ctx}}$ with a notion of equivalence popularized by Mason and Talcott (1991).

7.5.5   DEFINITION [CIU-EQUIVALENCE, $=_{\mathrm{ciu}}$]: Two closed F_{ML} terms belonging to the same (closed) type are *ciu-equivalent* if they have the same termination behavior when they are paired with any frame stack (a "<u>u</u>se" of the terms); the relation is extended to open terms via closing substitutions (or "<u>c</u>losed <u>i</u>nstantiations"—thus we arrive at an explanation of the rather cryptic name for this equivalence).

More formally, we define $=_{\mathrm{ciu}}$ to be the type-respecting relation R° (using the operation from Definition 7.5.2(v)), where R consists of quadruples $(\varnothing, \mathsf{t}, \mathsf{t}', \mathsf{T})$ satisfying $\varnothing \vdash \mathsf{t} : \mathsf{T}$, $\varnothing \vdash \mathsf{t}' : \mathsf{T}$, and $\forall S.\ \langle S, \mathsf{t} \rangle {\downarrow} \ \Leftrightarrow \ \langle S, \mathsf{t}' \rangle {\downarrow}$.   □

7.5.6   LEMMA: For any frame stack S and term t, define a term $S[\mathsf{t}]$ by induction of the length of the stack S as follows:

$$
\left.
\begin{array}{rcl}
Id[\mathsf{t}] & \overset{\mathrm{def}}{=} & \mathsf{t} \\
S \circ (\mathsf{x}.\mathsf{t}')[\mathsf{t}] & \overset{\mathrm{def}}{=} & S[\mathsf{let\ x=t\ in\ t}']
\end{array}
\right\}
\tag{7.14}
$$

Then $\langle S, \mathsf{t} \rangle {\downarrow}$ if and only if $S[\mathsf{t}] {\downarrow}$ (i.e., $\langle Id, S[\mathsf{t}] \rangle {\downarrow}$).   □

Proof: This is proved by induction on the length of S. The base case $S = Id$ is trivial. The induction step follows from the fact that $\langle S, \mathsf{let\ x=t\ in\ t}' \rangle {\downarrow}$ holds if and only if it was derived using rule (S-SEQ) in Figure 7-4, if and only if $\langle S \circ (\mathsf{x}.\mathsf{t}'), \mathsf{t} \rangle {\downarrow}$ holds.   □

7.5.7   THEOREM [CIU THEOREM FOR F_{ML}]: The contextual and ciu-equivalence relations coincide.   □

Proof: We first show that $=_{\mathrm{ctx}}$ is contained in $=_{\mathrm{ciu}}$. Suppose

$$\Gamma \vdash \mathsf{t} =_{\mathrm{ctx}} \mathsf{t}' : \mathsf{T}. \tag{7.15}$$

Since $=_{\mathrm{ctx}}$ satisfies the substitutivity and reflexivity properties from Figure 7-4, it follows that

$$\varnothing \vdash \sigma(\mathsf{t}) =_{\mathrm{ctx}} \sigma(\mathsf{t}') : \sigma(\mathsf{T}) \tag{7.16}$$

for any Γ-closing substitution σ. For any frame stack S, since $=_{ctx}$ satisfies the compatibility (and reflexivity) properties from Figure 7-4, from (7.16) we deduce that $\varnothing \vdash S[\sigma(t)] =_{ctx} S[\sigma(t')] : \sigma(T)$ (using the notation of (7.14)). Since $=_{ctx}$ is adequate, this means that $S[\sigma(t)]\downarrow$ if and only if $S[\sigma(t')]\downarrow$; hence by Lemma 7.5.6, $\langle S, \sigma(t)\rangle\downarrow$ if and only if $\langle S, \sigma(t')\rangle\downarrow$. As this holds for all σ and S, we have $\Gamma \vdash t =_{ciu} t' : T$, as required.

To complete the proof of the theorem we have to show conversely that $=_{ciu}$ is contained in $=_{ctx}$. We can deduce this as a corollary of a stronger characterisation of $=_{ctx}$ in terms of logical relations (Theorem 7.6.25) that we establish later; so we postpone the rest of this proof until then. $\square$

7.5.8   COROLLARY [CONVERSIONS]: The following are valid contextual equivalences:

(i) $\Gamma \vdash$ if true then t_1 else $t_2 =_{ctx} t_1 : T$ and
$\Gamma \vdash$ if false then t_1 else $t_2 =_{ctx} t_2 : T$, where $\Gamma \vdash t_i : T$ for $i = 1, 2$.

(ii) $\Gamma \vdash op(c_i{}^{i \in 1..n}) =_{ctx} c : Gnd$, where c is the value of $op(c_i{}^{i \in 1..n})$ and $Typeof(c) = Gnd$.

(iii) $\Gamma \vdash v_1 \, v_2 =_{ctx} [f \mapsto v_1][x \mapsto v_2]t : T_2$,
where $v_1 = $ fun $f(x{:}T_1){=}t{:}T_2$.

(iv) $\Gamma \vdash \{l_i{=}v_i{}^{i \in 1..n}\}.j =_{ctx} v_j : T_j$,
where $\Gamma \vdash \{l_i{=}v_i{}^{i \in 1..n}\} : \{l_i{:}T_i{}^{i \in 1..n}\}$.

(v) $\Gamma \vdash (\lambda X.v)T_1 =_{ctx} [X \mapsto T_1]v : [X \mapsto T_1]T$, where $\Gamma \vdash v : \forall X.T$.

(vi) $\Gamma \vdash$ let $\{{*}X,x\}{=}(\{{*}T_1,v_1\}$ as $\{\exists X,T\})$ in $t =_{ctx} [X \mapsto T_1][x \mapsto v_1]t : T_2$, where $\Gamma, X, x{:}T \vdash t : T_2$ with $X \notin ftv(\Gamma, T_2)$.

(vii) $\Gamma \vdash$ let $x{=}v$ in $t =_{ctx} [x \mapsto v]t : T_2$, where $\Gamma \vdash v : T_1$ and $\Gamma, x{:}T_1 \vdash t : T_2$.

(viii) $\Gamma \vdash$ let $x_1{=}t_1$ in (let $x_2{=}t_2$ in $t) =_{ctx}$
let $x_2{=}($let $x_1{=}t_1$ in $t_2)$ in $t : T$, where $\Gamma \vdash t_1 : T_1$,
$\Gamma, x_1{:}T_1 \vdash t_2 : T_2$ and $\Gamma, x_2{:}T_2 \vdash t : T$. $\square$

Proof: These are all ciu-equivalences, so we can just apply Theorem 7.5.7 (using the difficult half of the theorem whose proof we have postponed to §7.6!). The ciu-equivalences all follow easily from the definition of the termination relation (Figure 7-2) except for the last one, where one can apply property (7.7) from Exercise 7.4.2 to reduce proving (viii) for $=_{ciu}$ to the special case when t_1 is a value: see the following exercise. $\square$

7.5.9   EXERCISE [★, ⇀]: Given

$$\varnothing \vdash \mathsf{t}_1 : \mathsf{T}_1$$
$$\mathsf{x}_1{:}\mathsf{T}_1 \vdash \mathsf{t}_2 : \mathsf{T}_2$$
$$\mathsf{x}_2{:}\mathsf{T}_2 \vdash \mathsf{t} : \mathsf{T}$$

use property (7.7) to show for all frame stacks S that

$$\langle S \circ (\mathsf{x}_1.\mathtt{let}\ \mathsf{x}_2{=}\mathsf{t}_2\ \mathtt{in}\ \mathsf{t}), \mathsf{t}_1 \rangle{\downarrow} \quad \mathrm{iff} \quad \langle S \circ (\mathsf{x}_2.\mathsf{t}) \circ (\mathsf{x}_1.\mathsf{t}_2), \mathsf{t}_1 \rangle{\downarrow}.$$

Deduce part (viii) of Corollary 7.5.8.   □

7.5.10   EXERCISE [★★]: Recall from Definition 7.5.1 the notion of an adequate type-respecting binary relation. Let us call a type-respecting binary relation R true-*adequate* if, whenever $\varnothing \vdash \mathsf{t}\ R\ \mathsf{t}' : \mathtt{Bool}$ holds, $\langle \mathit{Id}, \mathsf{t} \rangle \longrightarrow^* \langle \mathit{Id}, \mathtt{true} \rangle$ holds if and only if $\langle \mathit{Id}, \mathsf{t}' \rangle \longrightarrow^* \langle \mathit{Id}, \mathtt{true} \rangle$ does. Here $\longrightarrow^*$ is the relation defined in Exercise 7.4.2. One can adapt the proof of Theorem 7.5.3 to show that there is a largest type-respecting binary relation $=^{\mathtt{true}}_{\mathrm{ctx}}$ between F_{ML} terms that is a congruence and true-adequate. Show that $=^{\mathtt{true}}_{\mathrm{ctx}}$ coincides with contextual equivalence, $=_{\mathrm{ctx}}$.   □

7.6   An Operationally Based Logical Relation

We now have a precise definition of contextual equivalence for F_{ML} terms. Before showing that the Extensionality Principle 7.3.6 holds for existential types in F_{ML}, we need a precise definition of the action of types on term-relations, $r \mapsto \mathsf{T}[r]$, mentioned in the principle. That is the topic of this section. We will end up with a characterisation of $=_{\mathrm{ctx}}$ in terms of a logical relation, yielding several useful extensionality properties of contextual equivalence.

7.6.1   NOTATION: Let *Typ* denote the set of closed F_{ML} types. Given $\mathsf{T} \in \mathit{Typ}$, let

- *Term*(T) denote the set of closed terms of type T, i.e., those terms t for which $\varnothing \vdash \mathsf{t} : \mathsf{T}$ holds;

- *Val*(T) denote the subset of *Term*(T) whose elements are values; and

- *Stack*(T) denote the set of closed frame stacks whose argument type is T, i.e., those frame stacks S for which $\varnothing \vdash S : \mathsf{T} \multimap \mathsf{T}'$ for some $\mathsf{T}' \in \mathit{Typ}$.

Given $\mathsf{T}, \mathsf{T}' \in \mathit{Typ}$, let

- *TRel*(T, T') denote the set of all subsets of *Term*(T) × *Term*(T'); we call its elements *term-relations*;

- *VRel*(T, T′) denote the set of all subsets of *Val*(T) × *Val*(T′); we call its elements *value-relations*;

- *SRel*(T, T′) denote the the set of all subsets of *Stack*(T) × *Stack*(T′); we call its elements *stack-relations*.   □

Note that every value-relation is also a term-relation (since values are particular sorts of term): *VRel*(T, T′) ⊆ *TRel*(T, T′). On the other hand we can obtain a value-relation from a term-relation just by restricting attention to values: given $r \in TRel(\mathsf{T}, \mathsf{T}')$, define $r^v \in VRel(\mathsf{T}, \mathsf{T}')$ by

$$r^v \stackrel{\text{def}}{=} \{(\mathsf{v}, \mathsf{v}') \in Val(\mathsf{T}) \times Val(\mathsf{T}') \mid (\mathsf{v}, \mathsf{v}') \in r\}. \tag{7.17}$$

We will be particularly interested in term-relations r that are indistinguishable, as far as termination properties are concerned, from their value restrictions, r^v. Definition 7.6.3 makes this precise, using a Galois connection between term-relations and stack-relations. The definition may appear to be rather mysterious; its nature will emerge as we develop the action of types on term-relations and its properties. First we recall for the reader what is meant in general by a "Galois connection."

7.6.2   DEFINITION: A *Galois connection* between partially ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$ is specified by a pair of functions $f : P \to Q$ and $g : Q \to P$ satisfying $q \leq_Q f(p)$ if and only if $p \leq_P g(q)$, for all $p \in P$ and $q \in Q$.   □

7.6.3   DEFINITION [CLOSED AND VALUABLE TERM-RELATIONS]: Let $\mathsf{T} \in Typ$ and $\mathsf{T}' \in Typ$ be closed types. Given a term-relation $r \in TRel(\mathsf{T}, \mathsf{T}')$, define a stack-relation $r^s \in SRel(\mathsf{T}, \mathsf{T}')$ by

$(S, S') \in r^s$ if and only if for all $(\mathsf{t}, \mathsf{t}') \in r$, $\langle S, \mathsf{t} \rangle \downarrow$ holds if and only if $\langle S', \mathsf{t}' \rangle \downarrow$ does.

Conversely, given a stack-relation $s \in SRel(\mathsf{T}, \mathsf{T}')$, define a term-relation $s^t \in TRel(\mathsf{T}, \mathsf{T}')$ by

$(\mathsf{t}, \mathsf{t}') \in s^t$ if and only if for all $(S, S') \in s$, $\langle S, \mathsf{t} \rangle \downarrow$ holds if and only if $\langle S', \mathsf{t}' \rangle \downarrow$ does.

Call a term-relation $r \in TRel(\mathsf{T}, \mathsf{T}')$ *closed* if it satisfies $r = r^{st}$ and *valuable* if it satisfies $r = r^{vst}$.   □

7.6.4   NOTE: The operator $(-)^{st}$ is denoted $(-)^{\top\top}$ in Pitts (1998; 2000).   □

7.6.5   LEMMA: The operations $(-)^s$ and $(-)^t$ for turning term-relations into stack-relations and *vice versa*, form a Galois connection:

$$s \subseteq r^s \quad \text{if and only if} \quad r \subseteq s^t. \tag{7.18}$$

Hence the operator $(-)^{st}$ on term-relations is monotone ($r_1 \subseteq r_2$ implies $(r_1)^{st} \subseteq (r_2)^{st}$), inflationary ($r \subseteq r^{st}$), and idempotent ($(r^{st})^{st} = r^{st}$).   □

Proof: If $s \subseteq r^s$, then for any $(\mathsf{t}, \mathsf{t}') \in r$ we have for all $(S, S') \in s$ that $(S, S') \in r^s$, so $\langle S, \mathsf{t} \rangle \downarrow$ iff $\langle S', \mathsf{t}' \rangle \downarrow$; hence $(\mathsf{t}, \mathsf{t}') \in s^t$. Thus $s \subseteq r^s$ implies $r \subseteq s^t$. The converse implication holds by a similar argument. Once we have (7.18), the other properties follow by standard arguments true of any Galois connection, which we give in case the reader has not seen them before.

Thus for any term-relation r, since $r^s \subseteq r^s$, from (7.18) we conclude that $r \subseteq r^{st}$; so $(-)^{st}$ is inflationary (and symmetrically, so is the operator $(-)^{ts}$ on stack-relations).

Now we can deduce that $(-)^s$ and $(-)^t$ are order-reversing. For if $r_1 \subseteq r_2$, then $r_1 \subseteq r_2 \subseteq r_2^{st}$, so by (7.18), $r_2^s \subseteq r_1^s$. Similarly, $s_1 \subseteq s_2$ implies $s_2^t \subseteq s_1^t$. Hence $(-)^{st}$ is monotone (and so is $(-)^{ts}$).

Finally, for idempotence, in view of the inflationary property we just have to show $(r^{st})^{st} \subseteq r^{st}$. But applying (7.18) to $r^{st} \subseteq r^{st}$ we get $r^s \subseteq (r^{st})^s$; applying the order-reversing operator $(-)^t$ to this yields $(r^{st})^{st} \subseteq r^{st}$, as required.   □

7.6.6   COROLLARY: Every valuable term-relation is—in particular—a closed term-relation.   □

Proof: Note that because $(-)^{st}$ is idempotent (by the above lemma), any term-relation of the form r^{st} is closed. Thus valuable term-relations (ones satisfying $r = r^{vst}$) are in particular closed.   □

The following exercise establishes a supply of valuable term-relations that we will need later.

7.6.7   EXERCISE [RECOMMENDED, ★★]: Given any value-relation $r \in VRel(\mathsf{T}, \mathsf{T}')$, show that r^{st} is valuable, i.e., satisfies $r^{st} = (r^{st})^{vst}$.   □

Closed term-relations (and hence also valuable term-relations) have excellent "admissibility" properties that we record in the following lemma.

7.6.8   LEMMA: If $r \in TRel(\mathsf{T}, \mathsf{T}')$ satisfies $r = r^{st}$ (and in particular if it is valuable), then it has the following properties.

Equivalence-respecting If $(\mathsf{t}, \mathsf{t}') \in r$, $\varnothing \vdash \mathsf{t} =_{\text{ciu}} \mathsf{t}_1 : \mathsf{T}$, and $\varnothing \vdash \mathsf{t}' =_{\text{ciu}} \mathsf{t}_1' : \mathsf{T}$, then $(\mathsf{t}_1, \mathsf{t}_1') \in r$.

Admissibility Given recursive function values $\mathsf{F} \stackrel{\text{def}}{=} \mathsf{fun}\ \mathsf{f(x:T_1)=u:T_2}$ and $\mathsf{F'} \stackrel{\text{def}}{=} \mathsf{fun}\ \mathsf{f(x:T_1)=u':T_2}$, let F_n and F'_n $(n = 0, 1, \ldots)$ be their "unwindings," as in Theorem 7.4.4. If $([\mathsf{x} \mapsto \mathsf{F}_n]\mathsf{t}, [\mathsf{x} \mapsto \mathsf{F}'_n]\mathsf{t'}) \in r$ for all $n = 0, 1, \ldots$, then $([\mathsf{x} \mapsto \mathsf{F}]\mathsf{t}, [\mathsf{x} \mapsto \mathsf{F}']\mathsf{t'}) \in r$. $\qquad\qquad\square$

Proof: Suppose $(\mathsf{t}, \mathsf{t'}) \in r$, $\varnothing \vdash \mathsf{t} =_{\text{ciu}} \mathsf{t}_1 : \mathsf{T}$ and $\varnothing \vdash \mathsf{t'} =_{\text{ciu}} \mathsf{t}'_1 : \mathsf{T}$. To see that $(\mathsf{t}_1, \mathsf{t}'_1) \in r$, since $r = (r^s)^t$, it suffices to show for all $(S, S') \in r^s$ that $\langle S, \mathsf{t}_1 \rangle \downarrow$ iff $\langle S', \mathsf{t}'_1 \rangle \downarrow$. But

$$
\begin{array}{ll}
\langle S, \mathsf{t}_1 \rangle \downarrow \text{ iff } \langle S, \mathsf{t} \rangle \downarrow & (\text{since } \varnothing \vdash \mathsf{t} =_{\text{ciu}} \mathsf{t}_1 : \mathsf{T}) \\
\quad\quad\quad \text{iff } \langle S', \mathsf{t'} \rangle \downarrow & (\text{since } (S, S') \in r^s \text{ and } (\mathsf{t}, \mathsf{t'}) \in r) \\
\quad\quad\quad \text{iff } \langle S', \mathsf{t}'_1 \rangle \downarrow & (\text{since } \varnothing \vdash \mathsf{t'} =_{\text{ciu}} \mathsf{t}'_1 : \mathsf{T}).
\end{array}
$$

For the **Admissibility** property we apply the Unwinding Theorem. Suppose $([\mathsf{x} \mapsto \mathsf{F}_n]\mathsf{t}, [\mathsf{x} \mapsto \mathsf{F}'_n]\mathsf{t'}) \in r$ holds for all $n = 0, 1, \ldots$. Then for any $(S, S') \in r^s$ we have

$$
\begin{array}{ll}
\langle S, [\mathsf{x} \mapsto \mathsf{F}]\mathsf{t} \rangle \downarrow & \\
\text{iff for some } n,\ \langle S, [\mathsf{x} \mapsto \mathsf{F}_n]\mathsf{t} \rangle \downarrow & (\text{by Theorem 7.4.4}) \\
\text{iff for some } n,\ \langle S', [\mathsf{x} \mapsto \mathsf{F}'_n]\mathsf{t'} \rangle \downarrow & (\text{since } (S, S') \in r^s \text{ and} \\
 & \quad ([\mathsf{x} \mapsto \mathsf{F}_n]\mathsf{t}, [\mathsf{x} \mapsto \mathsf{F}'_n]\mathsf{t'}) \in r) \\
\text{iff } \langle S, [\mathsf{x} \mapsto \mathsf{F}']\mathsf{t'} \rangle \downarrow & (\text{by Theorem 7.4.4 again})
\end{array}
$$

and therefore $([\mathsf{x} \mapsto \mathsf{F}]\mathsf{t}, [\mathsf{x} \mapsto \mathsf{F}']\mathsf{t'}) \in (r^s)^t$; but $r^{st} = r$. $\qquad\square$

7.6.9   DEFINITION [ACTION OF TYPES ON TERM-RELATIONS]: The action of types on term-relations takes the following form: if $\mathsf{T}(\overline{\mathsf{X}})$ is a type whose free type variables lie among the list $\overline{\mathsf{X}} = \mathsf{X}_1, \ldots, \mathsf{X}_n$, then given a corresponding list of term relations $r_1 \in \mathit{TRel}(\mathsf{T}_1, \mathsf{T}'_1), \ldots, r_n \in \mathit{TRel}(\mathsf{T}_n, \mathsf{T}'_n)$, we define a term relation $\mathsf{T}[\overline{r}] \in \mathit{TRel}([\overline{\mathsf{X}} \mapsto \overline{\mathsf{T}}]\mathsf{T}, [\overline{\mathsf{X}} \mapsto \overline{\mathsf{T}'}]\mathsf{T})$. The definition is by induction on the structure of T as follows.

$$
\begin{array}{rcl}
\mathsf{X}_i[\overline{r}] & \stackrel{\text{def}}{=} & (r_i)^{vst} \\
\mathsf{Gnd}[\overline{r}] & \stackrel{\text{def}}{=} & (\mathit{Id}_{\mathsf{Gnd}})^{st} \\
(\mathsf{T}_1 {\to} \mathsf{T}_2)[\overline{r}] & \stackrel{\text{def}}{=} & \mathsf{fun}(\mathsf{T}_1[\overline{r}], \mathsf{T}_2[\overline{r}])^{st} \\
\{\mathsf{l}_i {:} \mathsf{T}_i{}^{\ i \in 1..n}\}[\overline{r}] & \stackrel{\text{def}}{=} & \{\mathsf{l}_i {=} \mathsf{T}_i[\overline{r}]{}^{\ i \in 1..n}\}^{st} \\
(\forall \mathsf{X}.\mathsf{T})[\overline{r}] & \stackrel{\text{def}}{=} & (\lambda r . \mathsf{T}[r, \overline{r}])^{st} \\
\{\exists \mathsf{X}, \mathsf{T}\}[\overline{r}] & \stackrel{\text{def}}{=} & \{\exists r, \mathsf{T}[r, \overline{r}]\}^{st}
\end{array}
$$

$\boxed{Id_{\mathsf{Gnd}}}\ \in VRel(\mathsf{Gnd},\mathsf{Gnd})$
is $\{(\mathsf{c},\mathsf{c})\mid Typeof(\mathsf{c}) = \mathsf{Gnd}\}$.

$\boxed{\mathsf{fun}(r_1,r_2)}\ \in VRel(\mathsf{T}_1{\to}\mathsf{T}_2,\mathsf{T}_1'{\to}\mathsf{T}_2')$,
given $r_1 \in TRel(\mathsf{T}_1,\mathsf{T}_1')$ and $r_2 \in TRel(\mathsf{T}_2,\mathsf{T}_2')$,
is defined by:

$\quad(\mathsf{v},\mathsf{v}') \in \mathsf{fun}(r_1,r_2)$ if and only if for all
$\quad(\mathsf{v}_1,\mathsf{v}_1') \in (r_1)^v$, it is the case that
$\quad(\mathsf{v}\ \mathsf{v}_1,\mathsf{v}'\ \mathsf{v}_1') \in r_2$.

$\boxed{\{\mathsf{l}_i{=}r_i{}^{\ i\in 1..n}\}}\ \in VRel(\{\mathsf{l}_i{:}\mathsf{T}_i{}^{\ i\in 1..n}\},$
$\qquad\qquad\qquad\qquad\{\mathsf{l}_i{:}\mathsf{T}_i'{}^{\ i\in 1..n}\})$
given $(r_i \in TRel(\mathsf{T}_i,\mathsf{T}_i')^{\ i\in 1..n})$,
is defined by:

$\quad(\mathsf{v},\mathsf{v}') \in \{\mathsf{l}_i{=}r_i{}^{\ i\in 1..n}\}$ if and only if for all
$\quad i \in 1..n$, it is the case that
$\quad(\mathsf{v}.\mathsf{l}_i,\mathsf{v}'.\mathsf{l}_i) \in r_i$.

$\boxed{\lambda r\,.\,R(r)}\ \in VRel(\forall\mathsf{X}.\mathsf{T},\forall\mathsf{X}.\mathsf{T}')$,
given $R(r) \in TRel([\mathsf{X} \mapsto \mathsf{T}_1]\mathsf{T},[\mathsf{X} \mapsto \mathsf{T}_1']\mathsf{T}'))$ for
$r \in TRel(\mathsf{T}_1,\mathsf{T}_1')$ and $\mathsf{T}_1,\mathsf{T}_1' \in Typ$,
is defined by:

$\quad(\mathsf{v},\mathsf{v}') \in \lambda r\,.\,R(r)$ if and only if for all
$\quad\mathsf{T}_1,\mathsf{T}_1' \in Typ$ and all $r \in TRel(\mathsf{T}_1,\mathsf{T}_1')$, it
$\quad$is the case that $(\mathsf{v}\ \mathsf{T}_1,\mathsf{v}'\ \mathsf{T}_1') \in R(r)$.

$\boxed{\{\exists r,R(r)\}}\ \in VRel(\{\exists\mathsf{X},\mathsf{T}\},\{\exists\mathsf{X},\mathsf{T}'\})$,
given $R(r) \in TRel([\mathsf{X} \mapsto \mathsf{T}_1]\mathsf{T},[\mathsf{X} \mapsto \mathsf{T}_1']\mathsf{T}'))$ for
$r \in TRel(\mathsf{T}_1,\mathsf{T}_1')$ and $\mathsf{T}_1,\mathsf{T}_1' \in Typ$,
is defined by:

$\quad(\mathsf{v},\mathsf{v}') \in \{\exists r,R(r)\}$ if and only if there
$\quad$exist $\mathsf{T}_1,\mathsf{T}_1' \in Typ, r \in TRel(\mathsf{T}_1,\mathsf{T}_1')$ and
$\quad(\mathsf{v}_1,\mathsf{v}_1') \in R(r)$ with
$\quad\mathsf{v} = \{{*}\mathsf{T}_1,\mathsf{v}_1\}$ as $\{\exists\mathsf{X},\mathsf{T}\}$ and
$\quad\mathsf{v}' = \{{*}\mathsf{T}_1',\mathsf{v}_1'\}$ as $\{\exists\mathsf{X},\mathsf{T}'\}$.

Figure 7-5: Type-directed constructions on term-relations

In addition to the operations on term-, value- and stack-relations given in Definition 7.6.3, these definitions make use of the operations for constructing value-relations from term-relations given in Figure 7-5.   □

We can use the action of types on term-relations to define a type-respecting binary relation between *open* terms (in the sense of Definition 7.5.1) by insisting that if we substitute related terms for the free value variables, the resulting terms are still related. This "mapping related things to related things" property is the common characteristic of the wide variety of constructs called *logical relations* that have arisen since the seminal work of Plotkin (1973) and Statman (1985) concerning simply typed λ-calculus; see also Chapter 6.

7.6.10   DEFINITION [LOGICAL RELATION, Δ]: Given $\Gamma \vdash \mathsf{t} : \mathsf{T}$ and $\Gamma \vdash \mathsf{t}' : \mathsf{T}$, with $\Gamma = \mathsf{X}_1,\dots,\mathsf{X}_m, \mathsf{x}_1 : \mathsf{T}_1,\dots,\mathsf{x}_n : \mathsf{T}_n$ say, we write $\Gamma \vdash \mathsf{t} \ \Delta\ \mathsf{t}' : \mathsf{T}$ to mean that for all Γ-closing substitutions σ,σ' (cf. Definition 7.5.2(v)) and all families of term-relations $\overline{r} = (r_i \in TRel(\sigma(\mathsf{X}_i),\sigma'(\mathsf{X}_i))^{\ i\in 1..m}$, if $(\sigma(\mathsf{x}_j),\sigma'(\mathsf{x}_j)) \in \mathsf{T}_j[\overline{r}]^v$ holds for each $j = 1,\dots,n$, then $(\sigma(\mathsf{t}),\sigma'(\mathsf{t}')) \in \mathsf{T}[\overline{r}]$.   □

7.6.11   REMARK: Since it is far from straightforward, the form of Definitions 7.6.9 and 7.6.10 deserves some explanation. These definitions embody certain ex-

tensionality and parametricity properties (see §7.7 and Theorem 7.7.8) that we wish to show hold for F_{ML} contextual equivalence: eventually we show that the above logical relation Δ coincides with contextual equivalence (Theorem 7.6.25). To get that coincidence we have to formulate the definition of Δ so that it satisfies the crucial property of Lemma 7.6.17 below (the so-called fundamental property of the logical relation) and is adequate (Lemma 7.6.24). The definition of the action of types on term-relations in Definition 7.6.9 is carefully formulated to ensure these properties hold.

First of all, note the use of closing substitutions to reduce the logical relation for open terms to that for closed ones. This builds in the "instantiation" aspect of ciu-equivalence that we wish to prove of contextual equivalence. (It also means that the logical relation has the "monotonicity" property *monotonicity property of logical relations* considered in Chapter 6.)

Secondly, we want $T[\overline{r}]$ to always be a closed term-relation, because then it has the equivalence-respecting and admissibility properties noted in Lemma 7.6.8. This accounts for the use of $(-)^{st}$ in the definition. The $(-)^{s}$ and $(-)^{t}$ operators build into the logical relation a delicate interplay between terms and frame stacks. Of course this relies on the formulation of the operational semantics of F_{ML} in §7-3: although more traditional "big-step" or "small-step" operational semantics lead to the same termination relation (cf. Exercise 7.4.2), the pairing between frame stacks and terms defined in Figure 7-2 is ideal for our purposes.

Lastly, the call-by-value nature of F_{ML} dictates that relational parametricity properties of polymorphic types should be with respect to term-relations that are valuable; but instead of letting r range over such relations in the definition of $(\forall X.T)[\overline{r}]$ and $\{\exists X,T\}[\overline{r}]$ we have used an equivalent formulation in which r ranges over all term-relations (of appropriate type), but type variables X are interpreted using the closure of the value-restriction operator $(-)^{v}$: for in fact as r ranges over all term-relations, r^{vst} ranges over all valuable term-relations. □

The rest of this section is devoted to showing that contextual equivalence and ciu-equivalence coincide with the logical relation.

7.6.12   LEMMA: Each of the term relations $T[\overline{r}]$ defined in Definition 7.6.9 is valuable, i.e., satisfies $T[\overline{r}] = T[\overline{r}]^{vst}$, and hence in particular by Corollary 7.6.6 is closed. □

Proof: It is immediate from the definition that each $T[\overline{r}]$ is of the form r^{st} for some *value*-relation r; so just apply Exercise 7.6.7. □

The following lemma helps with calculations involving the action on term-relations of function types. We give its proof in detail since it typifies the kind

of reasoning needed when working with the Galois connection given by the $(-)^s$ and $(-)^t$ operators. (For related properties for record and $\forall$-types, see Exercise 7.6.14.)

7.6.13   LEMMA: The operation $\mathsf{fun}(-,-)$ from Definition 7.6.9(ii) satisfies

$$\mathsf{fun}(r_1, (r_2)^{st})^{stv} = \mathsf{fun}(r_1, (r_2)^{st}) \tag{7.19}$$

$$\mathsf{fun}((r_1)^{vst}, (r_2)^{st}) = \mathsf{fun}(r_1, (r_2)^{st}). \tag{7.20}$$

Proof: To prove (7.19), first note that since $(-)^{st}$ is inflationary (Lemma 7.6.5) we have $\mathsf{fun}(r_1, (r_2)^{st}) \subseteq \mathsf{fun}(r_1, (r_2)^{st})^{st}$; and since $\mathsf{fun}(r_1, (r_2)^{st})$ is a value-relation, it follows that $\mathsf{fun}(r_1, (r_2)^{st}) \subseteq \mathsf{fun}(r_1, (r_2)^{st})^{stv}$. For the reverse inclusion it suffices to prove

$$\mathsf{fun}(r_1, (r_2)^{st})^{st} \subseteq \mathsf{fun}(r_1, (r_2)^{st}) \tag{7.21}$$

and then apply $(-)^v$ to both sides (noting that $\mathsf{fun}(r_1, (r_2)^{st})$, being a value-relation, is equal to $\mathsf{fun}(r_1, (r_2)^{st})^v$). For (7.21) we use the following simple property of the termination relation (Figure 7-2) with respect to application:

$$\langle S \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1), \mathsf{v} \rangle {\downarrow} \;\;\Leftrightarrow\;\; \langle S, \mathsf{v}\,\mathsf{v}_1 \rangle {\downarrow}$$

and hence

$$(\langle S, \mathsf{v}\,\mathsf{v}_1 \rangle {\downarrow} \Leftrightarrow \langle S', \mathsf{v}'\,\mathsf{v}_1' \rangle {\downarrow}) \;\;\Leftrightarrow\;\;$$
$$(\langle S \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1), \mathsf{v} \rangle {\downarrow} \Leftrightarrow \langle S' \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1'), \mathsf{v}' \rangle {\downarrow}) \quad \text{(7.22)}$$

If $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})$ and $(\mathsf{v}_1, \mathsf{v}_1') \in (r_1)^v$, then we have $(\mathsf{v}\,\mathsf{v}_1, \mathsf{v}'\,\mathsf{v}_1') \in (r_2^s)^t$ by definition of the $\mathsf{fun}(-,-)$ operation on term-relations (Figure 7-5). So if $(S, S') \in (r_2)^s$, then

$$\langle S, \mathsf{v}\,\mathsf{v}_1 \rangle {\downarrow} \;\;\Leftrightarrow\;\; \langle S', \mathsf{v}'\,\mathsf{v}_1' \rangle {\downarrow}$$

and hence by (7.22)

$$\langle S \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1), \mathsf{v} \rangle {\downarrow} \;\;\Leftrightarrow\;\; \langle S' \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1'), \mathsf{v}' \rangle {\downarrow}.$$

Since this holds for all $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})$, we deduce that

$$(S, S') \in (r_2)^s \;\&\; (\mathsf{v}_1, \mathsf{v}_1') \in (r_1)^v \;\Rightarrow$$
$$(S \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1), S' \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1')) \in \mathsf{fun}(r_1, (r_2)^{st})^s.$$

So for any $(S, S') \in (r_2)^s$ and $(\mathsf{v}_1, \mathsf{v}_1') \in (r_1)^v$, since

$$(S \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1), S' \circ (\mathsf{f}.\mathsf{f}\,\mathsf{v}_1')) \in \mathsf{fun}(r_1, (r_2)^{st})^s$$

it follows that if

$$(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})^{st} \tag{7.23}$$

then $\langle S \circ (\mathsf{f}.\mathsf{f}\ \mathsf{v}_1), \mathsf{v} \rangle \downarrow \ \Leftrightarrow \ \langle S' \circ (\mathsf{f}.\mathsf{f}\ \mathsf{v}_1'), \mathsf{v}' \rangle \downarrow$, and hence by (7.22) it follows that $\langle S, \mathsf{v}\ \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S', \mathsf{v}'\ \mathsf{v}_1' \rangle \downarrow$. Since this holds for all $(S, S') \in (r_2)^s$, it follows that $(\mathsf{v}\ \mathsf{v}_1, \mathsf{v}'\ \mathsf{v}_1') \in (r_2)^{st}$ whenever $(\mathsf{v}_1, \mathsf{v}_1') \in (r_1)^v$. So $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})$ whenever (7.23) holds; thus we have proved the inclusion in (7.21), as required.

Turning to the proof of (7.20), first note that since since $(-)^{st}$ is inflationary, we have $(r_1)^v \subseteq (r_1)^{vst}$. So since $\mathsf{fun}(-,-)$ is clearly order-reversing in its first argument, we have $\mathsf{fun}((r_1)^{vst}, (r_2)^{st}) \subseteq \mathsf{fun}((r_1)^v, (r_2)^{st})$; and $\mathsf{fun}((r_1)^v, (r_2)^{st}) = \mathsf{fun}(r_1, (r_2)^{st})$, because $\mathsf{fun}(-,-)$ only depends upon the values related by its first argument. Thus to prove (7.20), we just have to show

$$\mathsf{fun}(r_1, (r_2)^{st}) \subseteq \mathsf{fun}((r_1)^{vst}, (r_2)^{st}). \tag{7.24}$$

For this we use the following fact about termination

$$\langle S \circ (\mathsf{x}.\mathsf{v}\ \mathsf{x}), \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S, \mathsf{v}\ \mathsf{v}_1 \rangle \downarrow$$

which is immediate from the definition in Figure 7-2. From this it follows that

$$(\langle S, \mathsf{v}\ \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S', \mathsf{v}'\ \mathsf{v}_1' \rangle \downarrow) \ \Leftrightarrow$$
$$(\langle S \circ (\mathsf{x}.\mathsf{v}\ \mathsf{x}), \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S' \circ (\mathsf{x}.\mathsf{v}'\ \mathsf{x}), \mathsf{v}_1' \rangle \downarrow) \tag{7.25}$$

If $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})$ and $(\mathsf{v}_1, \mathsf{v}_1') \in (r_1)^v$, then by definition of $\mathsf{fun}(-,-)$ we have $(\mathsf{v}\ \mathsf{v}_1, \mathsf{v}'\ \mathsf{v}_1') \in (r_2)^{st}$. So if $(S, S') \in (r_2)^s$, then

$$\langle S, \mathsf{v}\ \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S', \mathsf{v}'\ \mathsf{v}_1' \rangle \downarrow$$

and hence by (7.25) we have

$$\langle S \circ (\mathsf{x}.\mathsf{v}\ \mathsf{x}), \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S' \circ (\mathsf{x}.\mathsf{v}'\ \mathsf{x}), \mathsf{v}_1' \rangle \downarrow.$$

Since this holds for all $(\mathsf{v}_1, \mathsf{v}_1') \in (r_1)^v$, we deduce that

$$(S, S') \in (r_2)^s \ \& \ (\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st}) \ \Rightarrow$$
$$(S \circ (\mathsf{x}.\mathsf{v}\ \mathsf{x}), S' \circ (\mathsf{x}.\mathsf{v}'\ \mathsf{x})) \in (r_1)^{vs}.$$

So for any $(S, S') \in (r_2)^s$ and $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})$, since $(S \circ (\mathsf{x}.\mathsf{v}\ \mathsf{x}), S' \circ (\mathsf{x}.\mathsf{v}'\ \mathsf{x})) \in (r_1)^{vs}$, it follows for any $(\mathsf{v}_1, \mathsf{v}_1') \in ((r_1)^{vst})^v \subseteq ((r_1)^{vs})^t$ that we have $\langle S \circ (\mathsf{x}.\mathsf{v}\ \mathsf{x}), \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S' \circ (\mathsf{x}.\mathsf{v}'\ \mathsf{x}), \mathsf{v}_1' \rangle \downarrow$, and hence by (7.25) that $\langle S, \mathsf{v}\ \mathsf{v}_1 \rangle \downarrow \ \Leftrightarrow \ \langle S', \mathsf{v}'\ \mathsf{v}_1' \rangle \downarrow$. Since this holds for all $(S, S') \in (r_2)^s$, it follows that $(\mathsf{v}\ \mathsf{v}_1, \mathsf{v}'\ \mathsf{v}_1') \in (r_2)^{st}$. Hence $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}((r_1)^{vst}, (r_2)^{st})$ whenever $(\mathsf{v}, \mathsf{v}') \in \mathsf{fun}(r_1, (r_2)^{st})$, as required for (7.24). $\qquad\square$

7.6.14   EXERCISE [RECOMMENDED, ⋆]: Show that constructions (iii) and (iv) in Definition 7.6.9 satisfy

$$\{l_i = (r_i)^{st\ i \in 1..n}\}^{stv} = \{l_i = (r_i)^{st\ i \in 1..n}\} \tag{7.26}$$

$$(\lambda r . R(r)^{st})^{stv} = \lambda r . R(r)^{st}. \tag{7.27}$$

(Cf. the proof of Lemma 7.6.13.)   □

7.6.15   LEMMA: For all ground types Gnd, $(Id_{\mathsf{Gnd}})^{stv} = Id_{\mathsf{Gnd}}$.   □

Proof: Since $(-)^{st}$ is idempotent (Lemma 7.6.5), we have $Id_{\mathsf{Gnd}} \subseteq (Id_{\mathsf{Gnd}})^{st}$; and since Id_{Gnd} is a value-relation it follows that $Id_{\mathsf{Gnd}} \subseteq (Id_{\mathsf{Gnd}})^{stv}$. To prove the reverse inclusion, for each constant c of type Gnd consider

$$\mathsf{diverge} \stackrel{\text{def}}{=} (\mathsf{fun\ f(b:Bool)} = \mathsf{f\ b : Bool)true}$$

$$S_{\mathsf{c}} \stackrel{\text{def}}{=} Id \circ (\mathsf{x.\ if\ x=c\ then\ true\ else\ diverge}).$$

Note that for all constants c′ of type Gnd

$$\langle S_{\mathsf{c}}, \mathsf{c}' \rangle \downarrow \ \Leftrightarrow \ \mathsf{c} = \mathsf{c}'. \tag{7.28}$$

Furthermore, since $(\mathsf{c}', \mathsf{c}'') \in Id_{\mathsf{Gnd}}$ iff $\mathsf{c}' = \mathsf{c}''$, we have that $(S_{\mathsf{c}}, S_{\mathsf{c}}) \in (Id_{\mathsf{Gnd}})^s$; so if the constants c and c′ satisfy $(\mathsf{c}, \mathsf{c}') \in (Id_{\mathsf{Gnd}})^{st}$, then we have $\langle S_{\mathsf{c}}, \mathsf{c} \rangle \downarrow \Leftrightarrow \langle S_{\mathsf{c}}, \mathsf{c}' \rangle \downarrow$. So by (7.28), $(\mathsf{c}, \mathsf{c}') \in (Id_{\mathsf{Gnd}})^{st}$ implies $\mathsf{c} = \mathsf{c}'$; thus $(Id_{\mathsf{Gnd}})^{stv} \subseteq Id_{\mathsf{Gnd}}$.   □

7.6.16   LEMMA: The action of types on term-relations of Definition 7.6.9 has the following substitution property. For any types T and T′ with $ftv(\mathsf{T}) \subseteq \mathsf{X}, \overline{\mathsf{X}}$ and $ftv(\mathsf{T}') \subseteq \overline{\mathsf{X}}$, it is the case that $([\mathsf{X} \mapsto \mathsf{T}']\mathsf{T})[\overline{r}] = \mathsf{T}[\mathsf{T}'[\overline{r}], \overline{r}]$.   □

Proof: This follows by induction on the structure of the type T; for the base case when T = X, use Lemma 7.6.12.   □

7.6.17   LEMMA [FUNDAMENTAL PROPERTY OF THE LOGICAL RELATION]: The logical relation Δ of Definition 7.6.10 has the substitutivity and compatibility properties defined in Figure 7-4.   □

Proof: The first substitutivity property in Figure 7-4 (closure under substituting values for value variables) holds for Δ because of the way it is defined in terms of closing substitutions. The second substitutivity property (closure under substituting types for types variables) holds for Δ because of Lemma 7.6.16.

Now consider the compatibility properties given in Figure 7-4. There is one for each clause in the grammar of F_{ML} terms and values (Figure 7-1). We consider each in turn, giving the details in some cases and setting the others as exercises (with solutions).

Value variables: This case is immediate from the definition of Δ in Definition 7.6.10.

Constants: We have to show for each constant c, with $Typeof(c) = $ Gnd say, that $(c, c) \in $ Gnd$[\overline{r}] = (Id_{Gnd})^{st}$. But by definition of Id_{Gnd} (Figure 7-5), $(c, c) \in Id_{Gnd}$; and $Id_{Gnd} \subseteq (Id_{Gnd})^{st}$ by Lemma 7.6.5.

Recursive functions: Using property (7.19) and the fact that each $T[\overline{r}]$ is valuable and hence closed (Lemma 7.6.12), the compatibility property for recursive functions reduces to proving the property in Exercise 7.6.18.

Record values: This case follows from the property in Exercise 7.6.19.

Type abstractions: This case follows from the property in Exercise 7.6.20.

Package values: This case follows easily from the definition of $\{\exists r, R(r)\}$ in Figure 7-5, using Lemma 7.6.16.

Conditionals: This case follows from the property in Exercise 7.6.21.

Operations: In view of Lemma 7.6.15, this compatibility property follows once we prove $(op(c_i{}^{i\in 1..n}), op(c_i{}^{i\in 1..n})) \in (Id_{Gnd})^{st}$ for any (suitably typed) constants c_i and operator op. But if the value of $op(c_i{}^{i\in 1..n})$ is the constant c say, then for any S

$$\langle S, op(c_i{}^{i\in 1..n})\rangle \downarrow \;\Leftrightarrow\; \langle S, c\rangle \downarrow.$$

Hence for any $(S, S') \in (Id_{Gnd'})^s$ (where Gnd$' = Typeof(c)$), we have

$$
\begin{aligned}
\langle S, op(c_i{}^{i\in 1..n})\rangle \downarrow &\Leftrightarrow \langle S, c\rangle \downarrow \\
&\Leftrightarrow \langle S', c\rangle \downarrow && \text{(since } (c, c) \in Id_{Gnd'}) \\
&\Leftrightarrow \langle S', op(c_i{}^{i\in 1..n})\rangle \downarrow.
\end{aligned}
$$

So we do indeed have $(op(c_i{}^{i\in 1..n}), op(c_i{}^{i\in 1..n})) \in (Id_{Gnd})^{st}$.

Applications: This case amounts to proving that if recursive function values v and v' satisfy $(v, v') \in fun(r_1, r_2)^{st}$ for some closed term-relations r_1 and r_2, then for any $(v_1, v'_1) \in r_1$ it is the case that $(v\, v_1, v'\, v'_1) \in r_2$. But this property follows immediately from the definition of $fun(-, -)$ using the first part of Lemma 7.6.13: for

$$
\begin{aligned}
(v, v') \in fun(r_1, r_2)^{stv} \\
= fun(r_1, (r_2)^{st})^{stv} && \text{(since } r_2 \text{ is closed)} \\
= fun(r_1, (r_2)^{st}) && \text{(by (7.19))} \\
= fun(r_1, r_2) && \text{(since } r_2 \text{ is closed).}
\end{aligned}
$$

Projections: This case is similar to the previous one, but using property (7.26) from Exercise 7.6.14 rather than (7.19).

Type applications: This case is similar to the previous one, using property (7.27) from Exercise 7.6.14.

Unpacking: This case follows from the property in Exercise 7.6.22.

Sequencing: This case follows from the property in Exercise 7.6.23.   □

7.6.18   EXERCISE [RECOMMENDED, ★★★]: Suppose

$$F \stackrel{\text{def}}{=} \mathsf{fun}\ f(x{:}T_1){=}t{:}T_2 \in \mathit{Val}(T_1{\rightarrow}T_2)$$

$$F' \stackrel{\text{def}}{=} \mathsf{fun}\ f(x{:}T_1'){=}t'{:}T_2' \in \mathit{Val}(T_1'{\rightarrow}T_2')$$

$$r_1 \in \mathit{TRel}(T_1, T_1')$$

$$r_2 \in \mathit{TRel}(T_2, T_2')$$

satisfy $r_2 = (r_2)^{st}$ and

$$([\mathsf{f} \mapsto \mathsf{v}][\mathsf{x} \mapsto \mathsf{v}_1]\mathsf{t}, [\mathsf{f} \mapsto \mathsf{v}'][\mathsf{x} \mapsto \mathsf{v}_1']\mathsf{t}') \in r_2, \tag{7.29}$$
$$\text{for all } (\mathsf{v},\mathsf{v}') \in \mathsf{fun}(r_1,r_2) \text{ and } (\mathsf{v}_1,\mathsf{v}_1') \in (r_1)^v.$$

Use the admissibility property of valuable term-relations established in Lemma 7.6.8 to show that $(F, F') \in \mathsf{fun}(r_1, r_2)$.   □

7.6.19   EXERCISE [★★]: Suppose for $i \in 1..n$ that $\mathsf{v}_i \in \mathit{Val}(T_i)$, $\mathsf{v}_i' \in \mathit{Val}(T_i')$ and $r_i \in \mathit{TRel}(T_i, T_i')$ with $r_i = (r_i)^{st}$. Putting

$$\mathsf{v} \stackrel{\text{def}}{=} \{\mathsf{l}_i{=}\mathsf{v}_i\ ^{i \in 1..n}\} \in \mathit{Val}(\{\mathsf{l}_i{:}T_i\ ^{i \in 1..n}\})$$

$$\mathsf{v}' \stackrel{\text{def}}{=} \{\mathsf{l}_i{=}\mathsf{v}_i'\ ^{i \in 1..n}\} \in \mathit{Val}(\{\mathsf{l}_i{:}T_i'\ ^{i \in 1..n}\})$$

show that if $(\mathsf{v}_i, \mathsf{v}_i') \in r_i$ for $i \in 1..n$, then $(\mathsf{v}, \mathsf{v}')$ is in the value-relation $\{\mathsf{l}_i{=}r_i\ ^{i \in 1..n}\}$ defined in Figure 7-5.   □

7.6.20   EXERCISE [★★]: Let T and T′ be types with at most X free. For each $T_1, T_1' \in \mathit{Typ}$ and $r \in \mathit{TRel}(T_1, T_1')$ suppose we are given a closed term-relation $R(r)$ in $\mathit{TRel}([X \mapsto T_1]T, [X \mapsto T_1']T'))$ (i.e., $R(r) = R(r)^{st}$). Show that if the values v and v' satisfy

$$X \vdash \mathsf{v} : T$$

$$X \vdash \mathsf{v}' : T'$$

$$\forall T_1, T_1' \in \mathit{Typ}, r \in \mathit{TRel}(T_1, T_1'). \ ([X \mapsto T_1]\mathsf{v}, [X \mapsto T_1']\mathsf{v}') \in R(r)$$

then $(\lambda X.\mathsf{v}, \lambda X.\mathsf{v}')$ is in the value-relation $\lambda r.R(r)$ defined in Figure 7-5.   □

7.6.21   EXERCISE [★★]: Suppose $(v, v') \in (Id_{\text{Bool}})^{st}$ and $(t_1, t'_1), (t_2, t'_2) \in r$, where $r \in TRel(T, T')$ is closed (i.e., $r = (r)^{st}$). Show that

$$(\text{if } v \text{ then } t_1 \text{ else } t_2, \text{if } v' \text{ then } t'_1 \text{ else } t'_2)$$

is in r. □

7.6.22   EXERCISE [★★]: Let T and T' be types with at most X free. For each $T_1, T'_1 \in Typ$ and $r_1 \in TRel(T_1, T'_1)$ suppose we are given a closed term-relation $R(r_1) = R(r_1)^{st}$ in $TRel([X \mapsto T_1]T, [X \mapsto T'_1]T')$. Suppose we are also given a closed term-relation $r_2 = (r_2)^{st} \in TRel(T_2, T'_2)$ for some closed types $T_2, T'_2 \in Typ$. Show that if the terms t, t' satisfy

$$X, x : T \vdash t : T_2$$
$$X, x : T' \vdash t' : T'_2$$
$$\forall T_1, T'_1 \in Typ, r_1 \in TRel(T_1, T'_1), (v_1, v'_1) \in (r_1)^v.$$
$$([X \mapsto T_1][x \mapsto v_1]t, [X \mapsto T_1][x \mapsto v_1]t) \in r_2$$

then whenever $(v, v') \in \{\exists r_1, R(r_1)\}^{stv}$, it is also the case that

$$(\text{let } \{*X, x\} = v \text{ in } t, \text{let } \{*X, x\} = v' \text{ in } t')$$

is in r_2. □

7.6.23   EXERCISE [★★]: Suppose we are given $r_1 \in TRel(T_1, T'_1)$, $r_2 \in TRel(T_2, T'_2)$ with r_1 valuable (i.e., $r_1 = (r_1)^{vst}$) and r_2 closed (i.e., $r_2 = (r_2)^{st}$). Show that if the terms t_2, t'_2 satisfy

$$x : T_1 \vdash t_2 : T_2$$
$$x : T'_1 \vdash t'_2 : T'_2$$
$$\forall (v_1, v'_1) \in (r_1)^v. ([x \mapsto v_1]t_2, [x \mapsto v'_1]t'_2) \in r_2$$

then whenever $(t_1, t'_1) \in r_1$, it is also the case that

$$(\text{let } x = t_1 \text{ in } t_2, \text{let } x = t'_1 \text{ in } t'_2)$$

is in r_2. □

7.6.24   LEMMA [ADEQUACY]: The logical relation Δ is adequate (Definition 7.5.1). □

Proof: Suppose $\varnothing \vdash t \, \Delta \, t' : T$; we have to show that $t \downarrow$ holds iff $t' \downarrow$ does, or equivalently that

$$\langle Id, t \rangle \downarrow \quad \text{iff} \quad \langle Id, t' \rangle \downarrow. \tag{7.30}$$

Unraveling Definition 7.6.10, the assumption that the closed terms t and t' of closed type T are Δ-related means that $(t, t') \in T[]$, the latter being the action of the type T on the empty list of term-relations. By Lemma 7.6.12, $T[]$ is valuable; so $(t, t') \in T[]^{vst}$. Hence to prove (7.30), it suffices to show that $(Id, Id) \in (T[]^v)^s$; but for any $(v, v') \in T[]^v$,

$$\langle Id, v \rangle \!\downarrow \quad \text{iff} \quad \langle Id, v' \rangle \!\downarrow$$

holds trivially by axiom (S-NilVal) in Figure 7-2.   □

We are finally able to put all the pieces together and prove the main result of this section. At the same time we complete the proof of Theorem 7.5.7.

7.6.25   THEOREM [$=_{ctx}$ EQUALS Δ EQUALS $=_{ciu}$]: F_{ML} contextual equivalence, $=_{ctx}$, (as defined in Theorem 7.5.3) coincides with the logical relation Δ of Definition 7.6.10 and with ciu-equivalence, $=_{ciu}$ (Definition 7.5.5): $\Gamma \vdash t =_{ctx} t' : T$ holds if and only if $\Gamma \vdash t \, \Delta \, t' : T$ does, if and only if $\Gamma \vdash t =_{ciu} t' : T$ does. □

Proof: It suffices to show that the following chain of inclusions holds:

$$=_{ctx} \overset{(1)}{\subseteq} =_{ciu} \overset{(3)}{\subseteq} \Delta \overset{(2)}{\subseteq} =_{ctx}.$$

(1) This is the half of Theorem 7.5.7 that we have already proved in §7.5.

(2) We have not yet shown that Δ is an equivalence relation; and in fact we will only deduce this once we have shown that it coincides with $=_{ctx}$ and $=_{ciu}$ (which are easily seen to be equivalence relations). However, we have shown that Δ is compatible, substitutive and adequate (Lemmas 7.6.17 and 7.6.24). In the proof of Theorem 7.5.3 we constructed $=_{ctx}$ as the union of all such type-respecting relations, without regard to whether they were also equivalence relations; therefore Δ is contained in $=_{ctx}$.

(3) Noting how $=_{ciu}$ and Δ are defined on open terms via substitutions, we can combine the first part of Lemma 7.6.8 with Lemma 7.6.12 to give

$$\Gamma \vdash t =_{ciu} t' : T \;\&\; \Gamma \vdash t' \, \Delta \, t'' : T \;\Rightarrow\; \Gamma \vdash t \, \Delta \, t'' : T. \tag{7.31}$$

We noted in the proof of Theorem 7.5.3 that every compatible term-relation is reflexive. (This is easily proved by induction on the structure of terms.) So since Δ is compatible (Lemma 7.6.17) it is in particular reflexive. So we can take $t' = t''$ in (7.31) to deduce that $\Gamma \vdash t =_{ctx} t' : T$ implies $\Gamma \vdash t \, \Delta \, t' : T$.   □

7.7 Operational Extensionality

In this section we develop some of the consequences of Theorem 7.6.25. Now that we know that contextual equivalence coincides with ciu-equivalence (Theorem 7.5.7), when giving general properties of $=_{ctx}$ we restrict attention to closed terms of closed type where possible, since the corresponding property for open terms can be obtained via closing substitutions.

7.7.1 THEOREM [EXTENSIONALITY FOR VALUES]: We now give extensionality principles for the various types of value; for package values, the principle is a formalization of the final one discussed in the Introduction (Principle 7.3.6).

1. CONSTANTS: Given constants c, c′ of the same ground type, Gnd say, $\varnothing \vdash$ c $=_{ctx}$ c′ : Gnd holds if and only if c = c′.

2. FUNCTIONS: Given f:$T_1 \to T_2$, x:$T_1 \vdash$ t : T_2 and f:$T_1 \to T_2$, x:$T_1 \vdash$ t′ : T_2, writing v and v′ for the recursive function values fun f(x:T_1)=t:T_2 and fun f(x:T_1)=t′:T_2 respectively, then $\varnothing \vdash$ v $=_{ctx}$ v′ : $T_1 \to T_2$ if and only if for all $\varnothing \vdash$ v_1 : T_1, it is the case that $\varnothing \vdash$ [f $\mapsto$ v][x $\mapsto$ v_1]t $=_{ctx}$ [f $\mapsto$ v′][x $\mapsto$ v_1]t′ : T_2.

3. RECORDS: Given values $\varnothing \vdash$ v_i : T_i and $\varnothing \vdash$ v_i' : T_i for $i \in 1..n$, then $\varnothing \vdash$ {l_i=v_i $^{i \in 1..n}$} $=_{ctx}$ {l_i=v_i' $^{i \in 1..n}$} : {l_i:T_i $^{i \in 1..n}$} if and only if for each $i \in 1..n$, $\varnothing \vdash$ v_i $=_{ctx}$ v_i' : T_i.

4. TYPE ABSTRACTIONS: Given X $\vdash$ v : T and X $\vdash$ v′ : T, then $\varnothing \vdash$ λX.v $=_{ctx}$ λX.v′ : ∀X.T if and only if for all closed types T′, $\varnothing \vdash$ [X $\mapsto$ T′]v $=_{ctx}$ [X $\mapsto$ T′]v′ : [X $\mapsto$ T′]T.

5. PACKAGES: For any closed existential type {∃X,T}, closed types T_1, T_2, and values $\varnothing \vdash$ v_i : [X $\mapsto$ T_i]T ($i = 1, 2$),

$$\varnothing \vdash \{*T_1, v_1\} \text{ as } \{\exists X, T\} =_{ctx} \{*T_2, v_2\} \text{ as } \{\exists X, T\} : \{\exists X, T\}$$

holds if there is some term-relation $r \in TRel(T_1, T_2)$ with $(v_1, v_2) \in T[r]$. □

Proof:

1. The property for constants follows from Lemma 7.6.15 combined with Theorem 7.6.25.

2. Suppose for all $\varnothing \vdash$ v_1 : T_1 that

$$\varnothing \vdash [f \mapsto v][x \mapsto v_1]t =_{ctx} [f \mapsto v'][x \mapsto v_1]t' : T_2 \tag{7.32}$$

where v and v' are as in part 2 of the theorem. To show $\varnothing \vdash v =_{\text{ctx}} v' : T_1 {\to} T_2$, by Theorem 7.6.25 it suffices to show $\varnothing \vdash v \mathbin{\Delta} v' : T_1 {\to} T_2$, i.e., that $(v, v') \in (T_1 {\to} T_2)[\,] = \mathsf{fun}(T_1[\,], T_2[\,])^{st}$. In fact we show that $(v, v') \in \mathsf{fun}(T_1[\,], T_2[\,])$. For this we have to prove that if $(v_1, v'_1) \in T_1[\,]^v$, then $(v\, v_1, v'\, v'_1) \in T_2[\,]$. By Theorem 7.6.25 again, this is the same as showing: if $\varnothing \vdash v_1 =_{\text{ctx}} v'_1 : T_1$, then $\varnothing \vdash v\, v_1 =_{\text{ctx}} v'\, v'_1 : T_2$. As noted in Corollary 7.5.8, we can turn the primitive reduction for function application into a ciu-equivalence and hence by Theorem 7.6.25 into a contextual equivalence:

$$\varnothing \vdash v\, v_1 =_{\text{ctx}} [f \mapsto v][x \mapsto v_1]t : T_2 \tag{7.33}$$

and similarly for $v'\, v'_1$. Therefore we just need to show: if $\varnothing \vdash v_1 =_{\text{ctx}} v'_1 : T_1$, then $\varnothing \vdash [f \mapsto v][x \mapsto v_1]t =_{\text{ctx}} [f \mapsto v'][x \mapsto v'_1]t' : T_2$. But this follows from the assumption (7.32) using the reflexivity and substitutivity properties of $=_{\text{ctx}}$. So we have established one half (the difficult half) of the property in 2. For the converse, if $\varnothing \vdash v =_{\text{ctx}} v' : T_1 {\to} T_2$, then for any $\varnothing \vdash v_1 : T_1$, the compatibility properties of $=_{\text{ctx}}$ give $\varnothing \vdash v\, v_1 =_{\text{ctx}} v'\, v_1 : T_2$; and then as before, we can compose with (7.33) to get (7.32).

3. We leave the extensionality property for records as an exercise (7.7.2).

4. For the property for type abstractions, suppose

$$\forall T' \in \mathit{Typ}. \quad \varnothing \vdash [X \mapsto T']v =_{\text{ctx}} [X \mapsto T']v' : [X \mapsto T']T. \tag{7.34}$$

Note that since Δ coincides with $=_{\text{ctx}}$ (Theorem 7.6.25) it is reflexive and hence $X \vdash v \mathbin{\Delta} v : T$ holds. According to Definition 7.6.10 this means that for all $T_1, T'_1 \in \mathit{Typ}$ and $r \in \mathit{TRel}(T_1, T'_1)$, $([X \mapsto T_1]v, [X \mapsto T'_1]v) \in T[r]$. Since $T[r]$ is closed (Lemma 7.6.12), we can combine (7.34) with the first part of Lemma 7.6.8 (using $=_{\text{ctx}}$ in place of $=_{\text{ciu}}$ by virtue of Theorem 7.6.25) to conclude that $([X \mapsto T_1]v, [X \mapsto T'_1]v') \in T[r]$ for all r. Then using the equivalence in Corollary 7.5.8(v), we have

$$\forall T_1, T'_1 \in \mathit{Typ}, r \in \mathit{TRel}(T_1, T'_1). \quad ((\lambda X.v)T_1, (\lambda X.v')T'_1) \in T[r]$$

and hence $(\lambda X.v, \lambda X.v')$ is in $\lambda r.T[r]$. Since $\lambda r.T[r] \subseteq (\lambda r.T[r])^{st}$ and the latter is equal to $(\forall X.T)[\,]$ by definition, we have $\varnothing \vdash \lambda X.v \mathbin{\Delta} \lambda X.v' : \forall X.T$, and hence by Theorem 7.6.25, $\varnothing \vdash \lambda X.v =_{\text{ctx}} \lambda X.v' : \forall X.T$. So we have established one half (the difficult half) of the property in 4. The argument for the other half is similar to that for property 2, using Corollary 7.5.8(v) and the congruence properties of $=_{\text{ctx}}$.

5. Finally, let us consider the extensionality property for package values. (Note that unlike the other four, this only gives a sufficient condition for contextual equivalence; Example 7.7.4 below shows that the condition is not necessary.) If $(v_1, v_2) \in T[r]$, then from Definition 7.6.9 we have

$$(\{*T_1, v_1\} \text{ as } \{\exists X, T\}, \{*T_2, v_2\} \text{ as } \{\exists X, T\}) \in \{\exists r, T[r]\}$$
$$\subseteq \{\exists r, T[r]\}^{st}$$
$$= \{\exists X, T\}[].$$

Thus $\varnothing \vdash \{*T_1, v_1\} \text{ as } \{\exists X, T\} \ \Delta \ \{*T_2, v_2\} \text{ as } \{\exists X, T\} : \{\exists X, T\}$ and we can apply Theorem 7.6.25 to get the desired contextual equivalence. □

7.7.2 EXERCISE [★★, ↛]: Use Theorem 7.6.25, Corollary 7.5.8 and the definition of the term-relation $\{l_i = r_i^{\ i \in 1..n}\}$ in Definition 7.6.9 to deduce extensionality property 3 of Theorem 7.7.1. □

To see how Theorem 7.7.1(5) can be used in practice, we will apply it to establish the contextual equivalence of Example 7.3.5 from the Introduction.

7.7.3 EXAMPLE: Recall the type Semaphore and its values $semaphore_1$, $semaphore_2$ from Example 7.3.5. To show $\varnothing \vdash semaphore_1 =_{ctx} semaphore_2 : $ Semaphore using Theorem 7.7.1(5), it suffices to show that $(v_1, v_2) \in T[r]$ where

$T \stackrel{\text{def}}{=} \{\text{bit:X, flip:X→X, read:X→Bool}\}$

$v_1 \stackrel{\text{def}}{=} \{\text{bit=true, flip=}\lambda\text{x:Bool.not x, read=}\lambda\text{x:Int.x}\}$

$v_2 \stackrel{\text{def}}{=} \{\text{bit=1, flip=}\lambda\text{x:Int.0-2*x, read=}\lambda\text{x:Int.x >= 0}\}$

and $r \in VRel(\text{Bool}, \text{Int})$ is

$r \stackrel{\text{def}}{=} \{(\text{true}, m) \mid m = (-2)^n \text{ for some even } n \geq 0\} \cup$
$\quad\quad \{(\text{false}, m) \mid m = (-2)^n \text{ for some odd } n \geq 0\}.$

Since r is a value-relation, we can use Lemma 7.6.13 to slightly simplify $T[r]$:

$T[r] \quad \stackrel{\text{def}}{=} \quad \{\text{bit=}r^{st}, \text{flip=fun}(r^{st}, r^{st})^{st}, \text{read=fun}(r^{st}, Id^{st}_{\text{Bool}})^{st}\}^{st}$
$\quad\quad = \quad \{\text{bit=}r^{st}, \text{flip=fun}(r, r^{st})^{st}, \text{read=fun}(r, Id^{st}_{\text{Bool}})^{st}\}^{st}.$

So since $(-)^{st}$ is inflationary, to prove $(v_1, v_2) \in T[r]$, it suffices to show

$$(\text{true}, 1) \ \in \ r$$
$$(\lambda\text{x:Bool.not x}, \lambda\text{x:Int.0-2*x}) \ \in \ \text{fun}(r, r^{st})$$
$$(\lambda\text{x:Int.x}, \lambda\text{x:Int.x >= 0}) \ \in \ \text{fun}(r, Id^{st}_{\text{Bool}}).$$

These follow from the definition of r—the first trivially and the second two once we combine the definition of $\mathsf{fun}(-,-)$ with the fact (Lemma 7.6.8) that closed relations such as r^{st} and Id^{st}_{Bool} respect ciu-equivalence. For example, if $(v_1, v'_1) \in r$, then $(\lambda\mathsf{x{:}Bool.not\ x})v_1$ and $(\lambda\mathsf{x{:}Int.0{-}2{*}x})v'_1$ are ciu-equivalent to r-related values v_2 and v'_2; then since $(v_2, v'_2) \in r \subseteq r^{st}$ and the latter is closed, we have $((\lambda\mathsf{x{:}Bool.not\ x})v_1, (\lambda\mathsf{x{:}Int.0{-}2{*}x})v'_1) \in r^{st}$. As this holds for all $(v_1, v'_1) \in r$, we have $(\lambda\mathsf{x{:}Bool.not\ x}, \lambda\mathsf{x{:}Int.0{-}2{*}x})$ in $\mathsf{fun}(r, r^{st})$.   □

Theorem 7.7.1(5) gives a sufficient condition for contextual equivalence of package values, but the condition is not necessary: it can be the case that $\{{*}\ \mathsf{T}_1,\ v_1\}\ \mathsf{as}\ \{\exists\mathsf{X, T}\}$ is contextually equivalent to $\{{*}\ \mathsf{T}_2,\ v_2\}\ \mathsf{as}\ \{\exists\mathsf{X, T}\}$ even though there is no $r \in \mathit{TRel}(\mathsf{T}_1, \mathsf{T}_2)$ with $(v_1, v_2) \in \mathsf{T}[r]$. The rest of this section is devoted to giving an example of this unpleasant phenomenon (based on a suggestion of Ian Stark arising out of our joint work on logical relations for functions and dynamically allocated names in Pitts and Stark, 1993).

7.7.4   EXAMPLE: Consider the following types and terms.

$$\mathsf{P} \stackrel{\text{def}}{=} (\mathsf{X}{\to}\mathsf{Bool}){\to}\mathsf{Bool}$$
$$\mathsf{Q} \stackrel{\text{def}}{=} \{\exists\mathsf{X,P}\}$$
$$\mathsf{N} \stackrel{\text{def}}{=} \forall\mathsf{X.X}$$
$$\mathsf{diverge} \stackrel{\text{def}}{=} (\mathsf{fun}\ \mathsf{f(b{:}Bool)} = \mathsf{f\ b : Bool})\mathsf{true}$$
$$\mathsf{G} \stackrel{\text{def}}{=} \mathsf{fun}\ \mathsf{g(f{:}N{\to}Bool)} = \mathsf{diverge : Bool}$$
$$\mathsf{G'} \stackrel{\text{def}}{=} \mathsf{fun}\ \mathsf{g(f{:}Bool{\to}Bool)} =$$
$$\quad(\mathsf{if\ f\ true\ then}$$
$$\qquad\mathsf{if\ f\ false\ then\ diverge\ else\ true}$$
$$\quad\mathsf{else\ diverge) : Bool.}$$

Thus N is a type with no values (Exercise 7.7.6); G is a function that diverges when applied to any value of type $\mathsf{N{\to}Bool}$; and $\mathsf{G'}$ is a function that diverges when applied to any value of type $\mathsf{Bool{\to}Bool}$ except ones (such as the identity function) that map true to true and false to false, in which case it returns true. We claim that

(i) there is no $r \in \mathit{TRel}(\mathsf{N}, \mathsf{Bool})$ for which $(\mathsf{G}, \mathsf{G'}) \in \mathsf{P}[r]$ holds,

(ii) but nevertheless $\varnothing \vdash \{{*}\mathsf{N,G}\}\ \mathsf{as}\ \mathsf{Q} =_{\text{ctx}} \{{*}\mathsf{Bool,G'}\}\ \mathsf{as}\ \mathsf{Q} : \mathsf{Q}$.   □

Proof: For (i) note that the definition of N implies that $Val(N) = \varnothing$, i.e., there are no closed values of type N (Exercise 7.7.6). So any $r \in TRel(N, Bool)$ satisfies $r^v = \varnothing$. Now

$$
\begin{aligned}
P[r]^v &\overset{def}{=} ((X \to Bool) \to Bool)[r]^v \\
&\overset{def}{=} fun((X \to Bool)[r], Id_{Bool}^{st})^{stv} \\
&= fun((X \to Bool)[r], Id_{Bool}^{st}) &&\text{using (7.19)} \\
&\overset{def}{=} fun(fun(r^{vst}, Id_{Bool}^{st})^{st}, Id_{Bool}^{st}) \\
&= fun(fun(r^{vst}, Id_{Bool}^{st})^{stv}, Id_{Bool}^{st}) &&\text{by definition of } fun(-,-) \\
&= fun(fun(r^{vst}, Id_{Bool}^{st}), Id_{Bool}^{st}) &&\text{using (7.19)} \\
&= fun(fun(r, Id_{Bool}^{st}), Id_{Bool}^{st}) &&\text{using (7.20)} \\
&= fun(fun(r^v, Id_{Bool}^{st}), Id_{Bool}^{st}) &&\text{by definition of } fun(-,-).
\end{aligned}
$$

Since $r^v = \varnothing$, we have $fun(r^v, Id_{Bool}^{st}) = Val(N \to Bool) \times Val(Bool \to Bool)$; and we know by Theorem 7.6.25 that Id_{Bool}^{st} is the relation $\{(t, t') \mid \varnothing \vdash t =_{ctx} t' : Bool\}$. Therefore

$$
\begin{aligned}
P[r]^v = \{(v, v') \mid \quad &\varnothing \vdash v\, v_1 =_{ctx} v'\, v_1' : Bool \\
&\text{for all } v_1 \in Val(N \to Bool) \text{ and } v_1' \in Val(Bool \to Bool)\,\}.
\end{aligned}
$$

However, $\varnothing \vdash G\, v_1 =_{ctx} G'\, v_1' : Bool$ does not hold if we take v_1 and v_1' to be the values

$$v_1 \overset{def}{=} \text{fun } f(x{:}N) = \text{diverge} : Bool$$

$$v_1' \overset{def}{=} \text{fun } f(x{:}Bool) = x : Bool$$

since evaluation of $G\, v_1$ does not terminate, whereas evaluation of $G'\, v_1'$ does. Therefore $(G, G') \notin P[r]^v$, for any $r \in TRel(N, Bool)$.

Turning to the proof of (ii), now we know that it cannot be deduced from the extensionality principle for package values in Theorem 7.7.1, we have to prove this contextual equivalence by brute force. The termination relation defined in Fig. 7-2 provides a possible strategy (if rather a tedious one) for proving ciu-equivalences and hence contextual equivalences—by what one might call *termination induction*. Thus to prove (ii) it suffices to prove that the two terms are ciu-equivalent:

$$\forall S. \quad \langle S, \{*N, G\} \text{ as } Q \rangle \downarrow \ \Leftrightarrow \ \langle S, \{*Bool, G'\} \text{ as } Q \rangle \downarrow.$$

Attempting to do this by induction on the derivation of terminations $\langle -, - \rangle \downarrow$ (for all S simultaneously), one rapidly realizes that a stronger induction hypothesis is needed: prove for all frame stacks S and terms t that

$\langle [x \mapsto \{*N,G\} \text{ as } Q]S, [x \mapsto \{*N,G\} \text{ as } Q]t \rangle \downarrow$
if and only if $\langle [x \mapsto \{*Bool,G'\} \text{ as } Q]S, [x \mapsto \{*Bool,G'\} \text{ as } Q]t \rangle \downarrow$.

It is possible to prove this by induction on the definition of the termination relation in Fig. 7-2 (for all S and t simultaneously). We omit the details except to note that the only difficult induction step is for the primitive reduction (R-UNPACKPACK) in Fig. 7-3 in the case that t is the form $\text{let}\{*X,g\}=x \text{ in } t'$. For that step, one can first show for all frame stacks S and terms t that

$\langle [X \mapsto N][g \mapsto G]S, [X \mapsto N][g \mapsto G]t \rangle \downarrow$
if and only if $\langle [X \mapsto Bool][g \mapsto G']S, [X \mapsto Bool][g \mapsto G']t \rangle \downarrow$.

This also is proved by induction on the definition of the termination relation. Once again we omit the details except to note that now the only difficult induction step is for the primitive reduction (R-APPABS) in the case that t is of the form $g\ v$ for some value v. To prove that step one can use Lemma 7.7.5 below. This lemma lies at the heart of the reason why the contextual equivalence in (ii) is valid: if an argument supplied to G' is sufficiently polymorphic (which is guaranteed by the existential abstraction), then when specialized to $Bool$ it cannot have the functionality ($\text{true} \mapsto \text{true}, \text{false} \mapsto \text{false}$) needed to distinguish G' from the divergent behavior of G.   □

7.7.5   LEMMA: For any value v satisfying $X, g{:}P \vdash v : X{\to}Bool$, evaluation of $G'([X \mapsto Bool][g \mapsto G']v)$ does not terminate.   □

Proof: To prove this we can use the logical relation from the previous section. Consider the following value-relation in *VRel*($Bool, Bool$):

$r \stackrel{\text{def}}{=} \{(\text{true},\text{true}), (\text{false},\text{false}), (\text{true},\text{false})\}$.

Note that

$$(X{\to}Bool)[r]^v \stackrel{\text{def}}{=} \text{fun}(r^{vt}, Id^{st}_{Bool})^{stv}$$
$$\stackrel{(7.20)}{=} \text{fun}(r, Id^{st}_{Bool})^{stv} \stackrel{(7.19)}{=} \text{fun}(r, Id^{st}_{Bool}) \quad (7.35)$$

and hence

$$P[r]^v \stackrel{\text{def}}{=} \text{fun}((X{\to}Bool)[r], Id^{st}_{Bool})^{stv} = \text{fun}((X{\to}Bool)[r]^v, Id^{st}_{Bool})^{stv}$$
$$\stackrel{(7.35)}{=} \text{fun}(\text{fun}(r, Id^{st}_{Bool}), Id^{st}_{Bool})^{stv} \stackrel{(7.19)}{=} \text{fun}(\text{fun}(r, Id^{st}_{Bool}), Id^{st}_{Bool}). \quad (7.36)$$

If $(v_1, v_1') \in \text{fun}(r, Id^{st}_{Bool})$, since $(\text{true},\text{true}), (\text{false},\text{false}) \in r$ and Id^{st}_{Bool} is contextual equivalence (Theorem 7.6.25) we get

$\varnothing \vdash v_1 \text{ true} =_{\text{ctx}} v_1' \text{ true} : Bool$
$\varnothing \vdash v_1 \text{ false} =_{\text{ctx}} v_1' \text{ false} : Bool$.

So using Corollary 7.5.8(iii) and the congruence properties of $=_{ctx}$, we have

$$
\begin{aligned}
\mathsf{G}'\, \mathsf{v}_1 \;=_{ctx}\; &(\texttt{if}\; \mathsf{v}_1 \;\texttt{true}\; \texttt{then} \\
&\qquad \texttt{if}\; \mathsf{v}_1 \;\texttt{false}\;\texttt{then}\;\texttt{diverge}\;\texttt{else}\;\texttt{true} \\
&\quad\; \texttt{else}\;\texttt{diverge}) \\
=_{ctx}\; &(\texttt{if}\; \mathsf{v}_1' \;\texttt{true}\; \texttt{then} \\
&\qquad \texttt{if}\; \mathsf{v}_1' \;\texttt{false}\;\texttt{then}\;\texttt{diverge}\;\texttt{else}\;\texttt{true} \\
&\quad\; \texttt{else}\;\texttt{diverge}) \\
=_{ctx}\; &\mathsf{G}'\, \mathsf{v}_1'
\end{aligned}
$$

Therefore $(\mathsf{G}'\, \mathsf{v}_1, \mathsf{G}'\, \mathsf{v}_1') \in Id_{\mathsf{Bool}}^{st}$ whenever $(\mathsf{v}_1, \mathsf{v}_1') \in \mathsf{fun}(r, Id_{\mathsf{Bool}}^{st})$; and so $(\mathsf{G}', \mathsf{G}') \in \mathsf{P}[r]^v$, by (7.36). Hence using Lemma 7.6.17 we have

$$
\begin{aligned}
([\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v}, [\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v}) &\in (\mathsf{X}{\to}\mathsf{Bool})[r]^v \\
&= \mathsf{fun}(r, Id_{\mathsf{Bool}}^{st}) \quad \text{by (7.35).}
\end{aligned}
$$

So since $(\texttt{true}, \texttt{false}) \in r$, we get

$$
([\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v}\;\texttt{true}, [\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v}\;\texttt{false}) \in Id_{\mathsf{Bool}}^{st}.
$$

Thus $([\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v})\texttt{true}$ and $([\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v})\texttt{false}$ are contextually equivalent closed terms of type Bool. Therefore it cannot be the case that the first evaluates to $\texttt{true}$ and the second to $\texttt{false}$ (cf. Exercise 7.5.10); but in that case, by definition of G', it must be that evaluation of $\mathsf{G}'([\mathsf{X} \mapsto \mathsf{Bool}][\mathsf{g} \mapsto \mathsf{G}']\mathsf{v})$ does not terminate. $\square$

7.7.6   EXERCISE [$\star$, $\nrightarrow$]: By considering the possible typing derivations from the rules in Figure 7-1, show that there is no value v satisfying $\varnothing \vdash \mathsf{v} : \forall\mathsf{X}.\mathsf{X}$. (Note that the syntactic restriction on values of universally quantified type mentioned in Remark 7.4.1 plays a crucial role here.) $\square$

7.7.7   REMARK [THE ROLE OF NON-TERMINATION]: Example 7.7.4 shows that the logical relation presented here is incomplete for proving contextual equivalence of F_{ML} values of existential type. The example makes use of the fact that, because of the presence of recursive function values, evaluation of F_{ML} terms need not terminate. However, it seems that the source of the incompleteness has more to do with the existence of types with no values (such as $\forall\mathsf{X}.\mathsf{X}$) than with non-termination. Eijiro Sumii (private communication) has suggested the

following, "terminating" version of Example 7.7.4:

$$P \stackrel{\text{def}}{=} \texttt{(X→Bool)→Bool}$$

$$Q \stackrel{\text{def}}{=} \texttt{\{∃X,P\}}$$

$$N \stackrel{\text{def}}{=} \texttt{∀X.X}$$

$$H \stackrel{\text{def}}{=} \texttt{λf:N→Bool. false}$$

$$H' \stackrel{\text{def}}{=} \texttt{λf:Bool→Bool.}$$

```
        (if f true then
            if f false then false else true
        else false) : Bool.
```

Consider a version of F_{ML} with only non-recursive function abstractions (i.e. with $\texttt{λx:T.t}$ rather than $\texttt{fun f(x:T) = t:T'}$). Evaluation is terminating in this version. So to be non-trivial, contextual equivalence should be formulated in terms of observing convergence to the same ground value in all contexts of ground type. Making corresponding changes to the definition of the operations $(-)^s$ and $(-)^t$ on term- and stack-relations, one could develop a logical relation for this terminating version of F_{ML}. It seems that properties (i) and (ii) in Example 7.7.4 are also true of H and H' in this version (the first by the same argument we gave, but the second by a different argument that nevertheless hinges on the observation at the end of the proof of Example 7.7.4). We leave investigating this as an extended exercise for the reader.   □

The proof of Lemma 7.7.5 exploits "relational parametricity" properties of polymorphic types in F_{ML}. In fact Theorem 7.6.25 tells us far more about the properties of type abstraction values than just the extensionality property of Theorem 7.7.1(4).

7.7.8   THEOREM [RELATIONAL PARAMETRICITY FOR ∀-TYPES]: Given $X \vdash v : T$ and $X \vdash v' : T$, then $\varnothing \vdash \texttt{λX.v} =_{\text{ctx}} \texttt{λX.v}' : \texttt{∀X.T}$ if and only if for all closed types $T_1, T_1' \in Typ$ and all term-relations $r \in TRel(T_1, T_1')$ it is the case that $([X \mapsto T_1]v, [X \mapsto T_1']v') \in T[r]$.   □

Proof: By Theorem 7.6.25, we have that $\varnothing \vdash \texttt{λX.v} =_{\text{ctx}} \texttt{λX.v}' : \texttt{∀X.T}$ iff $\varnothing \vdash \texttt{λX.v} \, \Delta \, \texttt{λX.v}' : \texttt{∀X.T}$, i.e., iff $(\texttt{λX.v}, \texttt{λX.v}') \in (\texttt{∀X.T})[] = (\lambda r.T[r])^{st}$. Since $\texttt{λX.v}$ and $\texttt{λX.v}'$ are values, the latter is the case iff $(\texttt{λX.v}, \texttt{λX.v}') \in (\lambda r.T[r])^{stv}$, and by Lemma 7.6.12 and Exercise 7.6.14 $(\lambda r.T[r])^{stv} = \lambda r.T[r]$. Hence $\varnothing \vdash \texttt{λX.v} =_{\text{ctx}} \texttt{λX.v}' : \texttt{∀X.T}$ iff $(\texttt{λX.v}, \texttt{λX.v}') \in \lambda r.T[r]$. By definition (Figure 7-5), this is the case iff for all for all closed types $T_1, T_1' \in Typ$ and all term-relations $r \in TRel(T_1, T_1')$, $((\texttt{λX.v})T_1, (\texttt{λX.v}')T_1') \in T[r]$; and the

latter holds iff $([X \mapsto T_1]v, [X \mapsto T_1']v') \in T[r]$, because $(\lambda X.v)T_1 =_{ciu} [X \mapsto T_1]v$ and $(\lambda X.v')T_1' =_{ciu} [X \mapsto T_1']v'$ (so that we can use Lemmas 7.6.8 and 7.6.12). □

The force of Theorem 7.7.1(4) is to give a method for establishing that two type abstraction values are contextually equivalent. By contrast, the force of Theorem 7.7.8 is to give us useful properties of families of values parameterized by type variables. Given such a value, $X \vdash v : T$, since $=_{ctx}$ is reflexive, we have $\varnothing \vdash \lambda X.v =_{ctx} \lambda X.v : \forall X.T$; hence the theorem has the following corollary.

7.7.9   COROLLARY: Given a value $X \vdash v : T$, for all $T_1, T_1' \in \textit{Typ}$ and all $r \in TRel(T_1, T_1')$, it is the case that $([X \mapsto T_1]v, [X \mapsto T_1']v) \in T[r]$. □

Such "relational parametricity" properties can often be exploited for proving contextual equivalences: we already saw an example in the proof of Lemma 7.7.5 and other examples can be found in Pitts (2000), Bierman, Pitts, and Russo (2000), and Johann (2002). However, the strict nature of function application and type abstraction in F_{ML} means that it does not satisfy all the parametricity properties one might expect. For example, in Pitts (2000), §7, it is shown that

$$\{\exists X, T\} \cong \forall Y.(\forall X.T \to Y) \to Y$$

holds in the polymorphic version of PCF (Plotkin, 1977) studied in that paper (where $\cong$ is "bijection up to contextual equivalence"—see Principle 7.3.4). However this bijection does not hold in general for F_{ML} (Exercise 7.7.10).

7.7.10   EXERCISE [$\star\star\star$]: Consider the type $N \stackrel{\text{def}}{=} \forall X.X$ from Example 7.7.4 that you showed has no closed values in Exercise 7.7.6. Show that there cannot exist values

$i \in \textit{Val}(\{\exists X, N\} \to \forall Y.(\forall X.N \to Y) \to Y)$

$j \in \textit{Val}((\forall Y.(\forall X.N \to Y) \to Y) \to \{\exists X, N\})$

that are mutually inverse, in the sense that

$$p:\{\exists X,N\} \vdash j(i\ p) =_{ctx} p : \{\exists X,N\}$$
$$y:\forall Y.(\forall X.N \to Y) \to Y \vdash i(j\ y) =_{ctx} y : \forall Y.(\forall X.N \to Y) \to Y.$$ □

7.7.11   EXERCISE [$\star\star\star$, ↛]: Verify the claim made in Note 7.3.7 that Principle 7.3.4 is a special case of Principle 7.3.6. To do so, you will first have to give a definition of the action of F_{ML} types on bijections mentioned in Principle 7.3.4. □

7.8   Notes

This chapter is a revised and expanded version of Pitts (1998) and also draws on material from Pitts (2000).

In discussing typed operational reasoning we have focused on reasoning about *contextual equivalence* of program phrases. Being by construction a congruence, contextual equivalence permits us to use the usual forms of equational reasoning (replacing equals by equals) when deriving equivalences between phrases. However, its definition does not lend itself to establishing the basic laws that are needed to get such reasoning going. We studied two characterisations of contextual equivalence in order to get round this problem: *ciu-equivalence* (Definition 7.5.5) and a certain kind of operationally based *logical relation* (Definition 7.6.10).

contextual equivalence!vs. bisimilarity The informal notion of contextual equivalence (Definition 7.3.2) has been studied for a wide variety of programming languages. If the language's operational semantics involves non-determinism—usually because the language supports some form of concurrent or interactive computation—then contextual equivalence tends to identify too many programs and various co-inductive notions of *bisimilarity* are used instead (see the textbook by Sangiorgi and David, 2001, for example). But even if we remain within the realm of languages with deterministic operational semantics, one may ask to what extent the results of this chapter are stable with respect to adding further features such as recursive datatypes, mutable state, and object-oriented features à la Objective Caml.

Ciu-equivalence has the advantage of being quite robust in this respect—it can provide a characterisation of contextual equivalence in the presence of such features (Honsell, Mason, Smith, and Talcott, 1995; Talcott, 1998). However, its usefulness is mainly limited to establishing basic laws such as the conversions in Corollary 7.5.8; it cannot be used directly to establish extensionality properties such as those in Theorem 7.7.1 without resorting to tedious "termination inductions" of the kind we sketched in the proof of Example 7.7.4. Ciu-equivalence is quite closely related to some notions of "applicative bisimilarity" that have been applied to functional and object-based languages (Gordon, 1995, 1998), in that their congruence properties can both be established using a clever technique due to Howe (1996). The advantage of applicative bisimilarity is that it has extensionality built into its definition; so when it does coincide with contextual equivalence, this provides a method of establishing some extensionality properties for $=_{\text{ctx}}$ (such as (1)–(4) in Theorem 7.7.1, but not, as far as I know, property (5) for package values).

The kind of operationally based logical relation we developed in this chapter provides a very powerful analysis of contextual equivalence. We used it

to prove not only conversions and simple extensionality principles for F_{ML}, but also quite subtle properties of $=_{ctx}$ such as Theorems 7.7.1(5) and 7.7.8. Similar logical relations can be used to prove some properties of ML-style references and of linear types: see Pitts and Stark (1998), Bierman, Pitts, and Russo (2000), and Pitts (2002). Unfortunately, the characteristic feature of logical relations—that functions are related iff they map related arguments to related results—makes it difficult to define them in the presence of "recursive features." I mean by the latter programming language features which in a denotational semantics lead one to have to solve domain equations in which the defined domain occurs both positively (to the left of an even number of function space constructions) and negatively (to the left of an odd number of function space constructions). Recursive datatypes involving function types can lead to such domain equations; as does the use of references to functions in ML. Suitable logical relations can be defined in the denotational semantics of languages with such features using techniques such as those in Pitts (1996), but they tell us properties of denotational equality, which is often a poor (if safe) approximation to contextual equivalence. For this reason people have tried to develop syntactical analogs of these denotational logical relations: see Birkedal and Harper (1999). The unwinding theorem (Theorem 7.4.4) provides the basis for such an approach. However, it seems like a fresh idea is needed to make further progress. Therefore I set a last exercise, whose solution is not included.

7.8.1 EXERCISE [★★★★..., ↛]: Extend F_{ML} with *isorecursive types*, μX.T, as in Figure 20-1 of *TAPL*, Chapter 20. By finding an operationally based logical relation as in §7.6 or otherwise, try to prove the kind of properties of contextual equivalence for this extended language that we developed for F_{ML} in this chapter. (For the special case of iso-recursive types μX.T for which T contains no negative occurrences of X, albeit for a non-strict functional language, see Johann (2002). The generalized ideal model of recursive polymorphic in Vouillon and Melliès (2004) uses the same kind of Galois connection as we used in §7.6 and may well shed light on this exercise. Recent work by Sumii and Pierce [2005] is also relevant.) □

come equipped with their own notion of logical equivalence that can be defined independently (*i.e.,* without reference to the general definition of logical equivalence). Thus, the definition of logical equivalence may refer to arbitrary candidates and remain well-founded.

7.4.2 HINT: First prove

$$\langle S_1, \mathsf{t}_1 \rangle \longrightarrow \langle S_2, \mathsf{t}_2 \rangle \Rightarrow (\forall S)(\langle S@S_2, \mathsf{t}_2 \rangle \downarrow \Rightarrow \langle S@S_1, \mathsf{t}_1 \rangle \downarrow)$$

by considering the different cases for $\longrightarrow$. Deduce the 'if' part of (7.7) from this. For the 'only if' part, show that

$$\{(S, \mathsf{t}) \mid (\exists S_1, S_2, \mathsf{v}) \ S = S_1@S_2 \ \& \ \langle S_2, \mathsf{t} \rangle \longrightarrow^* \langle \mathit{Id}, \mathsf{v} \rangle \ \& \ \langle S_1, \mathsf{v} \rangle \downarrow\}$$

is closed under the axiom and rules in Figure 7-2 inductively defining the termination relation.

7.5.4 SOLUTION: For property (iii), assuming R is compatible, argue by induction on the derivation of $\Gamma \vdash \mathsf{t} : \mathsf{T}$ that this typing judgment implies that $\Gamma \vdash \mathsf{t} \ R \ \mathsf{t} : \mathsf{T}$ holds. For property (v), if $R = \bigcup_{i \in I} R_i$ with $I \neq \varnothing$ and each R_i compatible, first note that by (iii), R is reflexive since it contains at least one relation R_i. For each of the compatibility properties in Figure 7-4 with a *single* hypothesis, it is clear that R has this property because each of the R_i does. For compatibility properties with multiple hypotheses, we can break them down into a chain of single-hypothesis compatibilities and appeal to the transitivity of R (which we are assuming). For example consider the compatibility property for function application. It suffices to show that R satisfies

$$\frac{\Gamma \vdash \mathsf{v}_1 \ R \ \mathsf{v}_1' : \mathsf{T}_1 {\rightarrow} \mathsf{T}_2 \qquad \Gamma \vdash \mathsf{v}_2 : \mathsf{T}_1}{\Gamma \vdash \mathsf{v}_1 \ \mathsf{v}_2 \ R \ \mathsf{v}_1' \ \mathsf{v}_2 : \mathsf{T}_2} \tag{A.1}$$

and

$$\frac{\Gamma \vdash \mathsf{v}_1 : \mathsf{T}_1 {\rightarrow} \mathsf{T}_2 \qquad \Gamma \vdash \mathsf{v}_2 \ R \ \mathsf{v}_2' : \mathsf{T}_1}{\Gamma \vdash \mathsf{v}_1 \ \mathsf{v}_2 \ R \ \mathsf{v}_1 \ \mathsf{v}_2' : \mathsf{T}_2}. \tag{A.2}$$

For then if $\Gamma \vdash \mathsf{v}_1 \ R \ \mathsf{v}_1' : \mathsf{T}_1 {\rightarrow} \mathsf{T}_2$ and $\Gamma \vdash \mathsf{v}_2 \ R \ \mathsf{v}_2' : \mathsf{T}_1$, we get

$\Gamma \vdash \mathsf{v}_1 \ \mathsf{v}_2 \ R \ \mathsf{v}_1' \ \mathsf{v}_2 : \mathsf{T}_2$ by (A.1), since $\Gamma \vdash \mathsf{v}_2 : \mathsf{T}_1$

$\Gamma \vdash \mathsf{v}_1' \ \mathsf{v}_2 \ R \ \mathsf{v}_1' \ \mathsf{v}_2' : \mathsf{T}_2$ by (A.2), since $\Gamma \vdash \mathsf{v}_1' : \mathsf{T}_1 {\rightarrow} \mathsf{T}_2$.

and hence $\Gamma \vdash \mathsf{v}_1 \ \mathsf{v}_2 \ R \ \mathsf{v}_1' \ \mathsf{v}_2' : \mathsf{T}_2$ by transitivity. Each of the single-hypothesis properties (A.1) and (A.2) holds of R because they hold for each R_i: each is a special case of the compatibility property for function application because each R_i, being compatible, is also reflexive by (iii).

7.5.10   SOLUTION: Consider the frame stacks

$$S \stackrel{\text{def}}{=} Id \circ (\text{x.}(\text{fun } f(\text{x}':\text{Bool}) = \text{if } \text{x}' \text{ then true else } f \text{ x}')\text{x})$$

$$S_\mathsf{T} \stackrel{\text{def}}{=} Id \circ (\text{x.}(\text{fun } f(\text{x}':\mathsf{T}) = \text{true})\text{x})$$

Note that $\varnothing \vdash S : \mathsf{Bool} \multimap \mathsf{Bool}$ and $\varnothing \vdash S_\mathsf{T} : \mathsf{T} \multimap \mathsf{Bool}$. It is not hard to see for all $\varnothing \vdash \mathsf{b} : \mathsf{Bool}$ that

$$S[\mathsf{b}]\!\downarrow \text{ iff } \langle Id, \mathsf{b} \rangle \longrightarrow^* \langle Id, \text{true} \rangle \tag{A.3}$$

and for all $\varnothing \vdash \mathsf{t} : \mathsf{T}$ that

$$\mathsf{t}\!\downarrow \text{ iff } \langle Id, S_\mathsf{T}[\mathsf{t}] \rangle \longrightarrow^* \langle Id, \text{true} \rangle \tag{A.4}$$

From (A.3) and the fact that $=_{\mathrm{ctx}}$ is a congruence (so that $\varnothing \vdash \mathsf{b} =_{\mathrm{ctx}} \mathsf{b}' : \mathsf{Bool}$ implies $\varnothing \vdash S[\mathsf{b}] =_{\mathrm{ctx}} S[\mathsf{b}'] : \mathsf{Bool}$) it follows that $=_{\mathrm{ctx}}$ is true-adequate; hence it is contained in $=_{\mathrm{ctx}}^{\text{true}}$. Similarly, (A.4) and the fact that $=_{\mathrm{ctx}}^{\text{true}}$ is a congruence implies that it is adequate and hence contained in $=_{\mathrm{ctx}}$.

7.6.7   SOLUTION: Since $(-)^{st}$ is inflationary we have $r \subseteq r^{st}$; and since r only relates values, this implies $r \subseteq r^{stv}$. Then since $(-)^{st}$ is monotone, we have $r^{st} \subseteq r^{stvst}$. Conversely, since $(r')^v \subseteq r'$ for any r', we have $r^{stv} \subseteq r^{st}$; and then since $(-)^{st}$ is monotone and idempotent, $r^{stvst} \subseteq r^{stst} = r^{st}$.

7.6.14   HINT: The proof of (7.26) is just like the proof of (7.21), using the following property of the termination relation:

$$(\langle S, \mathsf{v}.\mathsf{1} \rangle\!\downarrow \Leftrightarrow \langle S', \mathsf{v}'.\mathsf{1} \rangle\!\downarrow) \text{ iff } (\langle S \circ (\text{x.x.1}), \mathsf{v} \rangle\!\downarrow \Leftrightarrow \langle S' \circ (\text{x.x.1}), \mathsf{v}' \rangle\!\downarrow).$$

Similarly, the proof of (7.27) follows from:

$$(\langle S, \mathsf{v}\,\mathsf{T} \rangle\!\downarrow \Leftrightarrow \langle S', \mathsf{v}'\,\mathsf{T}' \rangle\!\downarrow) \text{ iff } (\langle S \circ (\text{x.x}\,\mathsf{T}), \mathsf{v} \rangle\!\downarrow \Leftrightarrow \langle S' \circ (\text{x.x}\,\mathsf{T}'), \mathsf{v}' \rangle\!\downarrow).$$

7.6.18   SOLUTION: It suffices to show

$$(\forall n = 0, 1, \ldots)\ (\mathsf{F}_n, \mathsf{F}'_n) \in \mathsf{fun}(r_1, r_2) \tag{A.5}$$

where F_n and F'_n are the unwindings associated with F and F' respectively, as in Theorem 7.4.4. For if (A.5) holds, then using the fact that $(-)^{st}$ is inflationary

$$(\mathsf{F}_n, \mathsf{F}'_n) \in \mathsf{fun}(r_1, r_2) \subseteq \mathsf{fun}(r_1, r_2)^{st}$$

for each n; so by the *Admissibility* property in Lemma 7.6.8 we have $(\mathsf{F}, \mathsf{F}') \in \mathsf{fun}(r_1, r_2)^{st}$. Thus $(\mathsf{F}, \mathsf{F}') \in \mathsf{fun}(r_1, r_2)^{stv} = \mathsf{fun}(r_1, r_2)$ by Lemma 7.6.13, since $(r_2)^{st} = r_2$. (A.5) is proved by induction on n:

Base case $n = 0$: By definition of F_0, $\langle S, F_0 \, v_1 \rangle \downarrow$ does not hold for any $S \in$ *Stack*(T_2) and $v_1 \in Val(T_1)$; similarly for F_0'. Hence for all $(v_1, v_1') \in (r_1)^v$, $(F_0 \, v_1, F_0' \, v_1') \in s^t$ for any $s \in SRel(T_2, T_2')$ and hence in particular for $s = (r_2)^s$. So $(F_0 \, v_1, F_0' \, v_1') \in (r_2)^{st} = r_2$ for all $(v_1, v_1') \in (r_1)^v$. Therefore $(F_0, F_0') \in \mathsf{fun}(r_1, r_2)$.

Induction step: Suppose $(F_n, F_n') \in \mathsf{fun}(r_1, r_2)$. Then for any $(v_1, v_1') \in (r_1)^v$, from (7.29) we have

$$([f \mapsto F_n][x \mapsto v_1]t, [f \mapsto F_n'][x \mapsto v_1']t') \in r_2.$$

By definition of F_{n+1} and Corollary 7.5.8 we have $\varnothing \vdash F_{n+1}v_1 =_{\text{ctx}} [f \mapsto F_n][x \mapsto v_1]t$; and similarly, $\varnothing \vdash F_{n+1}'v_1' =_{\text{ctx}} [f \mapsto F_n'][x \mapsto v_1']t'$. So since r_2 is closed, we can apply the *Equivalence-respecting* property in Lemma 7.6.8 to conclude that $(F_{n+1}v_1, F_{n+1}'v_1') \in r_2$. Since this holds for any $(v_1, v_1') \in (r_1)^v$, we have $(F_{n+1}, F_{n+1}') \in \mathsf{fun}(r_1, r_2)$.

7.6.19 SOLUTION: To show $(v, v') \in \{l_i = r_i {}^{i \in 1..n}\}$ we must show $(v.l_i, v'.l_i) \in r_i$ for each $i \in 1..n$. Since each r_i is closed, this is equivalent to showing $(v.l_i, v'.l_i) \in (r_i)^{st}$, i.e. that $\langle S, v.l_i \rangle \downarrow \Leftrightarrow \langle S', v'.l_i \rangle \downarrow$ holds for all (S, S') in $(r_i)^s$. But by definition of v, $\langle S, v.l_i \rangle \downarrow \Leftrightarrow \langle S, v_i \rangle \downarrow$; and similarly for v'. So it suffices to show $\langle S, v_i \rangle \downarrow \Leftrightarrow \langle S', v_i' \rangle$; and this holds because by assumption $(v_i, v_i') \in r_i$ and $(S, S') \in (r_i)^s$.

7.6.20 SOLUTION: To show $(\lambda X.v, \lambda X.v') \in \lambda r. R(r)$ we have to show for each $T_1, T_1' \in$ *Typ* and $r \in TRel(T_1, T_1')$ that $((\lambda X.v)T, (\lambda X.v')T') \in R(r)$. Since each $R(r)$ is closed, this is equivalent to showing $((\lambda X.v)T, (\lambda X.v')T') \in R(r)^{st}$, i.e. that $\langle S, (\lambda X.v)T \rangle \downarrow \Leftrightarrow \langle S', (\lambda X.v')T' \rangle \downarrow$ holds for all $(S, S') \in R(r)^s$. But $\langle S, (\lambda X.v)T \rangle \downarrow \Leftrightarrow \langle S, [X \mapsto T_1]v \rangle \downarrow$; and similarly for v'. So it suffices to show $\langle S, [X \mapsto T_1]v \rangle \downarrow \Leftrightarrow \langle S, [X \mapsto T_1']v' \rangle \downarrow$; and this holds because by assumption $([X \mapsto T_1]v, [X \mapsto T_1]v) \in R(r)$ and $(S, S') \in R(r)^s$.

7.6.21 HINT: To show (if v then t_1 else t_2, if v' then t_1' else t_2') $\in r = (r)^{st}$ it suffices to show for all $(S, S') \in (r)^s$ that

$$\langle S, \text{if } v \text{ then } t_1 \text{ else } t_2 \rangle \downarrow \Leftrightarrow \langle S', \text{if } v' \text{ then } t_1' \text{ else } t_2' \rangle \downarrow$$

or equivalently that

$$\langle S \circ (x.\text{if } x \text{ then } t_1 \text{ else } t_2), v \rangle \downarrow \Leftrightarrow$$
$$\langle S' \circ (x.\text{if } x \text{ then } t_1' \text{ else } t_2'), v' \rangle \downarrow.$$

Do this by proving that

$$(S \circ (x.\text{if } x \text{ then } t_1 \text{ else } t_2), S' \circ (x.\text{if } x \text{ then } t_1' \text{ else } t_2')) \in (Id_{\text{Bool}})^s.$$

7.6.22   SOLUTION: For any $(S, S') \in (r_2)^s$ it follows from the assumptions on t, t' and the definition of $\{\exists r_1, R(r_1)\}$ (Figure 7-5) that

$(S \circ (\text{y.let } \{^*X, x\} = y \text{ in } t), S' \circ (\text{y.let } \{^*X, x\} = y \text{ in } t'))$

is in $\{\exists r_1, R(r_1)\}^s$. Hence if $(v, v') \in \{\exists r_1, R(r_1)\}^{stv} \subseteq (\{\exists r_1, R(r_1)\}^s)^t$, then

$\langle S \circ (\text{y.let } \{^*X, x\} = y \text{ in } t), v \rangle \downarrow \Leftrightarrow \langle S' \circ (\text{y.let } \{^*X, x\} = y \text{ in } t'), v' \rangle \downarrow$

and so $\langle S, \text{let } \{^*X, x\} = v \text{ in } t \rangle \downarrow \Leftrightarrow \langle S', \text{let } \{^*X, x\} = v' \text{ in } t' \rangle \downarrow$. Since this is true for all $(S, S') \in (r_2)^s$, we deduce that

$(\text{let } \{^*X, x\} = v \text{ in } t, \text{let } \{^*X, x\} = v \text{ in } t) \in (r_2)^{st} = r_2.$

7.6.23   SOLUTION: For any $(S, S') \in (r_2)^s$ it follows from the assumptions on t, t' that $(S \circ (x. t_2), S' \circ (x. t'_2)) \in (r_1)^{vs}$. Since $((r_1)^{vs})^t = r_1$, if $(t_1, t'_1) \in r_1$ then we get $\langle S \circ (x. t_2), t_1 \rangle \downarrow \Leftrightarrow \langle S' \circ (x. t'_2), t'_1 \rangle \downarrow$, and hence that

$\langle S, \text{let } x = t_1 \text{ in } t_2 \rangle \downarrow \Leftrightarrow \langle S', \text{let } x = t'_1 \text{ in } t'_2 \rangle \downarrow.$

Since this holds for all $(S, S') \in (r_2)^s$, we deduce that

$(\text{let } x = t_1 \text{ in } t_2, \text{let } x = t'_1 \text{ in } t'_2) \in (r_2)^{st} = r_2.$

7.7.10   SOLUTION: Since N has no closed values, neither does $\{\exists X, N\}$. On the other hand

```
val v = λY.fun f(x:∀X.N→Y) = (f x):Y
```

is a closed value of type $\forall Y. (\forall X. N \rightarrow Y) \rightarrow Y$. If i and j were to exist with the stated properties we could use them to construct from v a closed value of type $\{\exists X, N\}$, which is impossible. (For i(j v) and v are ciu-equivalent (Theorem 7.5.7); so since $v \downarrow$, we also have i(j v) $\downarrow$. Hence by Exercise 7.4.2, $\langle Id, j v \rangle \longrightarrow^* \langle Id, v' \rangle$ for some v', which is a closed value of type $\{\exists X, N\}$, by Exercise 7.4.3.)

8.2.1   SOLUTION: As of this writing, the question of how far nominal module systems can be pushed is wide open. A step in this direction was recently taken by Odersky, Cremet, Rockl, and Zenger (2003).

8.5.3   SOLUTION: Define m_1 to be the module

```
module m₁ = mod {
  type X = Int
  val c = 0
  val f = succ
}
```

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