Notes on Categorical Logic

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Introduction

What is Categorical Logic? The syntax of traditional mathematical logics (such as the first-order predicate calculus) provides formal languages for describing mathematical structures in terms of properties of the constituent parts of the structures—elements of sets, functions between sets, relations on sets, and so on. This is a very straightforward approach: if you want to describe a structure, look inside it and describe the things of which it is composed. Category theory is based upon a less direct approach to describing properties of structures—one concentrating not on the internal composition of the structures themselves, but on the transformations, or 'morphisms', between them. One might guess that the kind of properties describable solely in terms of morphisms would be quite limited, but experience with category theory in the last 40 years has shown that this is far from true. Indeed, something quite surprising is the case: namely that it is possible to give an interpretation of the traditional syntax of mathematical logic based upon morphisms between structures rather than on their internal composition (their elements). Moreover, this categorical interpretation is equivalent to the traditional one when the structures involved are just sets (and gives rise to other known interpretations, such as forcing or realizability, by varying the category involved). This is the starting point of 'categorical' logic.

Is it relevant to Computer Science? The categorical interpretation of syntax gives explanations of various logical concepts (propositional connectives, variables, substitution, quantifiers and equality, to name some of the ones which will crop up in these notes) in terms of relatively few categorical ones—principal among which is the concept of *adjoint functor*. This results in a unification of concepts and aids the creation of new ones using analogies. This in part is why categorical logic has proved useful in devising and studying semantics for non-traditional logics of relevance to computer science. (Such as various kinds of lambda calculus.) So I would say that categorical logic is very relevant to semantic studies in computer science.

From a formal point of view, a categorical interpretation of some variety of logical syntax provides a translation of that syntax into a simpler (variable-free, for example) algebra of 'combinators'. Then rules for evaluating the categorical expressions induce an operational semantics for the original syntax. (The 'Categorical Abstract Machine' of Cousineau, Curien and Mauny is an example of this.) So categorical logic has a potential (only partly realized so far) for directly influencing notions of computation.

Another aspect of the categorical logic, which I have not mentioned yet, seems relevant to the development of specification methods in computer science. In categorical logic, logical theories of a particular kind are themselves identified with particular varieties of category (possibly with extra, categorical structure). The earliest explicit example of this was Lawvere's identification of algebraic theories with categories possessing finite products (see section 6 in Part B of these notes).

The motto of category theory is "remember the morphisms"; and here, identifying theories with categorical structures gives rise to a rich calculus of 'theory morphisms' (in terms of various kinds of functor) which includes the traditional notions of 'model' of a theory and 'interpretation' (or 'translation') of one theory in another. Thus we can apply categorical methods at this higher level of theories and theory morphisms to compare and combine different theories. A little domain-theoretic illustration of this is given in section 5 in Part A of these notes.

Contents of these notes. Some knowledge of the beginning ideas of category theory will be assumed. A Glossary collects together some definitions and fixes the notation.

Part A is concerned with the categorical semantics of propositions and the relation of logical entailment. In this semantics propositions ϕ are interpreted as objects $[\![\phi]\!]$ in some suitable category, and proofs of logical entailment $\phi \vdash \psi$ are interpreted as morphisms $[\![\phi]\!] \longrightarrow [\![\psi]\!]$ in the category. If we are not interested in the structure of proofs, but only in the *relation* of entailment, then we can restrict attention to categories which are *preorders*—ones in which there is at most one morphism between any pair of objects. This is the viewpoint of Part A. It is a good starting place, since the preorder versions of general categorical notions are easier to grasp. The crucial aspect of the categorical interpretation of the propositional connectives is that it characterizes their meaning in terms of operations in the preordered set built up by a series of *adjunctions*.

Part B treats (many-sorted) equational logic (section 6) and its realtion to categories with finite products, and then goes on to full first-order predicate logic (section 8) and its categorical semantics using indexed preorders (Lawvere's 'hyperdoctrines'). We concentrate first on the calculus without equality predicates, because its categorical semantics is less well known (and more relevant to some of the potential computer science applications). Equality predicates are discussed in section 10. What emerges is an *adjoint* formulation of first-order logic possessing an interesting symmetry (a symmetry different from that of the Gentzen Sequent Calculus for first-order logic—*cf.* the remarks of Girard in [Gi]). Section 9 illustrates the proof-theoretic application of these techniques by giving a categorical proof (due to Freyd) of the disjunction and explicit definability properties of intuitionistic predicate calculus.

If there were ever a Part C it would give the categorical approach to higher-order logics and type theories. (Meanwhile, see [LS] for the former at least.)

The four 'worlds' Ⓣ Ⓚ Ⓢ Ⓛ. Running through the notes are four extended examples which (with apologies to Barwise and Co.) we call

- *Tarski's World* Ⓣ
- *Kleene's World* Ⓚ
- *Scott's World* Ⓢ
- *Lawvere's World* Ⓛ

and whose appearance at various points in the notes is indicated by the corresponding symbol. Alas, we do little but scratch the surface of these worlds.

Ⓣ is the world of classical logic and set theory. Ⓚ is the world of the recursive 'realizability' semantics of intuitionistic logic; see [Hy] for an account of the categorical and topos-theoretic riches here. Ⓢ is the world of domain theory and L(ogic of)C(omputable)F(unctions); categorical investigations here are the subject of current research. Ⓛ is the world of a categorically axiomatized category of sets and functions. This last is really a variable example (ho!ho!†): requirements on an arbitrary category are built up embodying categorical versions of logical and set-theoretic properties of 'the' category of sets and functions, leading to the notion of 'elementary topos' (see [Joh1], [LS, Part II] and [Be]).

† See [La3].

Part A

Propositional Logic

1 Ordered Sets

1.1 Definitions. A *preorder* on a set X is a binary relation $\leq$ which is

reflexive: $x \leq x$ (all $x \in X$) and

transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$ (all $x, y \in X$).

If $x \leq y$ and $y \leq x$, then we write $x \cong y$ and say that x and y are *isomorphic* elements. Clearly $\cong$ is an eqivalence relation, *i.e.* is reflexive, transitive and

symmetric: $x \cong y$ implies $y \cong x$ (all$x, y \in X$).

A *partial order* is a preorder for which the relation $\cong$ coincides with equality, so that $\leq$ is

antisymmetric: $x \leq y$ and $y \leq x$ imply $x = y$ (all $x, y \in X$).

A *preordered set* $(X, \leq)$ is a set equipped with a preorder. Most of the time we will refer to X as the preordered set without mentioning the associated relation $\leq$ explicitly. A set equipped with a partial order is often referred to as a *poset*. If X is a preordered set, we can manufacture a poset by quotienting the set X by the equivalence relation $\cong$ of isomorphism determined by the preorder $\leq$ on X: the resulting set $X/\cong = \{[x] \mid x \in X\}$ of equivalence classes has a well-defined *partial order* on it given by

$[x] \leq [y]$ if and only if $x \leq y$.

The poset $X/\cong$ is called the *poset reflection* of the preorder X.

1.2 Exercise. Note that the quotient function $q: X \longrightarrow X/\cong$ given by $q(x) = [x]$, is *monotone*—meaning that $x \leq y$ implies $q(x) \leq q(y)$. Show that it is 'universal' amongst monotone functions from X into a poset, in the sense that if $f: X \longrightarrow Y$ is any monotone function to a poset Y, then there is a unique monotone function $X/\cong \longrightarrow Y$ whose composition with q is f.

1.3 Examples. (i) Ⓣ The set of subsets of a set X, $\mathbf{P}(X)$ is partially ordered by the relation of inclusion: $A \subseteq B$ if for all $x \in X$, $x \in A$ implies $x \in B$.

(ii) Ⓛ Here is the generalization of (i) to an arbitrary category **C**. (Some of the basic definitions of category theory and our choice of notation can be found in the Glossary.) For a given $X \in \mathrm{ob}\mathbf{C}$, the collection of monomorphisms with codomain X is preordered by the relation

$(A \rightarrowtail^{a} X) \leq (B \rightarrowtail^{b} X)$ if and only if there is some (necessarily mono-)morphism $A \xrightarrow{f} B$ with $b \circ f = a$.

It is a simple exercise to prove that the notion of isomorphism associated with this preorder is

$(A \rightarrowtail^{a} X) \cong (B \rightarrowtail^{b} X)$ if and only if there is some isomorphism $i: A \cong B$ in **C** with $b \circ i = a$.

The equivalence classes for this relation of isomorphism are called *subobjects* of X, and the collection of all such is denoted $\mathrm{Sub}_{\mathbf{C}}(X)$ (or just $\mathrm{Sub}(X)$ if **C** is understood). The preorder on monomorphisms induces a partial order on subobjects as in 1.1. It is a common notational convenience to confuse a subobject with any monomorphism $A \rightarrowtail X$ in its equivalence class.

When **C**=**Set** is the category of sets and functions there is a bijection between the collection of subobjects of X and the set $\mathbf{P}(X)$ of (i). This bijection is given one way by sending a subset $A \subseteq X$ to the equivalence class of the monomorphism $A \rightarrowtail X$ obtained by restricting the identity function on X to A; in the other direction, send a subobject to $\{f(x) \mid x \in A\}$ where $f: A \rightarrowtail X$ is any monomorphism representing the subobject. (**Exercise**: work through the details of this; on the way you will need to prove that a morphism in **Set** is mono just in case it is an injective function.)

Note that in general there may be a possibly large number of subobjects of any given object in a category **C**. However, the above remark shows that when **C**=**Set**, $\mathrm{Sub}_{\mathbf{C}}(X)$ is again a set.

(iii) Ⓚ Let N denote the set of natural numbers. For any set X, let $(X \to \mathbf{P}\mathrm{N})$ denote the set of functions from X to the powerset of N. Define a binary relation $\leq$ on this set by:

$p \leq q$ if and only if there is a partial recursive function $\varphi: \mathrm{N} \rightharpoonup \mathrm{N}$ such that for all $x \in X$ and $n \in \mathrm{N}$, if $n \in p(x)$, then $\varphi(n)$ is defined and belongs to $q(x)$; in this case we say that φ *witnesses* the fact that $p \leq q$ and write $\varphi: p \leq q$ to indicate this.

Since the identity function $\mathrm{N} \rightharpoonup \mathrm{N}$ is partial recursive we have that $\leq$ is reflexive; and since the composition of two partial recursive functions is another such, we have that $\leq$ is transitive. We will denote by $\mathbf{R}(X)$ the preordered set $((X \to \mathbf{P}\mathrm{N}), \leq)$. (**Exercise**: show that when $X=1$ a one element set, the poset reflection of $\mathbf{R}(1)$ only contains two distinct elements, call them $\bot$ and $\top$, with $\bot \leq \top$. Is $\mathbf{R}(2)$ as simple as this?)

(iv) If X and Y are preordered sets, we can make a new one, called their *product*, by taking the cartesian product of the sets

$$X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$$

and imposing on this set the reflexive, transitive relation given by

$$(x,y) \leq (x',y') \text{ if and only if } x \leq x' \text{ in } X \text{ and } y \leq y' \text{ in } Y.$$

(**Exercise**. Prove that this gives the categorical product (see Glossary) of X and Y in the category of preordered sets and monotone functions (see Definition 2.1).)

(v) If X is a preordered set, its *opposite* X^{op} is the preordered set obtained by changing the preorder $\leq$ on the set X to $\leq^{\mathrm{op}}$, where for all $x,y \in X$, $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$.

The **Principle of Duality** for preorders is that whenever we have an order-theoretic concept or property, we get another one (called its dual) by replacing the preorders involved with their opposites. Here are some examples:

1.4 Definitions. Let X be a preordered set and $S \subseteq X$ a subset. An element $x \in X$ is an *upper bound* for S if $s \leq x$ for all $s \in S$. Dually, $y \in X$ is a *lower bound* for S if it is an upper bound for S in X^{op}, in other words if $y \leq s$ in X for all $s \in S$. An element $x \in X$ is a *greatest* element of S if it an upper bound which belongs to S; similarly a lower bound for S which also belongs to S is called a *least element* of S.

1.5 Proposition. *Any two greatest elements of S are isomorphic: we say that 'the' greatest element of S is defined 'uniquely up to isomorphism'. Similarly for least elements.*

Proof. If x and x' are both greatest elements of S then $x \leq x'$ since $x \in S$ and x' is an upper bound of S; but symmetrically $x' \leq x$, so $x \cong x'$. □

1.6 Remark. Conventially the word 'maximal' (respectively 'minimal') is reserved for a different concept than that of 'greatest' (respectively 'least'): $x \in S$ is a *maximal* element of S if for all $s \in S$, $x \leq s$ implies $x \cong s$. (Dually, $y \in S$ is *minimal* if for all $s \in S$, $s \leq y$ implies $s \cong y$.) For example in the poset $\{0,1,1'\}$ with $0 \leq 0 \leq 1 \leq 1$ and $0 \leq 1' \leq 1'$, but no other relations, 1 and $1'$ are both maximal elements, but neither is greatest.

1.7 Definitions. Let X be a preordered set. The *meet* (or *inf*, or *glb*) of a subset $S \subseteq X$ is the greatest element in the set of lower bounds of S. If it exists, it is unique up to isomorphism by Proposition 1.5 and is denoted $\bigwedge S$. Thus $\bigwedge S \in X$ has the property that for all $x \in X$:

(*1.1*) $\quad x \leq \bigwedge S$ if and only if for all $s \in S$, $x \leq s$.

Note that the reflexivity and transitivity of $\leq$ imply that (*1.1*) is equivalent to the conjunction of

(*1.2*) $\quad$ for all $s \in S$, $\bigwedge S \leq s$ and
(*1.3*) $\quad$ ($x \leq s$, all $s \in S$) implies $x \leq \bigwedge S$.

Dually, the *join* (or *sup*, or *lub*) of S is the least element of the set of upper bounds of S and is denoted $\bigvee S$.

Some special cases:

(i) Case $S = \emptyset$. $\bigwedge\emptyset$ is written $\top$ and called the *top* of X; $\bigvee\emptyset$ is written $\bot$ and called the *bottom* of X. Thus $\top$ and $\bot$ are defined up to isomorphism by the properties

(*1.4*) $\quad$ for all $x \in X$, $x \leq \top$ and
(*1.5*) $\quad$ for all $x \in X$, $\bot \leq x$.

(ii) Case $S = \{x,y\}$. $\bigwedge\{x,y\}$ is written $x \wedge y$ and called the (*binary*) *meet* of x and y; $\bigvee\{x,y\}$ is written $x \vee y$ and called the (*binary*) *join* of x and y. They are defined up to isomorphism by the properties

(*1.6*) for all $z\in X$, $z\leq x\wedge y$ if and only if ($z\leq x$ and $z\leq y$);
(*1.7*) for all $z\in X$, $x\vee y\leq z$ if and only if ($x\leq z$ and $y\leq z$).

If X posseses a top element and all binary meets, then (by induction) it has meets of all finite subsets and we call X *finitely complete*. Dually, if X has joins of all finite subsets, we will say it is *finitely cocomplete*.

(iii) Case S is a *directed* subset—which by definition means that every finite subset of S has an upper bound in S. This is equivalent to saying that every pair of elements of S has an upper bound in S and that S is non-empty (because 'finite' includes the case 'empty' and examing the definition carefully, you will see that for the empty subset of S to have an upper bound in S it is necessary and sufficient just that there be some element in S.) We say that X is *directed-cocomplete* if it has joins of all directed subsets. We will call X a *dcpo* if it is a directed-cocomplete *poset*. (**N.B.** Since the empty set is not directed, a dcpo does not necessarily possess a bottom element.)

1.8 Remark. There is a clash of terminology at this point between domain theory and category theory. A '*complete* poset', or 'cpo', is usually taken in domain theory to mean a poset with joins of all directed subsets and a bottom element—in other words with certain types of joins. But in category theory, 'completeness' refers to the existence of meets (or, more generally, of limits) whereas existence of joins (or more generally, colimits) is referred to as '*co*completeness'. Thus to category-theorists, a 'complete' poset is one with meets of all subsets and a 'cocomplete' poset is one with all joins; however, they also know that:

1.9 Proposition. *A preordered set has all joins if and only if it has all meets.*

Proof. The interdefinability of meets and joins is given by the formulas

$$\bigwedge S \;=\; \bigvee\{x\in X \mid x\leq s,\ \text{all } s\in S\} \quad\text{and}\quad \bigvee S \;=\; \bigwedge\{x\in X \mid s\leq x,\ \text{all } s\in S\},$$

where $S\subseteq X$. □

1.10 Examples. (i) Ⓣ For a set X, the poset $\mathbf{P}(X)$ is complete: for $S\subseteq\mathbf{P}(X)$, $\bigwedge S$ is given by the *intersection* $\bigcap S=\{x\in X \mid x\in A,\ \text{all } A\in S\}$; and and $\bigvee S$ is given by the *union* $\bigcup S=\{x\in X \mid x\in A,\ \text{some } A\in S\}$. In particular $\top$ is X and $\bot$ is $\emptyset$.

(ii) Ⓚ *F*or a set X, the preordered set $\mathbf{R}(X)$ has a top element, namely the function $\top : X\longrightarrow \mathbf{PN}$ sending each $x\in X$ to $\mathbf{N}$ itself. (For any $p\in\mathbf{R}(X)$, the identity function $\mathbf{N}\longrightarrow\mathbf{N}$ witnesses the fact that $p\leq\top$.) Note that in $\mathbf{R}(X)$ there are in fact many different top elements, but they are all isomorphic (necessarily, by Proposition 1.5). For example, for each $k\in\mathbf{N}$ the function $x\in X\mapsto\{k\}\in\mathbf{PN}$ also gives a top element of $\mathbf{R}(X)$ (as witnessed by the recursive function $\mathbf{N}\longrightarrow\mathbf{N}$ which sends all $n\in\mathbf{N}$ to k). $\mathbf{R}(X)$ is truly a preorder and not a poset.

Next we wish to show that $\mathbf{R}(X)$ has binary meets. For this we use a *recursive pairing function* $\langle -,-\rangle : \mathbf{N}\times\mathbf{N}\longrightarrow\mathbf{N}$ (such as $\langle n,m\rangle=\frac{1}{2}(n+m)(n+m+1)+n$, for example) with corresponding *recursive projections* $(-)_0 : \mathbf{N}\longrightarrow\mathbf{N}$ and $(-)_1 : \mathbf{N}\longrightarrow\mathbf{N}$. These are recursive functions satisfying $(\langle n,m\rangle)_0 = n$ and $(\langle n,m\rangle)_1 = m$, all $n,m \in \mathbf{N}$. (The particular pairing mentioned is surjective, but we do not need to use that fact.) The pairing function determines a function $\wedge : \mathbf{PN} \times \mathbf{PN} \longrightarrow \mathbf{PN}$ sending $A \subseteq \mathbf{N}$ and $B \subseteq \mathbf{N}$ to

$$A \wedge B = \{\langle n,m\rangle \mid n \in A \text{ and } m \in B\}.$$

Then the meet of p and q in $\mathbf{R}(X)$ is given by $p \wedge q : X \longrightarrow \mathbf{PN}$ where for each $x \in X$

$$(p \wedge q)(x) = p(x) \wedge q(x) = \{\langle n,m\rangle \mid n \in p(x) \text{ and } m \in q(x)\}.$$

For $(-)_0 : p \wedge q \leq p$ and $(-)_1 : p \wedge q \leq q$; and if $r \in \mathbf{R}(X)$ with $\varphi : r \leq p$ and $\psi : r \leq q$ for partial recursive functions $\varphi : \mathbf{N} \rightharpoonup \mathbf{N}$ and $\psi : \mathbf{N} \rightharpoonup \mathbf{N}$, then the partial recursive function $n \longmapsto \langle \varphi(n), \psi(n)\rangle$ witnesses that $r \leq p \wedge q$. Thus each $\mathbf{R}(X)$ is finitely complete.

It is also the case that $\mathbf{R}(X)$ is finitely cocomplete. A bottom element is given by $\bot : X \longrightarrow \mathbf{PN}$ where $\bot(x) = \emptyset$, all $x \in X$. And given $p,q \in \mathbf{R}(X)$, their join is given by given by $p \vee q : X \longrightarrow \mathbf{PN}$ where for each $x \in X$

$$(p \vee q)(x) = \{\langle 0,n\rangle \in \mathbf{N} \mid n \in p(x)\} \cup \{\langle 1,m\rangle \mid m \in q(x)\}.$$

In other words $(p \vee q)(x) = p(x) \vee q(x)$, where $\vee : \mathbf{PN} \times \mathbf{PN} \longrightarrow \mathbf{PN}$ is the function

$$(A,B) \longmapsto A \vee B = \{\langle 0,n\rangle \in \mathbf{N} \mid n \in A\} \cup \{\langle 1,m\rangle \mid m \in B\}.$$

(**Exercise.** Prove that this does indeed give finite joins in $\mathbf{R}(X)$.)

(iii) Ⓛ If X is an object in a category $\mathbf{C}$ then $\mathrm{Sub}_{\mathbf{C}}(X)$ always has a top element, namely the subobject determined by the identity morphism $\mathrm{id}_X : X \longrightarrow X$ (which is of course a monomorphism). In general that is all we can say. However, if $\mathbf{C}$ has pullbacks (see Glossary) then each $\mathrm{Sub}(X)$ has binary meets: the meet of $A \rightarrowtail X$ and $B \rightarrowtail X$ is given by first forming the pullback square

$$\begin{array}{ccc} A \times_X B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \rightarrowtail & X \end{array}$$

and then using the easily verified facts (**Exercises!**) that

- the pullback of a monomorphism is a monomorphism
- the composition of two monomorphisms is a monomorphism

to conclude that $A \times_X B \rightarrowtail A \rightarrowtail X$ determines a subobject of X; and the universal property of the above pullback square implies in particular that this subobject is the meet of the subobjects $A \rightarrowtail X$ and $B \rightarrowtail X$.

(iv) Ⓢ Let D be a dcpo (see 1.7(iii)). A subset $C \subseteq D$ is called

- *inductive* if whenever $S \subseteq D$ is directed and $S \subseteq C$, then $\bigvee S \in C$;
- *downwards closed* if $x \leq y \in C$ implies $x \in C$;
- *upwards closed* if $x \geq y \in C$ implies $x \in C$;
- *convex* if $y \leq x \leq z$ with $y,z \in C$ implies $x \in C$.

(**Exercise.** Show that C is convex if and only if it is equal to the intersection of an upwards closed subset with a downwards closed one.)

The *Scott closed* subsets of D are those C which are both inductive and downwards closed. Let $\mathbf{F}(D)$ denote the poset of all Scott closed subsets of D partially ordered by subset inclusion. It is easy to see that the intersection of a collection of inductive (respectively upwards closed, downwards closed, or convex) subsets is another such. In particular, it follows that $\mathbf{F}(D)$ has all meets and that they are calculated as in $\mathbf{P}(D)$. Hence by Proposition 1.9, $\mathbf{F}(D)$ also has all joins—although these joins are not necessarily given by set theoretic union. In general, the most we can say is that *finite* joins are given by union. (**Exercise.** Prove that the union of two Scott closed subsets is again Scott closed.)

1.11 Remark: preordered sets are categories. If $\mathbf{C}$ is a category such that the function

$$\begin{array}{rcl} \mathrm{mor}\mathbf{C} & \longrightarrow & \mathrm{ob}\mathbf{C} \times \mathrm{ob}\mathbf{C} \\ f & \longmapsto & (\mathrm{dom} f, \mathrm{cod} f) \end{array}$$

is injective (in other words, for all parallel pairs of morphisms $f, g : ? \rightrightarrows ?$ in $\mathbf{C}$, $f = g$), then the morphisms of $\mathbf{C}$ amount to specifying a relation between the objects. That relation is reflexive (because of identity morphisms) and transitive (because of composition of morphisms), in other words, is a preorder.

Conversely, every preordered set X determines a category of this kind with $\mathrm{ob}\mathbf{C} = X$ by declaring that morphisms $x \longrightarrow y$ are instances of the preorder relation: there is a morphism from x to y, call it (x,y), just in case $x \leq y$. Thus the identity on x is (x,x) (reflexivity!) and the composition $(y,z) \circ (x,y)$ is (x,z) (transitivity!).

2 Monotone Functions

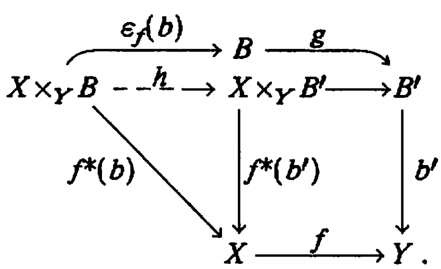
2.1 Definition. A function $f: X \longrightarrow Y$ between preordered sets is *monotone* if

$$x \leq x' \text{ in } X \quad \text{implies} \quad f(x) \leq f(x') \text{ in } Y.$$

2.2 Examples. (i) Ⓣ If $f: X \longrightarrow Y$ is a function between sets, we get a monotone function $f^{-1}: \mathbf{P}(Y) \longrightarrow \mathbf{P}(X)$ by taking *inverse images* of subsets: for $B \subseteq Y$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

(ii) Ⓛ Suppose that $\mathbf{C}$ is a category with pullbacks (see Glossary). The pullback along a morphism $f: X \longrightarrow Y$ of a monomorphism $b: B \rightarrowtail Y$ results in a morphism $f^*(b): X \times_Y B \longrightarrow X$ which is again a monomorphism. Thus 'pullback along f' gives a function from the monomorphisms with codomain Y to the monomorphisms with codomain X. This function is montone for the preorder defined in 1.3(ii). For suppose that $(B \overset{b}{\rightarrowtail} Y) \leq (B' \overset{b'}{\rightarrowtail} Y)$, so that there is some $g: B \longrightarrow B'$ with $b' \circ g = b$. We have to show that $(X \times_Y B \overset{f^*(b)}{\rightarrowtail} X) \leq (X \times_Y B' \overset{f^*(b')}{\rightarrowtail} X)$ by producing an $h: X \times_Y B \longrightarrow X \times_Y B'$ with $f^*(b') \circ h = f^*(b)$; but the universal property of the pullback square for f and b' furnishes such an h:



Since pullback along f respects the preorder on monomorphisms, it induces a well-defined monotone function between posets of subobjects which we denote by

$$f^{-1}: \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X).$$

It is convenient to denote the result of applying f^{-1} to (the subobject determined by) $B \rightarrowtail Y$ by $f^{-1}(B) \rightarrowtail X$.

When $\mathbf{C} = \mathbf{Set}$, it is easily verified that under the identification of subobjects with subsets established in 1.3(ii), the operation of pulling back subobjects along a function becomes identified with the operation defined in (i) of taking inverse images of subsets.

(iii) Ⓚ Every function $f: X \longrightarrow Y$ induces a function $f^*: (Y \rightarrow \mathbf{PN}) \longrightarrow (X \rightarrow \mathbf{PN})$ given by precomposition with f: $f^*(p) = p \circ f$. Given $p, q \in (Y \rightarrow \mathbf{PN})$, if $\varphi: p \leq q$ in $\mathbf{R}(Y)$, then clearly φ also witnesses that $f^*(p) \leq f^*(q)$ in $\mathbf{R}(X)$. So we have a monotone function

$$f^*: \mathbf{R}(Y) \longrightarrow \mathbf{R}(X).$$

(iv) Ⓢ By definition, a function $f: D \longrightarrow E$ between dcpo's is *continuous* if it is monotone and preserves directed joins: the latter means that for all directed subsets $S \subseteq D$

$$f(\bigvee S) = \bigvee\{f(s) \mid s \in S\}.$$

(Note that the join on the right hand side exists because the monotonicity of f implies that $\{f(s) \mid s \in S\} \subseteq E$ is directed.) In this case the inverse image function $f^{-1}: \mathbf{P}(E) \longrightarrow \mathbf{P}(D)$ sends Scott closed subsets of E to Scott closed subsets of E. (Preservation by f^{-1} of the property of being downwards closed is a consequence of monotonicity of f, whilst preservation by f^{-1} of the property of being inductive is a consequence of f preserving directed joins.) Hence f^{-1} restricts to give a monotone function

$$f^*: \mathbf{F}(E) \longrightarrow \mathbf{F}(D).$$

(**Exercise.** Show conversely that a function $f: D \longrightarrow E$ is continuous if $f^{-1}: \mathbf{P}(E) \longrightarrow \mathbf{P}(D)$ sends Scott closed subsets of E to Scott closed subsets of D.)

2.3 Definition. If $f, g: X \rightrightarrows Y$ are monotone functions between preordered sets, define

(*2.1*) $\quad f \leq g \quad$ if and only if $\quad$ for all $x \in X$, $f(x) \leq g(x)$.

Clearly, this establishes a preorder on the set of all monotone functions from X to Y: we denote the resulting preordered set by $[X, Y]$. (In contrast to $(X \to Y)$, which denotes the set of *all* functions from X to Y.)

2.4 Proposition. (i) *The collection of monotone functions between preorders contains the identity functions and is closed under composition.* (*Hence there is a category, denoted* **Preord**, *of preordered sets and monotone functions.*) *Moreover, composition* $(g, f) \longmapsto g \circ f$ *is a monotone function* $[Y, Z] \times [X, Y] \longrightarrow [X, Z]$.

(ii) *If X, Y and Z are preorders, a function $f: X \times Y \longrightarrow Z$ is monotone if and only if it is* monotone in each variable separately, *which means for all $x, x' \in X$ and $y, y' \in Y$*

$$x \leq x' \text{ implies } f(x, y) \leq f(x', y) \text{ and}$$
$$y \leq y' \text{ implies } f(x, y) \leq f(x, y').$$

Proof. Left as easy **Exercises.** □

As one might expect, two preorders X and Y are *isomorphic* if they are isomorphic objects (see Glossary) in the category **Preord**, which is to say that there are monotone functions $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. So in particular f is a bijection of sets. Conversely, a bijection f is an isomorphism in **Preord** if it both preserves and reflects order: $x \leq x'$ in X if and only if $f(x) \leq f(x')$ in Y.

Warning. A monotone bijection is not necessarily an isomorphism of preordered sets. (**Exercise.** What is the simplest example of this failure?)

In practice the more useful relation between preorders is not isomorphism, but rather equivalence:

2.5 Definition. A monotone function $f: X \longrightarrow Y$ between preordered sets is

(i) *full* if for all $x, x' \in X$, $f(x) \leq f(x')$ implies $x \leq x'$;

(ii) *essentially surjective* if for all $y \in Y$ there is some $x \in X$ with $f(x) \cong y$;

(iii) a *weak equivalence* if it is both full and essentially surjective;

(iv) an *equivalence* if it possesses an *essential inverse*—which is a monotone function $g: Y \longrightarrow X$ satisfying $g \circ f \cong \mathrm{id}_X$ in $[X, X]$ and $f \circ g \cong \mathrm{id}_Y$ in $[Y, Y]$; in this case we say that X and Y are *equivalent* and write $X \simeq Y$.

2.6 Example. The quotient function $X \longrightarrow X/\cong$ from a preordered set to its poset reflection is a weak equivalence. It is only an isomorphism when X is already a poset.

2.6 Remarks. (i) A full monotone function between posets is necessarily an injective function and is usually called an *embedding* of posets.

(ii) Each equivalence is a weak equivalence. Conversely, if $f: X \longrightarrow Y$ is a weak equivalence, we can use the Axiom of Choice to construct a function $g: Y \longrightarrow X$ which for each $y \in Y$ picks out some $g(y) \in X$ with $f(g(y)) \cong y$. (There is such an element because f is essentially surjective.) This g is automatically monotone since if $y \leq y'$, then $f(g(y)) \cong y \leq y' \cong f(g(y'))$, so that $g(y) \leq g(y')$ since f is full. Moreover for each $x \in X$, putting $y = f(x)$, we have $f(g(y)) \cong y = f(x)$, so that $g(y) \cong x$ by fullness of f, which is to say that $g(f(x)) \cong x$. So all in all f is an equivalence with essential inverse g. (Note that if X were a poset, the Axiom of Choice would not be needed in this construction since for each y there would be a unique x with $f(x) \cong y$.)

Thus modulo the Axiom of Choice, the notions of 'weak equivalence' and 'equivalence' coincide. In practice the weak equivalence of two preordered sets is enough for them to share similar order-theoretic properties.

2.7 Remark: monotone functions are functors. In 1.11 we noted that preordered sets can be regarded as particular kinds of category—ones for which there is at most one morphism between any pair of objects. Consequently category theoretic notions make sense for preordered sets. In particular, specifying a functor (see Glossary) between two such categories amounts to giving a monotone function. And given two monotone functions $f, g: X \rightrightarrows Y$ regarded as functors, there is at most one natural transformation (see Glossary) from f to g, and there is one just in case $f \leq g$ in $[X, Y]$.

The notions of fullness and equivalence given above coincide with the usual category theoretic ones because any functor between preordered sets is automatically faithful (see Glossary).

In the next section we turn our attention to what is probably the most important concept in category theory, that of 'adjoint functor', and see what it looks like for the particularly simple case of preorders.

3 Adjoint Functions

3.1 Definition. A pair of monotone functions $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ between preordered sets determines an *adjunction* if for all $x \in X$ and $y \in Y$

(*3.1*) $\quad f(x) \le y$ in Y if and only if $x \le g(y)$ in X.

In this case we write $f \dashv g$ and say '*f is left adjoint to g*', or '*g is right adjoint to f*' (Mnemonic: *left* adjoints appear on the *left* of $\le$ in the above definition and *right* adjoints on the *right*.)

The following lemmas establish the basic properties of adjoints.

3.2 Proposition. ('Unit and counit' characterization of adjunction.) *Given* $f \in [X,Y]$ *and* $g \in [Y,X]$, $f \dashv g$ *if and only if* $\mathrm{id}_X \le g \circ f$ *in* $[X,X]$ *and* $f \circ g \le \mathrm{id}_Y$ *in* $[Y,Y]$.

Proof. $\Rightarrow$: For all $x \in X$, $f(x) \le f(x)$, so $x \le g(f(x))$ by (*3.1*); so $\mathrm{id}_X \le g \circ f$ by (*2.1*). Similarly, since for all $y \in Y$ $g(y) \le g(y)$, we have $f(g(y)) \le y$ by (*3.1*), *i.e.* $f \circ g \le \mathrm{id}_Y$.

$\Leftarrow$: If $f(x) \le y$, then by monotonicity of g $g(f(x)) \le g(y)$; but $x \le g(f(x))$ by hypothesis, so $x \le g(y)$ by transitivity. A similar argument using $f \circ g \le \mathrm{id}_Y$ gives the converse implication in (*3.1*). □

3.3 Proposition. ('Pointwise' characterization of adjoints.) *A monotone function* $g\colon Y \longrightarrow X$ *between preordered sets possesses a left adjoint if and only if there is a function assigning to each* $x \in X$ *an element* $f(x) \in Y$ *satisfying*

- $x \le g(f(x))$ *and*
- *for all* $y \in Y$, *if* $x \le g(y)$ *then* $f(x) \le y$.

(*An element* $f(x)$ *satisfying the above two conditions is said to be a* 'value of the left adjoint to g at x' *even if such elements do not exist for all* $x \in X$. *Note that* $f(x)$ *is determined uniquely up to isomorphism by the two conditions.*)

Proof. In view of (*3.1*) and Proposition 3.2, if $f \dashv g$ then $x \longmapsto f(x)$ is a function satisfying the requirements. Conversely, given such a function it is enough to see that it is monotonic: for then the second hypothesis gives one half of (*3.1*) directly and the first hypothesis gives the other half indirectly, arguing as in Proposition 3.2. But if $x \le x'$ in X, then $x \le x' \le g(f(x'))$, so taking $y = f(x')$ in the second hypotheses we have $f(x) \le y = f(x')$, as required for monotonicity. □

3.4 Proposition. (Uniqueness of adjoints up to isomorphism.) *Suppose* $f, f' \in [X,Y]$ *and* $g, g' \in [Y,X]$ *with* $f \dashv g$ *and* $f' \dashv g'$. *Then* $f \le f'$ *if and only if* $g' \le g$. *Hence* $f \cong f'$ *if and only if* $g' \cong g$. *In particular, the adjoint of a monotone function is unique up to isomorphism if it exists; and if* X *and* Y *are* posets, *the adjoints are actually unique if they exist.*

Proof. We use Propositions 2.4(i) and 3.2. If $f \leq f'$, then

$$\begin{aligned} g' &\leq g \circ f \circ g' && \text{since } \mathrm{id}_X \leq g \circ f \\ &\leq g \circ f' \circ g' \\ &\leq g && \text{since } f' \circ g' \leq \mathrm{id}_Y. \end{aligned}$$

Converesly if $g' \leq g$, then

$$\begin{aligned} f &\leq f \circ g' \circ f' && \text{since } \mathrm{id}_X \leq g' \circ f' \\ &\leq f \circ g \circ f' \\ &\leq f' && \text{since } f \circ g \leq \mathrm{id}. \quad \Box \end{aligned}$$

3.5 Proposition. (Composition of adjoints.) *Given preordered sets and monotone functions*

$$X \underset{f}{\overset{g}{\rightleftarrows}} Y \underset{h}{\overset{k}{\rightleftarrows}} Z\ ,$$

if $f \dashv g$ and $h \dashv k$, then $h \circ f \dashv g \circ k$.

Proof. For all $x \in X$ and $z \in Z$, $h(f(x)) \leq z$ iff $f(x) \leq k(z)$ (since $h \dashv k$) iff $x \leq g(k(z))$ (since $f \dashv g$). $\Box$

3.6 Proposition. *Note that a monotone function $f\colon X \longrightarrow Y$ also determines a monotone function from X^{op} to Y^{op}, which we will denote by $f^{\mathrm{op}}\colon X^{\mathrm{op}} \longrightarrow Y^{\mathrm{op}}$. Then $g^{\mathrm{op}} \dashv f^{\mathrm{op}}$ if $f \dashv g$. (Thus the Principle of Duality (cf* 1.3(v)) *turns statements about left adjoints into ones about right adjoint and vice versa.)*

Proof. For all $x \in X$ and $y \in Y$, $g^{\mathrm{op}}(y) \leq^{\mathrm{op}} x$ iff $x \leq g(y)$, iff $f(x) \leq y$, iff $y \leq^{\mathrm{op}} f^{\mathrm{op}}(x)$. $\Box$

3.7 Theorem. *Let $f\colon X \longrightarrow Y$ be a monotone function between preordered sets.*

(i) *If f has a right adjoint then it* preserves all joins *which exist in X: in other words, if $S \subseteq X$ and $\bigvee S$ exists, then $\bigvee\{f(s) \mid s \in S\}$ exists in Y and is isomorphic to $f(\bigvee S)$. (Dually, if f has a left adjoint it preserves all meets.)*

(ii) **(Adjoint Functor Theorem for preorders.)** *Conversely, provided X has joins of all subsets, f has a right adjoint if it preserves them. (Dually, f has a left adjoint if X has and f preserves all meets. Recall from Proposition 1.9 that X has all meets if and only if it has all joins.)*

Proof. (i) Suppose $f \dashv g$ and that $\bigvee S$ exists for some $S \subseteq X$. Then for all $y \in Y$, $f(\bigvee S) \leq y$ iff $\bigvee S \leq g(y)$, iff for all $s \in S$ $s \leq g(y)$, iff for all $s \in S$ $f(s) \leq y$. Hence $f(\bigvee S)$ is indeed the join of $\{f(s) \mid s \in S\}$ in Y.

(ii) For each $y \in Y$, define

$$g(y) = \bigvee\{x \mid f(x) \leq y\}.$$

By hypothesis $f(g(y)) \cong \bigvee\{f(x) \mid f(x) \leq y\}$ and clearly this join is $\leq y$. Moreover, if $f(x) \leq y$ for some $x \in X$, then $x \leq g(y)$ by definition of $g(y)$. Hence the hypotheses of the dual of Proposition 3.3 hold and we can conclude that $y \longmapsto g(y)$ determines a right adjoint for f. $\Box$

3.8 Examples. (Adjoints abound!)

(i) Let 1 denote the poset {0} with $0 \leq 0$. (1 is a terminal object in **Preord**.) For each preordered set X, the unique function $X \longrightarrow 1$ (is monotone and) has a right adjoint if and only if X has a top element—in which case the right adjoint is the function $1 \longrightarrow X$ with $0 \longmapsto \top$. Dually, $X \longrightarrow 1$ has a left adjoint if and only if X has a bottom element.

(ii) For each preordered set X the *diagonal function* $\Delta: X \longrightarrow X \times X$ given by $\Delta(x)=(x,x)$ is monotone; it has a right adjoint if and only if X has binary meets—in which case the right adjoint is given by $(x,x') \in X \times X \longmapsto x \wedge x' \in X$. Dually, Δ has a left adjoint just when X has binary joins.

(**Exercise.** Generalize (i) and (ii) to an adjoint characterization of infinite meets and joins.)

(iii) Ⓣ (**Quantifiers as adjoints.**) The monotone function $f^{-1}: \mathbf{P}(Y) \longrightarrow \mathbf{P}(X)$ of 2.2(i) possesses both left and right adjoints. The left adjoint is denoted $\exists f: \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ and sends $A \subseteq X$ to

$$\exists f(A) = \{f(x) \mid x \in A\} = \{y \in Y \mid \exists x \in X\ (\, f(x)=y \text{ and } x \in A\,)\}.$$

(**Exercise.** Check that $\exists f(A) \subseteq B$ iff $A \subseteq f^{-1}(B)$ for any $B \subseteq Y$.) Because of the first of the above equalities $\exists f(A)$ is often called the *image* of A along f (and denoted just by $f(A)$); because of the second of the equalities, $\exists f$ is called *existential quantification along f*. Dually, the right adjoint to f^{-1} is denoted $\forall f: \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ and sends $A \subseteq X$ to

$$\forall f(A) = \{y \in Y \mid f^{-1}\{y\} \subseteq A\} = \{y \in Y \mid \forall x \in X\ (\, f(x)=y \text{ implies } x \in A\,)\}.$$

$\forall f(A)$ is sometimes called the *dual image* of A along f and $\forall f$ is called *universal quantification along f*. (**Exercise.** Check that $f^{-1}(B) \subseteq A$ iff $B \subseteq \forall f(A)$.)

The connection with quantification is even plainer when we take f to be a product projection, say $\pi_1: X \times Y \longrightarrow X$. Then for $A \subseteq X \times Y$ and $B \subseteq X$

$$\begin{aligned} \pi_1^{-1}(B) &= \{(x,y) \in X \times Y \mid x \in B\}, \\ \exists \pi_1(A) &= \{x \in X \mid \exists y \in Y\ (x,y) \in A\} \text{ and} \\ \forall \pi_1(A) &= \{x \in X \mid \forall y \in Y\ (x,y) \in A\}. \end{aligned}$$

Thus $\pi_1^{-1}(B)$ is the 'weakening' of the property B of elements $x \in X$ to a property of elements $(x,y) \in X \times Y$ by ignoring y. So we may say (with Lawvere) that "existential quantification is left adjoint to weakening and universal quantification is right adjoint to weakening". Another, formal justification for this slogan will emerge in section 8.

(iv) Ⓛ Generalizing (iii) from **Set** to an arbitrary category **C** with pullbacks, given a morphism $f: X \longrightarrow Y$ in **C**, the values of the left and right adjoints to $f^{-1}: \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ at a subobject $A \rightarrowtail X$, *if they exist*, will be denoted

$$\exists f(A) \rightarrowtail Y \quad \text{and} \quad \forall f(A) \rightarrowtail Y$$

and called the *existential* and *universal quantifications* of $A \rightarrowtail X$ along f.

(**Exercise.** Show that $f: X \longrightarrow Y$ is a cover (see Glossary) if and only if $\exists f(\top)$ exists and $\exists f(\top) = \top \in \mathrm{Sub}(Y)$. Show that the left adjoints $\exists f$ exist for all f if and only if **C** has image factorizations (see Glossary).)

(**Exercise.** Recall that the composition of two monomorphisms is again a monomorphism. Hence show that if $f: X \longrightarrow Y$ is a monomorphism, then $\exists f: \mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(Y)$ exists and is given by composition with f.)

(v) Ⓢ If D is a dcpo, we saw in 1.10(iv) that meets in $\mathbf{F}(D)$ are calculated as in $\mathbf{P}(D)$—by intersections; thus the inclusion $\mathbf{F}(D) \hookrightarrow \mathbf{P}(D)$ preserves all meets and hence by Theorem 3.7(ii) it has a left adjoint, $\mathbf{P}(D) \longrightarrow \mathbf{F}(D)$ whose value at a subset $S \subseteq D$ is the Scott closed set

$$\bar{S} = \bigcap\{C \in \mathbf{F}(D) \mid S \subseteq C\},$$

called the *closure* of S.

If $f: D \longrightarrow E$ is a continuous function between dcpo's, then we saw in 2.2(iv) that $f^{-1}: \mathbf{P}(E) \longrightarrow \mathbf{P}(D)$ restricts to give $f^*: \mathbf{F}(E) \longrightarrow \mathbf{F}(D)$. Since the former preserves all meets (since from (iii) it has a left adjoint) and meets for $\mathbf{F}(E)$ are calculated as in $\mathbf{P}(E)$, it follows that f^* preserves meets. Hence by 3.7(ii) it has a left adjoint, which will be denoted $f_!: \mathbf{F}(D) \longrightarrow \mathbf{F}(E)$. (**Exercise.** Prove that for $C \in \mathbf{F}(D)$, $f_!(C)$ is the closure of the image of C along f: $f_!(C) = \overline{\exists f(C)}$.)

It is not the case in general that f^* has a right adjoint. (**Execise.** Give an example of a continuous f for which f^* does not have a right adjoint.)

(vi) Ⓚ The montone function $f^*: \mathbf{R}(Y) \longrightarrow \mathbf{R}(X)$ of 2.2(iii) has both left and right adjoints, which will again be denoted respectively $\exists f$ and $\forall f$. Given $p \in (X \to \mathbf{PN})$, $\exists f(p) \in (Y \to \mathbf{PN})$ is the function

$$y \in Y \longmapsto \bigcup\{p(x) \mid f(x) = y\}.$$

In other words

$$n \in \exists f(p)(y) \text{ if and only if } n \in p(x) \text{ for some } x \in X \text{ with } f(x) = y.$$

To see that this does give the value of the left adjoint to f^* at p, first note that for any $q \in (Y \to \mathbf{PN})$ and $y \in Y$, $\exists f(f^*q)(y) = \bigcup\{q(f(x)) \mid f(x) = y\} \subseteq q(y)$ so that $\mathrm{id}: \exists f(f^*q) \leq q$ in $\mathbf{R}(Y)$. Moreover, if $p \in (X \to \mathbf{PN})$ and $\varphi: \exists f(p) \leq q$ in $\mathbf{R}(Y)$, for some partial recursive $\varphi: \mathrm{N} \rightharpoonup \mathrm{N}$, then we have

> for all $x \in X$, $y \in Y$ and $n \in \mathrm{N}$, if $f(x) = y$ and $n \in p(x)$, then $\varphi(n)$ is defined and belongs to $q(y)$,

which means that for each $x \in X$, φ maps $p(x)$ into $q(f(x))$. Hence $\varphi: p \leq f^*(q)$ in $\mathbf{R}(X)$. Thus $\exists f \dashv f^*$ by Proposition 3.3.

We turn now to a description of $\forall f: \mathbf{R}(X) \longrightarrow \mathbf{R}(Y)$. To do this we must consider codes of partial recursive functions. Let

$$\begin{aligned} &\mathrm{N} \times \mathrm{N} \rightharpoonup \mathrm{N} \\ &(n, m) \longmapsto n \cdot m \end{aligned}$$

denote a *partial recursive application*. In other words, $n \cdot m$ is the value at m of the n^{th} partial recursive function in some well-behaved enumeration, if the latter is defined there. (The notation $\{n\}(m)$ is the traditional one for $n \cdot m$.) For each partial recursive $\varphi: \mathrm{N} \rightharpoonup \mathrm{N}$ there is some (in fact, necessarily infinitely many) $n \in \mathrm{N}$ such that for all $m \in \mathrm{N}$ $\varphi(m)$ is defined if and only if $n \cdot m$ is and in that case they are equal; n is called a *code* for φ. For partially defined expressions t which may or may not denote elements of N we introduce the following abbreviations:

$t\downarrow$ means 't is defined';

$t \asymp t'$ means '$t\downarrow$ iff $t'\downarrow$ and in that case they are equal'.

Also we will tend to write just st for $s\cdot t$; and in multiple applications we will often omit brackets and use the convention that application associates to the left: thus rst stands for $(r\cdot s)\cdot t$ for example. We will also use the λ-notation for partial functions: if $t(x)$ is some expression involving n, $\lambda n.t(n)$ denotes the partial function which is defined at $n\in\mathrm{N}$ iff $t(n)\downarrow$ and in that case has that value there.

The partial resursive application makes N a 'partial combinatory algebra' in the sense that there are numbers ('combinators') $\mathsf{K},\mathsf{S}\in\mathrm{N}$ satisfying for all $x,y,z\in\mathrm{N}$ that

$\mathsf{K}x\downarrow$ and $\mathsf{K}xy \asymp x$

$\mathsf{S}x\downarrow$, $\mathsf{S}xy\downarrow$ and $\mathsf{S}xyz \asymp xz(yz)$.

Now given $p\in(X\to\mathbf{PN})$ and $y\in Y$, define $\forall f(p)(y)\subseteq\mathrm{N}$ by

$n\in\forall f(p)(y)$ if and only if for all $x\in X$ and $m\in\mathrm{N}$, if $f(x)=y$ then $n\cdot m\downarrow$ and $n\cdot m\in p(x)$.

Then it is the case that $f^*(\forall f(p))\leq p$ is witnessed by the partial recursive function $\lambda n.n\cdot 0$ (for which $\mathsf{S(SKK)(K0)}$ is a code); and if we have $\varphi{:}f^*(q)\leq p$ for some $q\in(Y\to\mathbf{PN})$ and partial recursive φ, then $q\leq\forall f(p)$ is witnessed by $\lambda n.\mathsf{K}\cdot\varphi(n)$ (for which $\mathsf{S(KK)}\ulcorner\varphi\urcorner$ is a code, if $\ulcorner\varphi\urcorner$ is a code for φ). Hence by Proposition 3.3 $\forall f(p)$ is indeed a value of the right adjoint to f^* at p.

(**Exercise.** By analogy with the case of $\exists$, one might expect the value of the right adjoint at p to be given by $y\longmapsto\bigcap\{p(x)\mid f(x)=y\}$. Why doesn't this work?)

(vii) Ⓢ The notion of 'embedding-projection pair' used in domain theory in connection with the solution of recursive domain equations is a particular kind of adjunction. Given dcpo's D and E a *(continuous) embedding* $i{:}D\hookrightarrow E$ is a monotone function with a *continuous* right adjoint $p{:}E\longrightarrow D$ which is also left inverse to i. (Note that since i has a right adjoint it must preserve existing joins and in particular is continuous.) Thus i and p are both monotone and directed join preserving, and satisfy $p\circ i=\mathrm{id}_D$ and $i\circ p\leq\mathrm{id}_E$. Such a p is called a *(continuous) projection.* Note that since adjoints to monotone functions between posets are unique, each of i and p determines the other.

3.9 Definitions. Let X be a preordered set with binary meets. If it exists, the *(Heyting) implication* of $x,y\in X$ is the element $x\Rightarrow y\in X$ defined uniquely up to isomorphism by the property:

for all $z\in X$, $z\leq x\Rightarrow y$ if and only if $z\wedge x\leq y$.

Thus $x\Rightarrow y$ is a value at y of the right adjoint to the monotone function $(-)\wedge x{:}X\longrightarrow X$. (The term 'relative pseudocomplement' is also used for $x\Rightarrow y$.)

A monotone function $f{:}X\longrightarrow Y$ which preserves binary meets also *preserves* the implication $x\Rightarrow y$ if $f(x\Rightarrow y)$ is a value of the right adjoint to $(-)\wedge f(x)$ at $f(y)$ (so that $f(x\Rightarrow y)\cong f(x)\Rightarrow f(y)$).

A preordered set with all finite meets, all finite joins and all implications is called *Heyting.* A *Heyting algebra* is a poset which is Heyting as a preordered set.

(**Exercise.** Show that a preordered set is Heyting if and only if its poset reflection is a Heyting algebra.) A *morphism* of Heyting preordered sets is a monotone function preserving finite meets, finite joins and implications. The category of Heyting preordered sets and their morphisms will be denoted **Heyt**.

3.10 Proposition. (**'Frobenius Reciprocity'**) *Suppose that $f\colon X \longrightarrow Y$ is a monotone, binary meet preserving function between preordered sets with binary meets and implications. Suppose also that f has a left adjoint $\ell\colon Y \longrightarrow X$. Then f preserves implications if and only if $\ell \dashv f$ satisfies 'Frobenius Reciprocity':*

$$\text{for all } x \in X \text{ and } y \in Y, \qquad \ell(y \wedge f(x)) \cong \ell(y) \wedge x.$$

Proof. In the following two diagrams of monotone functions

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle x\Rightarrow(-)}\downarrow & & \downarrow{\scriptstyle f(x)\Rightarrow(-)} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xleftarrow{\ \ell\ } & Y \\ {\scriptstyle (-)\wedge x}\uparrow & & \uparrow{\scriptstyle (-)\wedge f(x)} \\ X & \xleftarrow[\ \ell\ ]{} & Y \end{array}$$

the arrows in the right-hand diagram are left adjoint to the corresponding arrows in the left-hand diagram. Hence by Propositions 3.4 and 3.5, the left-hand diagram commutes up to isomorphism if and only if the right-hand one does. □

3.11 Proposition. (**Distributivity.**) *If X is a preordered set with binary meets and joins and with Heyting implications, then for all $x,y,z \in X$*

$$(y \vee z) \wedge x \cong (y \wedge x) \vee (z \wedge y) \qquad \text{('binary meet distributes over binary joins') and}$$
$$(y \wedge z) \vee x \cong (y \vee x) \wedge (z \vee x) \qquad \text{('binary join distributes over binary meets').}$$

(*So in particular, every Heyting algebra is a* distributive lattice*—which by definition is a poset with all finite meets and joins in which binary meets and joins distribute over each other.*)

Proof. The first isomorphism has a 'categorical' proof in terms of properties of adjoints: the monotone function $(-)\wedge x\colon X \longrightarrow X$ has a right adjoint (*viz.* $x \Rightarrow (-)$) and hence by 3.7(i) preserves any joins which exist, and binary ones in particular—which is just what the first isomorphism says.

The second isomorphism is true for order-theoretic rather than categorical reasons. First note the general property: $u \cong u \wedge v$ iff $u \leq v$ iff $u \vee v \cong v$. Then using the isomorphism we have already established, we have

$$\begin{aligned} (y \vee x) \wedge (z \vee x) &\cong (y \wedge (z \vee x)) \vee (x \wedge (z \vee x)) \\ &\cong (y \wedge (z \vee x)) \vee x && \text{since } x \leq z \vee x \\ &\cong ((y \wedge z) \vee (y \wedge x)) \vee x \\ &\cong (y \wedge z) \vee ((y \wedge x) \vee x) \\ &\cong (y \wedge z) \vee x && \text{since } y \wedge x \leq x \end{aligned}$$

as required. □

3.12 Definitions. In a preordered set X with finite meets and joins a *complement* for $x \in X$ is an element c satisfying $x \wedge c = \bot$ and $x \vee c = \top$.

(**Exercise.** In a distributive lattice, the complement of x is unique if it exists.)

A *Boolean algebra* is a distibutive lattice in which every element has a complement. A morphism of Boolean algebras is a monotone function preserving finite meets and joins (and hence also complements, since if $f: X \longrightarrow Y$ preserves finite meets and joins, then $f(c)$ is a complement for $f(x)$ in Y when c is a complement for x in X.) The category of Boolean algebras and their morphisms will be denoted **Ba**.

3.13 Remark. In a Heyting preordered set X, the *pseudocomplement* $\neg x$ of an element x is by definition $x \Rightarrow \bot$. Thus $\neg x$ is defined uniquely up to isomorphism by the property

$$\text{for all } y \in X,\ y \leq \neg x \quad \text{if and only if} \quad y \wedge x \leq \bot .$$

3.14 Exercises. (i) In a Heyting preordered set H show that $\neg x$ is actually a complement for x iff $\neg x \vee x \cong \top$ iff $\neg\neg x \cong x$. Hence X is a Boolean algebra iff it is a Heyting algebra satisfying $\forall x \in X(\neg\neg x = x)$.

(ii) If B is a Boolean algebra, show that pseudocomplements (exist and) are complements; furthermore the implication $x \Rightarrow y$ is given by $\neg x \vee y$. Thus each Boolean algebra is a Heyting algebra and each Boolean algebra morphism is a morphism for the Heyting structure.

(iii) Suppose that X is a preordered set with all joins, and hence also all meets (*cf.* 1.9). Show that it is Heyting iff binary meet distributes over *infinite* joins, that is, for all $x \in X$ and $S \subseteq X$

$$x \wedge \bigvee S \cong \bigvee\{x \wedge s \mid s \in S\}.$$

A poset with this property is called a *complete Heyting algebra*, a *frame*, or a *locale* (depending upon what kind of morphisms are being considered).

3.15 Examples. (i) Ⓣ If X is a set, $\mathbf{P}(X)$ is a Boolean algebra. If $f: X \longrightarrow Y$ is a function then $f^{-1}: \mathbf{P}(Y) \longrightarrow \mathbf{P}(X)$ is a Boolean algebra morphism (since it has both adjoints and hence preserves all meets and joins).

(ii) Ⓛ If **C** is a category with pullbacks and universal quantification of subobjects along morphisms (see 3.8(iv)), then in fact each Sub(X) has Heyting implication. For, given $A \overset{a}{\rightarrowtail} X$ and $B \rightarrowtail X$, the implication $(A \Rightarrow B) \rightarrowtail X$ in Sub(X) is given by $\forall a(a^{-1}(B)) \rightarrowtail X$. This is because for any $C \rightarrowtail X$, $C \wedge A \rightarrowtail X$ is given by a pullback square

$$\begin{array}{ccc} C \wedge A & \rightarrowtail & C \\ \downarrow & & \downarrow \\ A & \overset{a}{\rightarrowtail} & X \end{array}$$

so that (using the second Exercise in 3.8(iv)) $C \wedge A \in \text{Sub}(X)$ is $\exists a(a^{-1}(C))$. Hence

$$\begin{aligned} C \wedge A = \exists a(a^{-1}(C)) \leq B \text{ in Sub}(X) &\text{ iff } a^{-1}(C) \leq a^{-1}(B) \text{ in Sub}(A), &&\text{since } \exists a \dashv a^{-1}, \\ &\text{ iff } C \leq \forall a(a^{-1}(B)) \text{ in Sub}(X), &&\text{since } a^{-1} \dashv \forall a, \end{aligned}$$

so that $\forall a(a^{-1}(B))$ has the defining property of $(A \Rightarrow B)$.

(iii) Ⓢ If D is a dcpo then $\mathbf{F}(D)$ is a distributive lattice (because finite meets and joins are calculated as in $\mathbf{P}(D)$, by intersections and unions respectively) but not in general a Heyting algebra. (However, $\mathbf{F}(D)^{\mathrm{op}}$ is a complete Heyting algebra by Exercise 3.14(iii), since infinite joins in $\mathbf{F}(D)^{\mathrm{op}}$ are given as in $\mathbf{P}(D)^{\mathrm{op}}$, by intersections.)

(iv) Ⓚ Each $\mathbf{R}(X)$ is Heyting. We have already seen that it has finite meets and joins, so we have to establish the existence of Heyting implications. Let $\Rightarrow : \mathbf{PN}\times\mathbf{PN} \longrightarrow \mathbf{PN}$ be the function sending $A \subseteq \mathrm{N}$ and $B \subseteq \mathrm{N}$ to

$$A \Rightarrow B = \{n \in \mathrm{N} \mid \forall m \in A,\ n\cdot m \text{ is defined and a member of } B\}$$

and for $p,q \in (X \to \mathbf{PN})$, define $p \Rightarrow q \in (X \to \mathbf{PN})$ to be the function $x \longmapsto p(x) \Rightarrow q(x)$.

Then for any $r \in (X \to \mathbf{PN})$, if $\varphi : r \wedge p \leq q$ then $r \leq p \Rightarrow q$ is witnessed by the partial recursive function $\lambda n.\mathsf{S}(\mathsf{K}\ulcorner\varphi\urcorner)(\mathsf{P}n)$ (for which $\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{K}\ulcorner\varphi\urcorner)))\mathsf{P}$ is a code), where $\ulcorner\varphi\urcorner$ is a code for φ and P is a code for the pairing function in the sense that for all $n,m \in \mathrm{N}$, $\mathsf{P}nm \asymp \langle n,m\rangle$. Conversely, if $\psi : r \leq p \Rightarrow q$, then $r \wedge p \leq q$ is witnessed by the partial recursive function $\lambda n.\psi(\mathsf{P}_0 n)(\mathsf{P}_1 n)$ (for which $\mathsf{S}(\mathsf{S}(\mathsf{K}\ulcorner\psi\urcorner)\mathsf{P}_0)\mathsf{P}_1$ is a code, when $\ulcorner\psi\urcorner$ is a code for ψ), where P_0 and P_1 are codes for the projection functions $(-)_0,(-)_1 : \mathrm{N} \longrightarrow \mathrm{N}$. Thus $p \Rightarrow q \in \mathbf{R}(X)$ does indeed have the property required of an implication.

Note also from the way $\top,\wedge,\bot,\vee$ and $\Rightarrow$ are defined for $\mathbf{R}(X)$ that each $f^* : \mathbf{R}(Y) \longrightarrow \mathbf{R}(X)$ preserves the operations up to equality and is in particular a morphism of Heyting preordered sets.

Using $\Rightarrow$ we can restate the definition of $\forall f : \mathbf{R}(X) \longrightarrow \mathbf{R}(Y)$ from 3.8(vi) in a suggestive way:

$$\forall f(p)(y) = \bigcap_{x \in X} \big(\delta_Y(f(x),y) \Rightarrow p(x)\big),$$

where $\delta_Y \in (Y \times Y \to \mathbf{PN})$ is the 'indicator' function

$$\delta_Y(y,y') = \begin{cases} \mathrm{N} & \text{if } y = y' \\ \emptyset & \text{if } y \neq y'\ . \end{cases}$$

And we could have defined $\exists f$ analogously as

$$\exists f(p)(y) = \bigcup_{x \in X} \big(\delta_Y(f(x),y) \wedge p(x)\big),$$

but this is in fact isomorphic to the definition we gave in 3.8(vi). (Compare these formulas with those for $\exists f, \forall f : \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ in 3.8(iii).)

4 Propositional Theories

This section establishes the close relationship between preordered sets and theories in propositional logics.

4.1 Definitions. Starting with a set A, whose elements we will call *atomic propositions,* the *first-order propositions* ϕ over A are given by the following grammar

$$\phi ::= p \mid \top \mid \phi\wedge\phi \mid \bot \mid \phi\vee\phi \mid \phi\Rightarrow\phi \mid \neg\phi$$

where p runs over A. (There is a degree of redundancy in this definition: without loss of expressiveness, we could *define* $\neg\phi$ to be $\phi\Rightarrow\bot$ and $\top$ to be $\neg\bot$. However, it useful to give the definition as we have since later we will consider 'fragments' not containing $\Rightarrow$.)

If H is a Heyting preordered set (see 3.9), a *structure* M for a set A of atomic propositions is just a function $M: A\longrightarrow H$. The *denotation* of a first-order proposition ϕ over A in the structure M is an element $[\![\phi]\!]_M \in H$, defined inductively by the following clauses:

- if $p\in A$, then $[\![p]\!]_M = M(p)$;
- $[\![\top]\!]_M = \top$, the top element in H;
- $[\![\phi\wedge\psi]\!]_M = [\![\phi]\!]_M \wedge [\![\psi]\!]_M$, binary meet in H;
- $[\![\bot]\!]_M = \bot$, the bottom element in H;
- $[\![\phi\vee\psi]\!]_M = [\![\phi]\!]_M \vee [\![\psi]\!]_M$, binary join in H;
- $[\![\phi\Rightarrow\psi]\!]_M = [\![\phi]\!]_M \Rightarrow [\![\psi]\!]_M$, Heyting implication in H;
- $[\![\neg\phi]\!]_M = \neg[\![\phi]\!]_M$, pseudocomplement in H.

4.2 Examples. (i) Ⓣ $\mathbf{P}(1)=\{\emptyset,1\}$ with $\emptyset\le\emptyset\le 1\le 1$ is a (Boolean, hence a) Heyting algebra. The above semantics of propositions in $\mathbf{P}(1)$ coincides with the classical 2-valued semantics.

(ii) Ⓚ $\mathbf{R}(X)$ is Heyting. The semantics of propositions given in 4.1 when restricted to such Heyting preordered sets amounts to Kleene's '1945-realizability' explanation of the propositional connectives (see [Dum, 6.2]). Writing '$n \mathrel{\mathbf{r}} \phi$' instead of '$n\in[\![\phi]\!]_M$', we have:

- $n \mathrel{\mathbf{r}} p$ iff $n\in M(p)$;
- $n \mathrel{\mathbf{r}} \top$ for all $n\in\mathrm{N}$;
- $n \mathrel{\mathbf{r}} \phi\wedge\psi$ iff $(n)_0 \mathrel{\mathbf{r}} \phi$ and $(n)_1 \mathrel{\mathbf{r}} \psi$;
- $n \mathrel{\mathbf{r}} \bot$ for no $n\in\mathrm{N}$;
- $n \mathrel{\mathbf{r}} \phi\vee\psi$ iff $\big((n)_0=0$ and $(n)_1 \mathrel{\mathbf{r}} \phi\big)$ or $\big((n)_0=1$ and $(n)_1 \mathrel{\mathbf{r}} \psi\big)$;
- $n \mathrel{\mathbf{r}} \phi\Rightarrow\psi$ iff $\forall m\in\mathrm{N}\big(m \mathrel{\mathbf{r}} \phi$ implies $n\cdot m$ is defined and $n\cdot m \mathrel{\mathbf{r}} \psi\big)$;
- $n \mathrel{\mathbf{r}} \neg\phi$ iff $\forall m\in\mathrm{N}\big(m \mathrel{\mathbf{r}} \phi$ implies $n\cdot m$ is not defined$\big)$.

(iii) If X is a poset, recall that a subset $U\subseteq X$ is *downwards closed* if for all $x,y\in X$, $x\le y\in U$ implies $x\in U$. The set $\mathrm{D}(X)$ of all such subsets, partially ordered by subset inclusion, is a (complete) Heyting algebra. (**Exercise.** Show that $D(X)$ is isomorphic to $[X,2]$, where $2=\mathbf{P}(1)$ is as in (i).) Indeed meets and joins in $D(X)$ are given by set intersection and union respectively, whilst the Heyting implication of $U,V\in D(X)$ is $U\Rightarrow V=\{x\in X \mid \forall y\leq x(y\in U \text{ implies } y\in V)\}$. (**Exercise.** Prove this.)

When restricted to Heyting algebras of the form $D(X)$, the semantics of propositions given in 4.1 coincides with the explanation of the propositional connectives furnished by Kripke's 'forcing semantics'. Writing '$x \Vdash \phi$' instead of '$x\in[\![\phi]\!]_M$', we have:

- $x \Vdash p$ iff $n\in M(p)$;
- $x \Vdash \top$ for all $x\in X$;
- $x \Vdash \phi\wedge\psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$;
- $x \Vdash \bot$ for no $x\in X$;
- $x \Vdash \phi\vee\psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$;
- $x \Vdash \phi\Rightarrow\psi$ iff $\forall y\leq x(y \Vdash \phi$ implies $y \Vdash \psi)$;
- $x \Vdash \neg\phi$ iff there is no $y\leq x$ with $y \Vdash \phi$.

(Note that (i) is the special case of (iii) with $X=1$, since $D(1)=\mathbf{P}(1)$. On the other hand (iii) is a special case of the 'topological' semantics, which is based upon the complete Heyting algebra of open subsets (ordered by inclusion) of a topological space.)

We turn our attention now to calculi for deriving logical entailments between propositions in terms of 'sequents':

4.2 Definitions. A first-order *sequent* $\Gamma\vdash\phi$ over a set A of atomic propositions is a pair consisting of a finite set Γ of first-order propositions over A and a single first-order proposition ϕ over A. The notation $\Gamma,\phi\vdash\psi$ is used for $\Gamma\cup\{\phi\}\vdash\psi$; similarly $\Gamma,\Delta\vdash\psi$ means $\Gamma\cup\Delta\vdash\psi$, and $\phi\vdash\psi$ means $\{\phi\}\vdash\psi$.

A (*first-order propositional*) *theory* **T** is specified by a set of atomic propositions A and a set of sequents over A, which are called the *axioms* of **T**.

The intended meaning of the sequent $\Gamma\vdash\phi$ is that the propositions in Γ together logically entail the proposition ϕ. Rules for deriving sequents are given in Table 4.3. These rules constitute the Gentzen sequent calculus for *Intuitionistic Propositional Logic*, **IpC**. Each of the rules is of the form

$$\frac{\text{hypotheses}}{\text{conclusion}}$$

where 'hypotheses' is a finite (possibly empty) set of sequents and 'conclusion' is a sequent. A set of sequents is *closed* under the rules if whenever the hypotheses of a rule are contained in the set then the conclusion is an element of the set.

$$(\mathrm{Id})\frac{}{\phi \vdash \phi} \qquad (\mathrm{Cut})\frac{\Gamma \vdash \phi \quad \Delta, \phi \vdash \psi}{\Delta, \Gamma \vdash \psi}$$

$$(\mathrm{LWk})\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi} \qquad (\mathrm{RWk})\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}$$

$$(\mathrm{RT})\frac{}{\emptyset \vdash \top}$$

$$(\mathrm{L}\wedge)\frac{\Gamma, \phi, \psi \vdash \theta}{\Gamma, \phi \wedge \psi \vdash \theta} \qquad (\mathrm{R}\wedge)\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \wedge \psi}$$

$$(\mathrm{L}\bot)\frac{}{\bot \vdash \phi}$$

$$(\mathrm{L}\vee)\frac{\Gamma, \phi \vdash \theta \quad \Delta, \psi \vdash \theta}{\Gamma, \Delta, \phi \vee \psi \vdash \theta} \qquad (\mathrm{R}_1\vee)\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \qquad (\mathrm{R}_2\vee)\frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}$$

$$(\mathrm{L}\Rightarrow)\frac{\Gamma \vdash \phi \quad \Delta, \psi \vdash \theta}{\Gamma, \Delta, \phi \Rightarrow \psi \vdash \theta} \qquad (\mathrm{R}\Rightarrow)\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}$$

$$(\mathrm{L}\neg)\frac{\Gamma \vdash \phi}{\Gamma, \neg\phi \vdash \bot} \qquad (\mathrm{R}\neg)\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi}$$

4.3 Table: Gentzen Sequent Calculus for IpC

4.4 Definition. If **T** is a propositional theory, the *intuitionistic theorems* of **T** comprise the least set of sequents (over A) which contains the axioms of **T** and is closed under the rules in Table 4.3.

4.5 Definition. A structure in a Heyting preordered set M for a set of atomic propositions A, *satisfies* a sequent $\Gamma \vdash \phi$ over A if

$$\textstyle\bigwedge_{\gamma \in \Gamma} [\![\gamma]\!]_M \leq [\![\phi]\!]_M \quad \text{in } H.$$

(N.B. the right-hand side is a *finite* meet.)

If **T** is a propositional theory, then a structure M for the underlying set of atomic propositions of **T** is called a *model* of **T** if M satisfies all the axioms of **T**. The set of models of **T** in H will be denoted $\mathrm{Mod}(\mathbf{T}, H)$.

4.6 Remark. Call two structures M, N for A in H *isomorphic* and write $M \cong N$, if for all $p \in A$ $M(p) \cong N(p)$ in H. This gives an equivalence relation on $\mathrm{Mod}(\mathbf{T}, H)$. By induction on the structure of a proposition ϕ over A we have $[\![\phi]\!]_M \cong [\![\phi]\!]_N$ whenever $M \cong N$. Consequently the quotient function $q: H \longrightarrow H/\cong$ from H onto its poset reflection induces a function $\mathrm{Mod}(\mathbf{T}, H) \longrightarrow \mathrm{Mod}(\mathbf{T}, H/\cong)$ via composition with q, and this factors to give a bijection between $\mathrm{Mod}(\mathbf{T}, H)/\cong$ and $\mathrm{Mod}(\mathbf{T}, H/\cong)$.

(But note that because of the 'contavariant' properties of $(-) \Rightarrow \phi$ and $\neg(-)$, if we define $M \leq N$ to mean $\forall p \in A(M(p) \leq N(p))$, then we cannot conclude that $[\![\phi]\!]_M \leq [\![\phi]\!]_N$ for propositions involving $\Rightarrow$ or $\neg$.)

4.7 Proposition (Soundness). *If M is a model of a propositional theory* **T** *in a Heyting algebra H, then M satisfies any sequent which is an intuitionistic theorem of* **T**.

Proof. One has to show for each of the rules in Table 4.3 that if M satisfies the sequents in the hypothesis of the rule, it also satisfies the conclusion of the rule. But via the definition of satisfaction in 4.5, verifying this property for each rule involves straightforward calculations with the Heyting structure of H. For example rule (L$\Rightarrow$) corresponds to the statement

$$\text{if } u \leq x \text{ and } v \wedge y \leq z, \text{ then } u \wedge v \wedge (x \Rightarrow y) \leq z$$

which holds because $x \wedge (x \Rightarrow y) \leq y$ is always true (since $(x \Rightarrow y) \leq (x \Rightarrow y)$), so that the hypotheses imply $u \wedge (x \Rightarrow y) \leq x \wedge (x \Rightarrow y) \leq y$ and hence $u \wedge v \wedge (x \Rightarrow y) \leq v \wedge y \leq z$. □

The proof of Proposition 4.7 suggests that we could give an alternative formulation of the rules of inference for **IpC**—one which corresponds more closely to the semantics given to the connectives in 4.1. As we have seen in the previous section, this semantics uses operations on preordered sets which are built up in terms of a series of adjoint monotone functions. So we will call the corresponding logical system an 'Adjoint Calculus' for **IpC**: its rules are given in Table 4.8. The rules for the propositional operators take the form

$$\begin{array}{c}\text{sequents}\\ \hline\hline \text{sequent}\end{array}\ .$$

A set of sequents is closed under such a 'bi-rule' if the set contains the sequents above the double line if and only if it contains the sequent below the double line. (The rule is thus an abbreviation for several 'uni-directional' rules.)

$$(\mathrm{Id})\ \frac{}{\Gamma, \phi \vdash \phi} \qquad\qquad (\mathrm{Cut})\ \frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Delta, \Gamma \vdash \psi}$$

$$(\top)\ \begin{array}{c}\\ \hline\hline \Gamma \vdash \top\end{array} \qquad\qquad (\wedge)\ \begin{array}{c}\Gamma \vdash \phi \qquad \Gamma \vdash \psi\\ \hline\hline \Gamma \vdash \phi \wedge \psi\end{array}$$

$$(\bot)\ \begin{array}{c}\\ \hline\hline \Gamma, \bot \vdash \phi\end{array} \qquad\qquad (\vee)\ \begin{array}{c}\Gamma, \phi \vdash \theta \qquad \Gamma, \psi \vdash \theta\\ \hline\hline \Gamma, \phi \vee \psi \vdash \theta\end{array}$$

$$(\Rightarrow)\ \begin{array}{c}\Gamma, \phi \vdash \psi\\ \hline\hline \Gamma \vdash \phi \Rightarrow \psi\end{array} \qquad\qquad (\neg)\ \begin{array}{c}\Gamma, \phi \vdash \bot\\ \hline\hline \Gamma \vdash \neg\phi\end{array}$$

4.8 Table: Adjoint Calculus for IpC

Note that the presence of the extra propositions Γ in ($\vee$) mean that the character of disjunction is captured not just by a left adjunction, but one which has a certain 'stability' property, *viz.* distributivity as in 3.11; in the presence of ($\Rightarrow$) this stability is automatic (*i.e.* we could have given the rule without the extra propositions Γ). ($\perp$) is also 'stable' left adjunction rule, but in this case the stability gives rise to no extra condition.

The proof of Proposition 4.7 amounts to saying that the rules in Table 4.3 are derived rules of Table 4.8. Conversely, it is not hard to prove that the (uni-directional versions of) the rules in 4.8 are derived rules of Table 4.3. In this sense, the two tables give equivalent formulations of *what can be proved* in **IpC**. However, *proofs* themselves in the two formulations are very different, and in fact the Gentzen Calculus has much better proof-theoretic properties than the Adjoint Calculus. (For example, Gentzen's Cut Elimination Theorem (see [Dum, 4.3]) says that proofs using the rules in 4.3 can be transformed into equivalent ones not making use of the (Cut) rule; but I don't think the same is true of the rules in 4.8.)

The Adjoint Calculus formulation of **IpC** leads directly to the construction of Heyting preorders from propositional theories:

4.9 Construction: Classifying Preorder. Let **T** be a propositional theory over some set A of atomic propositions. Define a relation $\leq_{\mathbf{T}}$ on the set of first-order propositions over A by

$$\phi \leq_{\mathbf{T}} \psi \quad \text{if and only if} \quad \phi \vdash \psi \text{ is an intuitionistic theorem of } \mathbf{T} \ (cf.\ 4.4)$$

From the remarks above, we have that $\phi \leq_{\mathbf{T}} \psi$ holds if and only if $\phi \vdash \psi$ is in the closure of the set of axioms of **T** under the rules in Table 4.8. The form of those rules gives immediately that $\leq_{\mathbf{T}}$ makes the set of propositions into a Heyting preordered set, with the operations of meet, join and implication given by the corresponding propositional operations. We will denote this preordered set by Cl(**T**) and call it the *classifying preordered set* of the propositional theory **T**.

Since each $p \in A$ is a proposition, we get a structure for A in Cl(**T**) by sending p to itself. Definition 4.1 applied to this structure just gives $[\![\phi]\!] = \phi \in \mathrm{Cl}(\mathbf{T})$, for each proposition ϕ over A. Consequently the structure satisfies exactly those sequents which are intuitionistic theorems of **T**. In particular, the structure is a model of **T**—we will denote it by $G_{\mathbf{T}}$ and call it the *generic model of* **T** (for reasons which will become apparent below). This gives an easy converse to Proposition 4.7:

4.10 Corollary (Completeness). *The intuitionistic theorems of* **T** *are just those sequents which are satisfied in every model of* **T** *in Heyting preordered sets.*

(**Remark.** Sharper theorems result from restricting the class of Heyting preorders considered. For example, Kripke's Completeness Theorem says that the theorems of **T** are just those sequents satisfied by all models of **T** in the (proper) class of Heyting preorders of Example 4.2(iii). Unlike the situation for classical logic, one can prove (hard **Exercise**!) that there is no *single* Heyting preorder (in fact, no *set* of them) complete for the intuitionistic theorems of all propositional theories.)

The generic model of **T** in its classyfying preorder enjoys an important 'universal property' with respect to models of **T** in Heyting preorders. In order to state the property, we have to consider the *transport* of models along monotone functions. Recall that **Heyt** denotes the category of Heyting preordered sets and morphisms of such (see 3.9). Given $f: H \longrightarrow K$ in **Heyt** and $M \in \mathrm{Mod}(\mathbf{T}, H)$, we get a structure $f_*(M)$ for the atomic propositions in K by composing with f: $f_*(M)(p) = f(M(p))$. It is easy to prove by induction on the structure of a proposition ϕ that

$$f(\llbracket\phi\rrbracket_M) \cong \llbracket\phi\rrbracket_{f_*(M)}$$

(since f preserves up to isomorphism all the operations involved in the definition of $\llbracket - \rrbracket$ in 4.1). So if M satisfies $\Gamma \vdash \phi$, then

$$\textstyle\bigwedge_{\gamma\in\Gamma} \llbracket\gamma\rrbracket_{f_*(M)} \cong f\big(\bigwedge_{\gamma\in\Gamma} \llbracket\gamma\rrbracket_M\big) \leq f(\llbracket\phi\rrbracket_M) \cong \llbracket\phi\rrbracket_{f_*(M)}$$

so that $f_*(M)$ also satisfies $\Gamma \vdash \phi$. In particular, $f_*(M) \in \mathrm{Mod}(\mathbf{T}, K)$. Thus each morphism $f: H \longrightarrow K$ in **Heyt** gives rise to a function

$$f_*: \mathrm{Mod}(\mathbf{T}, H) \longrightarrow \mathrm{Mod}(\mathbf{T}, K).$$

Note that this function respects the equivalence relation of 4.6: if $M \cong N$, then $f_*(M) \cong f_*(N)$.

4.11 Theorem. *Let* $\mathbf{T}$ *be a propositional theory and* H *a Heyting preordered set. Each model* M *of* $\mathbf{T}$ *in* H *can be obtained by transporting the generic model of* $\mathbf{T}$ *along a morphism* $m: \mathrm{Cl}(\mathbf{T}) \longrightarrow H$ *in* **Heyt**, *that is,* $M = m_*(G_{\mathbf{T}})$. (*We say that* M *is 'classified' by* m.) *Moreover, such an* m *is determined uniquely up to isomorpism by* M. *In particular, if* H *is a Heyting algebra, the function* $m \longmapsto m_*(G_{\mathbf{T}})$ *induces a bijection*

$$\mathbf{Heyt}(\mathrm{Cl}(\mathbf{T}), H) \cong \mathrm{Mod}(\mathbf{T}, H)$$

(where the set on the left-hand side is the set of all morphisms from $\mathrm{Cl}(\mathbf{T})$ to H in **Heyt**.)

Proof. Given M, define $m: \mathrm{Cl}(\mathbf{T}) \longrightarrow H$ by $m(\phi) = \llbracket\phi\rrbracket_M$. Then m is monotone since if $\phi \leq_{\mathbf{T}} \psi$ in $\mathrm{Cl}(\mathbf{T})$, then $\phi \vdash \psi$ is a theorem of $\mathbf{T}$, so that by Proposition 4.7 M satisfies $\phi \vdash \psi$, which is to say that $m(\phi) = \llbracket\phi\rrbracket_M \leq \llbracket\psi\rrbracket_M = m(\psi)$. Furthermore, the clauses defining $\llbracket - \rrbracket$ in 4.1 imply that m preserves the Heyting structure of $\mathrm{Cl}(\mathbf{T})$. Finally, for each atomic proposition p, we have $m_*(G_{\mathbf{T}})(p) = m(p) = \llbracket p\rrbracket_M = M(p)$, so that $m_*(G_{\mathbf{T}}) = M$.

If $f: \mathrm{Cl}(\mathbf{T}) \longrightarrow H$ in **Heyt** also satisfies $M = f_*(G_{\mathbf{T}})$ (or indeed just $M \cong f_*(G_{\mathbf{T}})$), then $f(\phi) \cong \llbracket\phi\rrbracket_M$ can be proved by induction on the structure of ϕ starting with $f(p) = M(p)$ for each atomic proposition. Hence $f \cong m$. □

(**Exercise.** Show that $\mathrm{Cl}(\mathbf{T})$ is determined uniquely up to equivalence of preordered sets (*cf.* 2.5) and $G_{\mathbf{T}}$ uniquely up to isomorphism of $\mathbf{T}$-models by the property given in Theorem 4.11.)

4.12 Remark. We could restrict our attention entirely to Heyting *algebras* by taking the poset reflection of $\mathrm{Cl}(\mathbf{T})$ to obtain a Heyting algebra $\mathrm{H}(\mathbf{T}) = \mathrm{Cl}(\mathbf{T})/\cong$. Since for any Heyting algebra H we have $\mathbf{Heyt}(\mathrm{Cl}(\mathbf{T}), H) \cong \mathbf{Heyt}(\mathrm{Cl}(\mathbf{T})/\cong, H)$, the bijection in 4.11 becomes

$$\mathbf{Heyt}(\mathrm{H}(\mathbf{T}), H) \cong \mathrm{Mod}(\mathbf{T}, H),$$

that is, models of **T** in a Heyting algebra H are in bijection with Heyting algebra morphisms $\mathrm{H}(\mathbf{T}) \longrightarrow H$.

4.13 Proposition. *Every Heyting preordered set is equivalent to* $\mathrm{Cl}(\mathbf{T})$ *for some propositional theory* **T**.

Proof. If H is a Heyting preordered set, take a set of atomic propositions $A = \{\ulcorner h \urcorner \mid h \in H\}$ which is in bijection with H and consider the structure M with $M(\ulcorner h \urcorner) = h$. Let **T** be the theory over A whose axioms are precisely those sequents which are satisfied by M. Then M is automatically a model of **T** and so by Theorem 4.11 there is some Heyting morphism $m: \mathrm{Cl}(\mathbf{T}) \longrightarrow H$ with $m_*(G_{\mathbf{T}}) = M$; indeed from the proof of the theorem, we may take $m(\phi) = [\![\phi]\!]_M$, for each $\phi \in \mathrm{Cl}(\mathbf{T})$.

Then for each $h \in H$, $m(\ulcorner h \urcorner) = h$, so that m is surjective. Moreover, given $\phi, \psi \in \mathrm{Cl}(\mathbf{T})$, if $[\![\phi]\!]_M = m(\phi) \leq m(\psi) = [\![\psi]\!]_M$, then M satisfies $\phi \vdash \psi$, which is therefore an axiom of **T** by definition, and thus $\phi \leq_{\mathbf{T}} \psi$ in $\mathrm{Cl}(\mathbf{T})$. So m is both full (see 2.5) and surjective. Hence as in Remark 2.6(ii), it is an equivalence. Thus $H \simeq \mathrm{Cl}(\mathbf{T})$, as required. □

4.14 Remark: "categories are theories". Writing $\mathbf{T}_H$ for the propositional theory constructed from H in 4.13, we have functions in each direction between propostional theories and Heyting preordered sets:

$$\mathbf{T} \longmapsto \mathrm{Cl}(\mathbf{T})$$
$$\mathbf{T}_H \longleftarrow\!\!\dashv \; H$$

These functions are essentially inverse to each other in the following sense. We already have $\mathrm{Cl}(\mathbf{T}_H) \simeq H$; and conversely, **T** and $\mathbf{T}_{\mathrm{Cl}(\mathbf{T})}$ are 'equivalent' theories. *Equivalence* for theories **T** and **T′** means that there are 'interpretations' $I: \mathbf{T} \longrightarrow \mathbf{T}'$ and $J: \mathbf{T}' \longrightarrow \mathbf{T}$ with $J \circ I \cong \mathrm{Id}_{\mathbf{T}}$ and $I \circ J \cong \mathrm{Id}_{\mathbf{T}'}$; and an *interpretation* of **T** in **T′** is an assignment of propositions in **T′** to the atomic propositions of **T** in such a way that the axioms of **T** become theorems of **T′**. Thus giving an interpretation of **T** in **T′** amounts to giving a model of **T** in the Heyting preorder $\mathrm{Cl}(\mathbf{T}')$, and hence to giving a Heyting morphism $\mathrm{Cl}(\mathbf{T}) \longrightarrow \mathrm{Cl}(\mathbf{T}')$.

These facts establish a correspondence between propositional theories in **IpC** and Heyting preordered sets. Under this correspondence, models of a theory (and in particular, interpretations between theories) are identified with Heyting morphisms out of the classifying preorder of the theory. Taking poset reflections and working with Heyting algebras, we can sum the situation up as an equivalence of categories (see Glossary):

The category of propositional theories and interpretations in **IpC** *is equivalent to the category of Heyting algebras.*

This is a simple (because only propositional logic is involved) example of Lawvere's dictum that "categories are theories and models are functors". In this case the categories and functors involved are rather special, being just preorders and monotone functions (*cf.* 1.11 and 2.7), and the correspondence of theories with ordered structures goes back to Lindenbaum and Tarski, who observed the correspondence between theories in *classical* propositional logic and *Boolean* algebras. We will see more complicated examples of such logic-category correspondences later.

5 Example Ⓢ: theories as domains

This whole section is an extended example illustrating the use of the kind of correspondence between propositional theories and preordered sets which we saw in the last section. We look at consistent theories in the fragment of first-order propositional logic only involving $\top,\wedge$ and $\bot$. One the one hand, they correspond (via the classifying construction) to non-trivial, finitely complete posets with bottom; on the other hand, they correspond (via their posets of models) to Scott domains. This is the basis of a method for describing Scott domains, continuous functions and embeddings which is similar to the 'information system' approach (see [LW] for example), but with a more 'logical' flavour: see [Ab].

5.1 Definition. A *pointed meet-semilattice* is a poset with finite meets and a bottom element. An equivalent, algebraic description is that it is a set A equipped with constants $\top,\bot\in A$ and a binary operation $\wedge: A\times A\longrightarrow A$ satisfying the equations:

$$
\begin{aligned}
x\wedge y &= y\wedge x\\
(x\wedge y)\wedge z &= x\wedge(y\wedge z)\\
x\wedge\top &= x = x\wedge x\\
x\wedge\bot &= \bot\,.
\end{aligned}
$$

For the order-theoretic operations certainly obey these equations in a poset; and given such an algebraic structure, defining $x\leq y$ to mean $x = x\wedge y$, we get a partial order for which $\top$ is top element, $\bot$ is bottom and $\wedge$ is binary meet.

A *morphism* of pointed meet-semilattices is a monotone function which preserves finite meets and the bottom element (equivalently, which is a homomorphism for the above algebraic structure).

A pointed meet-semilattice A is *non-trivial* if it contains more than one element, which is to say $\bot\neq\top$ in A.

We can repeat the development of section 4, but restricting our attention to propositions not involving $\vee$, $\Rightarrow$, or $\neg$. This leaves the *non-strict Horn propositions*:

$$\phi ::= p \mid \top \mid \phi\wedge\phi \mid \bot\,.$$

A pointed meet-semilattice A has just the right structure to interpret such propositions as in 4.1, once we have a strucure M assigning elements $M(p)\in A$ to the atomic propositions p. A *non-strict propositional Horn theory* **T** is a propositional theory whose axioms are sequents only involving non-strict Horn propositions. As before, M is a *model* of **T** if it satisfies all the axioms of **T**. (N.B. the notion of satisfaction given in 4.5 only involves finite meets and so still makes sense in this restricted setting.) Mod(**T**,A) denotes the set of models of **T** in A. The *theorems* of **T** are those sequents derived using the relevant rules in Tables 4.3 or 4.8 (i.e. not those for $\vee$, $\Rightarrow$, or $\neg$).

5.2 Remark. It is not hard to see that we can change the set of axioms of a non-strict Horn theory without changing its collection of theorems, to obtain an axiomatization of the theory by a collection of axioms each of which is either of the form

$$p_1 \wedge \cdots \wedge p_n \vdash q$$

or of the form

$$p_1 \wedge \cdots \wedge p_n \vdash \bot$$

where the p_i and q are atomic propositions and $n \geq 0$. The first form is a 'strict Horn clause' and the second a 'non-strict Horn clause'.

5.3 Construction. A *classifying* pointed meet-semilattice A(T) can be constructed for each non-strict propositional Horn theory **T** just as in 4.9 and 4.12: preorder the non-strict Horn propositions by defining $\phi \leq_{\mathbf{T}} \psi$ to mean that $\phi \vdash \psi$ is a theorem of **T**, and then take the poset relection to obtain A(T). Thus the elements of A(T) are equivalence classes of non-strict Horn propositions (over the atomic propositions of T) under the equivalence relation: $\phi \cong_{\mathbf{T}} \psi$ iff $\phi \vdash \psi$ and $\psi \vdash \phi$ are theorems of **T**.

Then for any pointed meet-semilattice A, there is a bijection between the set of monotone functions $\mathrm{A(T)} \longrightarrow A$ preserving finite meets and $\bot$, and the set of models of **T** in A. (And this bijection is induced by transporting the generic model of **T** along the monotone function.)

Similarly, the construction of 4.13 when restricted to non-strict Horn propositions gives rise, for each pointed meet-semilattice A, to a non-strict propositional Horn theory $\mathbf{T}_A$ with $\mathrm{A}(\mathbf{T}_A) \cong A$.

In this way the correspondence of 4.14 restricts to one between pointed meet-semilattices and non-strict propositional Horn theories. Under this correspondence, the property given in 5.1 of A being 'non-trivial' means for A(T) that $\top \not\leq_{\mathbf{T}} \bot$, which is to say that $\top \vdash \bot$ is not a theorem of **T**.

5.4 Definition. A non-strict Horn theory **T** is *consistent* if $\top \vdash \bot$ is not a theorem of **T**.

5.5 Definitions. (i) The category $\mathbf{M}_\bot$ has for objects all non-trivial, pointed meet-semilattices and for morphisms monotone functions preserving finite meets and bottom element.

(ii) Given $A, B \in \mathrm{ob}\mathbf{M}_\bot$, the set $\mathbf{M}_\bot(A,B)$ of morphisms from A to B inherits a partial ordering from $[A,B]$: $f \leq g$ if and only if for all $a \in A$, $f(a) \leq g(a)$.

5.6 Summary. Let us summarize the relation between consistent, non-strict propositional Horn theories and the category $\mathbf{M}_\bot$:

- Each consistent, non-strict propositional Horn theory **T** has a classifying poset $\mathrm{A}(\mathbf{T}) \in \mathrm{ob}\mathbf{M}_\bot$ (whose elements are **T**-provable equivalence classes of non-strict Horn propositions over the atomic propositions of T). A(T) comes equipped with a generic model $G_{\mathbf{T}} \in \mathrm{Mod}(\mathbf{T}, A(\mathbf{T}))$ which has the property that for any $A \in \mathrm{ob}\mathbf{M}_\bot$, the function $f \longmapsto f_*(G_{\mathbf{T}})$ is a bijection

$$\mathbf{M}_\perp(\mathrm{A}(\mathbf{T}), A) \cong \mathrm{Mod}(\mathbf{T}, A).$$

In fact, we can order the models of $\mathbf{T}$ in A by defining $M \leq N$ to mean $M(p) \leq N(p)$ for all atomic propositions p (and hence by induction on the structure of ϕ, $M(\phi) \leq N(\phi)$ holds for all non-strict Horn propositions ϕ). Then the above bijection is actually an isomorphism of posets.

- For each $A \in \mathrm{ob}\mathbf{M}_\perp$ there is some consistent, non-strict propositional Horn theory $\mathbf{T}_A$ with $A \cong \mathrm{A}(\mathbf{T}_A)$ in $\mathbf{M}_\perp$.

Define an *interpretation* $I: \mathbf{T} \longrightarrow \mathbf{T}'$ between non-strict propositional Horn theories to be a function assigning a non-strict Horn proposition $I(p)$ in $\mathbf{T}'$ to each atomic proposition in $\mathbf{T}$ with the property that $\{I(\gamma) \mid \gamma \in \Gamma\} \vdash I(\phi)$ is a $\mathbf{T}'$-theorem whenever $\Gamma \vdash \phi$ is a $\mathbf{T}$-axiom (where I is extended from atomic to all non-strict Horn propositions in the obvious, strucure-preserving way). Such interpretations can be composed and there are identity interpretations: so we get a category of consistent, non-strict propositional Horn theories. Then the above properties imply that

> $\mathbf{T} \longmapsto \mathrm{A}(\mathbf{T})$ and $A \longmapsto \mathbf{T}_A$ *are the object parts of functors which together make this category of theories equivalent* (see Glossary) *to the category* $\mathbf{M}_\perp$.

5.7 Definitions. (i) An element $d \in D$ in a dcpo is *finite* (or *compact*, or *isolated*) if for all directed subsets $S \subseteq D$

$$d \leq \bigvee S \quad \text{implies} \quad d \leq s, \text{ for some } s \in S.$$

We will denote by D_f the subposet of finite elements of D. D is *algebraic* if for each $x \in D$ the subset $\{d \in D_f \mid d \leq x\}$ is directed and its join is x.

(ii) A dcpo D is a *Scott domain* if it is algebraic, has a bottom element and has meets of all *non-empty* subsets. (**Exercise.** Show that an algebraic dcpo is a Scott domain iff it is non-empty and has joins of all subsets which possess upper bounds.) We will denote by **SD** the category of Scott domains and continuous (= monotone and directed-join preserving) functions; and $\mathbf{SD}_\wedge$ will denote the category with the same objects, but whose morphisms are continuous functions which also preserve meets of non-empty subsets.

Given Scott domains D and E, the sets $\mathbf{SD}(D,E)$ and $\mathbf{SD}_\wedge(D,E)$ of all morphisms from D to E in **SD** and $\mathbf{SD}_\wedge$ respectively, inherit a partial order from $[D,E]$: $p \leq p'$ iff for all $x \in D$, $p(x) \leq p'(x)$ in E.

Let 2 denote the two element poset $\{\bot, \top\}$ with $\bot \leq \top$. Note that it is both a non-trivial pointed meet-semilattice (it is in fact the initial object in $\mathbf{M}_\perp$) and is also a Scott domain. Although it has almost no structure, this 'schizophrenic' object, which lives both in $\mathbf{M}_\perp$ and in **SD**, is the key to the connection between pointed meet-semilattices and Scott domains.

5.8 Theorem (Duality for Scott Domains). (i) *If* $A \in \mathrm{ob}\mathbf{M}_\perp$, *then* $\mathbf{M}_\perp(A,2)$ *is a Scott domain. Moreover, given* $f: A \longrightarrow B$ *in* $\mathbf{M}_\perp$, *composition with* f *induces a function*

$$\begin{aligned} \mathbf{M}_\perp(f,2) = f^*: \mathbf{M}_\perp(B,2) &\longrightarrow \mathbf{M}_\perp(A,2) \\ g &\longmapsto g \circ f \end{aligned}$$

which is in $\mathbf{SD}_\wedge$. *Consequently, we have a* (contravariant !) *functor*

$$\mathbf{M}_\perp(-,2):\mathbf{M}_\perp^{\mathrm{op}}\longrightarrow\mathbf{SD}_\wedge\,.$$

(ii) *If D is a Scott domain, then $\mathbf{SD}_\wedge(D,2)$ is a pointed meet-semilattice. Moreover, given $p:D\longrightarrow E$ in $\mathbf{SD}_\wedge$, composition with p induces a function*

$$\begin{array}{rl}\mathbf{SD}_\wedge(p,2)=p^*:\mathbf{SD}_\wedge(E,2) & \longrightarrow\mathbf{SD}_\wedge(D,2)\\ q & \longmapsto\quad q\circ p\end{array}$$

which is in $\mathbf{M}_\perp$. *Consequently we have a functor*

$$\mathbf{SD}_\wedge(-,2):\mathbf{SD}_\wedge\longrightarrow\mathbf{M}_\perp^{\mathrm{op}}.$$

(iii) *For $A\in\mathbf{M}_\perp$, the function*

$$\mathrm{ap}:A\longrightarrow\mathbf{SD}_\wedge\big(\mathbf{M}_\perp(A,2),2\big)$$

defined by $\mathrm{ap}(a)(f)=f(a)$ (*all $a\in A$ and $f\in\mathbf{M}_\perp(A,2)$*) *is an isomorphism in* $\mathbf{M}_\perp$.

(iv) *For D a Scott domain, the function*

$$\mathrm{ap}:D\longrightarrow\mathbf{M}_\perp\big(\mathbf{SD}_\wedge(D,2),2\big)$$

defined by $\mathrm{ap}(x)(p)=p(x)$ (*all $x\in D$ and $p\in\mathbf{SD}_\wedge(D,2)$*) *is an isomorphism in* $\mathbf{SD}_\wedge$.

(v) $\mathbf{M}_\perp$ *and* $\mathbf{SD}_\wedge$ *are equivalent categories* (see Glossary).

Proof. (i) Since 2 is a complete poset, so is $[A,2]$ (with meets and joins being calculated pointwise: $\big(\bigvee S\big)(a)=\bigvee\{s(a)\mid s\in S\}$, *etc.*). It is easy to see that the meet $\bigwedge S$ of a set S of finite meet preserving functions is again finite meet preserving; and if the functions in S also preserve $\perp$, then provided S is non-empty, say $s\in S$, then $\bigwedge S$ also preserves $\perp$, since $\big(\bigwedge S\big)(\perp)\le s(\perp)=\perp$. Similarly, it is not hard to show that the join of a directed set of finite meet preserving functions is again finite meet preserving, and trivial to see that this join also preserves $\perp$ if all the constituent functions do. So $\mathbf{M}_\perp(A,2)$ is a dcpo with meets of non-empty subsets. It also has a bottom element, namely the function

$$a\in A\longmapsto\begin{cases}\top & \text{if } a=\top\\ \perp & \text{if } a\ne\top\end{cases}$$

(which always preserves finite meets and preserves bottom because $\top\ne\perp$ in A). So to see that $\mathbf{M}_\perp(A,2)$ is a Scott domain, it only remains to prove that it is algebraic.

For each $a\ne\perp$ in A, define $f_a\in\mathbf{M}_\perp(A,2)$ by

$$f_a(b)\;=\;\begin{cases}\top & \text{if } a\le b\\ \perp & \text{if } a\not\le b\,.\end{cases}$$

Then for all $g\in\mathbf{M}_\perp(A,2)$, $f_a\le g$ if and only if $\top=g(a)$. In particular $f_a\le f_b$ iff $\top=f_b(a)$, iff $b\le a$. Hence $a\longmapsto f_a$ defines an embedding (see 2.6(i)) of posets

$$\{a\in A\mid a\ne\perp\}^{\mathrm{op}}\longrightarrow\mathbf{M}_\perp(A,2).$$

Moreover, for any directed subset $S\subseteq\mathbf{M}_\perp(A,2)$, $f_a\le\bigvee S$ iff $\top=\big(\bigvee S\big)(a)=\bigvee\{s(a)\mid s\in S\}$, iff $\top=s(a)$ for some $s\in S$, iff $f_a\le s$ for some $s\in S$. Thus the f_a $(a\ne\perp)$ are finite elements of $\mathbf{M}_\perp(A,2)$. Any $g\in\mathbf{M}_\perp(A,2)$ can be expressed as a directed join of such elements: for $S=\{a\in A\mid g(a)=\top\}$ is a directed subset of A^{op}, so $\{f_a\mid a\in S\}$ is a directed subset of $\mathbf{M}_\perp(A,2)$, and $g=\bigvee_{a\in S}f_a$ (since for all $b\in A$, $\left(\bigvee_{a\in S}f_a\right)(b)=\top$ iff $f_a(b)=\top$ for some $a\in S$, iff $a\leq b$ for some a with $g(a)=\top$, iff $g(b)=\top$). It follows that $\mathbf{M}_\perp(A,2)$ is algebraic with $\mathbf{M}_\perp(A,2)_{\mathrm{f}}=\{f_a\mid a\neq\perp\}\cong\{a\in A\mid a\neq\perp\}^{\mathrm{op}}$.

The second sentence of (i) follows from the fact that directed joins and meets of non-empty subsets in $\mathbf{M}_\perp(A,2)$ are calculated as in $[B,2]$ (*i.e.* pointwise from those in 2), together with the fact that for any monotone function $f:A\longrightarrow B$, the induced monotone function $f^*:[B,2]\longrightarrow[A,2]$ preserves *all* meets and joins.

(ii) As in (i), $[D,2]$ is a complete poset with meets and joins calculated pointwise from 2. Since the bottom element $\perp\in[D,2]$ (*viz.* $d\longmapsto\perp$) preserves non-empty meets and directed joins, it is also the bottom element in $\mathbf{SD}_\wedge(D,2)$. Similarly, the meet in $[D,2]$ of finitely many functions preserving directed joins and (non-empty) meets is another such (**Exercise**), so $\mathbf{SD}_\wedge(D,2)$ also has finite meets, calculated as in $[D,2]$. In particular the top element of $\mathbf{SD}_\wedge(D,2)$ is $\top\in[D,2]$, *viz.* $d\longmapsto\top$; hence $\mathbf{SD}_\wedge(D,2)$ is non-trivial, that is $\perp\neq\top$, since these functions differ in value at $\perp\in D$. Thus $\mathbf{SD}_\wedge(D,2)\in\mathbf{M}_\perp$.

Just as for (i), the second sentence of (ii) follows from the fact that finite meets and bottom in $\mathbf{SD}_\wedge(D,2)$ are calculated as in $[D,2]$ (*i.e.* pointwise from those in 2).

(iii) First note that if $a\leq a'$ in A, then for all $f\in\mathbf{M}_\perp(A,2)$

$$\mathrm{ap}(a)(f)=f(a)\leq f(a')=\mathrm{ap}(f)(a'),$$

so that $\mathrm{ap}(a)\leq\mathrm{ap}(a')$ in $\mathbf{SD}_\wedge(\mathbf{M}_\perp(A,2),2)$. Conversely, $\mathrm{ap}(a)\leq\mathrm{ap}(a')$ implies $a\leq a'$: for either $a=\perp$, in which case $a\leq a'$ automatically, or else $a\neq\perp$, in which case we have $f_a\in\mathbf{M}_\perp(A,2)$ as in the proof of (i) and then

$$\top=f_a(a)=\mathrm{ap}(a)(f_a)\leq\mathrm{ap}(a')(f_a)=f_a(a')$$

and hence $a\leq a'$. Therefore $\mathrm{ap}:A\longrightarrow\mathbf{SD}_\wedge(\mathbf{M}_\perp(A,2),2)$ is an order embedding, and it only remains to prove that it is also surjective. So given $\alpha\in\mathbf{SD}_\wedge(\mathbf{M}_\perp(A,2),2)$, we have to find $a\in A$ with $\mathrm{ap}(a)=\alpha$. If $\alpha=\perp=\mathrm{ap}(\perp)$ we are done; so suppose $\alpha\neq\perp$. Hence $\{f\in\mathbf{M}_\perp(A,2)\mid\alpha(f)=\top\}$ is non-empty, and since $\mathbf{M}_\perp(A,2)$ is a Scott domain we can take the meet of this subset:

$$m=\bigwedge\{f\in\mathbf{M}_\perp(A,2)\mid\alpha(f)=\top\}\in\mathbf{M}_\perp(A,2).$$

Since α preserves non-empty meets, $\alpha(m)=\bigwedge\{\alpha(f)\mid\alpha(f)=\top\}=\top$. Now as in the proof of (i) $m=\bigvee\{f_a\mid m(a)=\top\}$ a directed join. Since α also preserves directed joins, we get

$$\top=\alpha(m)=\bigvee\{\alpha(f_a)\mid m(a)=\top\}$$

and hence $\alpha(f_a)=\top$ for some $a\in A$ with $m(a)=\top$. But $\alpha(f_a)=\top$ implies $m\leq f_a$ by definition of m; and $m(a)=\top$ implies $f_a\leq m$ by definition of f_a. Therefore $m=f_a$. Thus for all $f\in\mathbf{M}_\perp(A,2)$, $\alpha(f)=\top$ iff $f_a=m\leq f$, iff $\mathrm{ap}(a)(f)=f(a)=\top$; hence $\mathrm{ap}(a)=\alpha$, as required.

(iv) We need to identify the elements of $\mathbf{SD}_\wedge(D,2)$. For $d\in D$, define $p_d:D\longrightarrow 2$ by

$$p_d(x) = \begin{cases} \top & \text{if } d \leq x \\ \bot & \text{if } d \not\leq x \,. \end{cases}$$

Then p_d automatically preserves meets and preserves directed joins if d is finite. Thus we get $p_d \in \mathbf{SD}_\wedge(D,2)$ when $d \in D_f$, and as before $p_d \leq p \in \mathbf{SD}_\wedge(D,2)$ iff $p(d) = \top$. Hence $d \longmapsto p_d$ defines an order embedding $(D_f)^{op} \longrightarrow \mathbf{SD}_\wedge(D,2)$. We claim that its image consists of exactly the non-bottom elements. For, given $p \neq \bot$ in $\mathbf{SD}_\wedge(D,2)$, we must have $p(x) \neq \bot$ for some $x \in D$; hence $\{x \in D \mid p(x) = \top\}$ is non-empty and we can form $d = \bigwedge\{x \in D \mid p(x) = \top\}$. Then since p preserves non-empty meets, $p(d) = \top$ and hence for all $x \in D$, $d \leq x$ iff $p(x) = \top$. This, together with the fact that p is continuous imply that $d \in D_f$. (For if S is directed, then $d \leq \bigvee S$ iff $\top = p(\bigvee S) = \bigvee\{p(s) \mid s \in S\}$, iff $\top = p(s)$ for some $s \in S$, iff $d \leq s$ for some $s \in S$.) Hence we have $p_d \in \mathbf{SD}_\wedge(D,2)$ and $p = p_d$ since for all $x \in D$

$$p(x) = \top \quad \text{iff } d \leq x \quad \text{iff} \quad p_d(x) = \top.$$

Thus $(D_f)^{op} \cong \mathbf{SD}_\wedge(D,2) \setminus \{\bot\}$.

We can now prove (iv). Arguing just as in (iii), we have that $\mathrm{ap}: D \longrightarrow \mathbf{M}_\bot(\mathbf{SD}_\wedge(D,2),2)$ is monotonic. To see that it also reflects the ordering, suppose $x, x' \in D$ with $\mathrm{ap}(x) \leq \mathrm{ap}(x')$. Then for all $d \in D_f$, $d \leq x$ implies

$$\top = p_d(x) = \mathrm{ap}(x)(p_d) \leq \mathrm{ap}(x')(p_d) = p_d(x')$$

and hence $d \leq x'$. Thus $\{d \in D_f \mid d \leq x\} \subseteq \{d \in D_f \mid d \leq x'\}$ and so the fact that D is algebraic gives

$$x = \bigvee\{d \in D_f \mid d \leq x\} \leq \bigvee\{d \in D_f \mid d \leq x'\} = x'.$$

Thus $\mathrm{ap}: D \longrightarrow \mathbf{M}_\bot(\mathbf{SD}_\wedge(D,2),2)$ is an order embedding, and it only remains to prove that it is also surjective. But given $\delta \in \mathbf{M}_\bot(\mathbf{SD}_\wedge(D,2),2)$, consider $S = \{d \in D_f \mid \delta(p_d) = \top\}$. Since $\bot \in D_f$ and $p_\bot = \top \in \mathbf{SD}_\wedge(D,2)$, we have $\bot \in S$—so S is non-empty. Furthermore, given $d, d' \in S$, since δ preserves $\wedge$, $\delta(p_d \wedge p_{d'}) = \delta(p_d) \wedge \delta(p_{d'}) = \top$; therefore we must have $p_d \wedge p_{d'} \neq \bot$ (since δ also preserves $\bot$), and hence for some $x \in D$ $(p_d \wedge p_d)(x) = \top$, *i.e.* $d \leq x$ and $d' \leq x$. Then because D is a Scott domain $d \vee d' = \bigwedge\{x \in D \mid d \leq x$ and $d' \leq x\}$ exists; and it is easy to see that $d \vee d' \in D_f$ because $d, d' \in D_f$. Thus S is non-empty and $d, d' \in S$ implies there is some $d'' \in S$ (the join of d and d', in fact) with $d \leq d''$and $d' \leq d''$. In other words S is a directed set, and so we can form $x = \bigvee S$ in D. Then for all $d \in D_f$, $\mathrm{ap}(x)(p_d) = p_d(x) = \top$ iff $d \leq x$, iff $d \in S$, iff $\delta(p_d) = \top$. Thus $\mathrm{ap}(x)$ and δ agree at all non-bottom elements of $\mathbf{SD}_\wedge(D,2)$; and they agree at $\bot$ since they both preserve $\bot$. Hence $\delta = \mathrm{ap}(x)$ and we have that ap is a surjective order embedding, hence an isomorphism.

(v) It is easy to see that the isomorphism of (iii) is natural (see Glossary) in A and that the isomorphism in (iv) is natural in D. Hence the functor $\mathbf{M}_\bot(-,2): \mathbf{M}_\bot^{op} \longrightarrow \mathbf{SD}_\wedge$ is an equivalence with essential inverse $\mathbf{SD}_\wedge(-,2): \mathbf{SD}_\wedge \longrightarrow \mathbf{M}_\bot^{op}$. □

Phew! Theorem 5.8 contains a lot of information. It is typical of a number of duality theorems† which arise because of a 'schizophrenic' object: see [Joh2, VI.3] for other, topological examples and see [MP] for a related categorical example.

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†A duality theorem asserts the equivalence of some category of mathematical structures with the opposite of some other, apparently unrelated category.

Combining the Theorem 5.8 with the remarks in 5.6, we have that up to isomorphism, *every Scott domain is the poset* Mod(**T**,2) *of models of a consistent, non-strict propositional Horn theory* **T** (*and interpretations* $I:\mathbf{T}\longrightarrow\mathbf{T}'$ *between such theories correspond precisely to functions between the Scott domains which are continuous and preserve non-empty meets*). This enables one to present Scott domains by specifying propositional theories (and proving their consistency). We give some examples and leave the claims made in them as **Exercises.**

5.9 Example: flat domains. The flat domain of natural numbers $\mathbf{N}_\bot$ (= $\mathbf{N}\cup\{\bot\}$ partially ordered by: $x\leq y$ iff $x=\bot$ or $x=y$) is isomorphic Mod(**T**,2) where **T** has one atomic proposition

$$`X=n'$$

for each $n\in\mathbf{N}$, and has one axiom

$$`X=n' \wedge `X=m' \vdash \bot$$

for each pair of distinct numbers $n\neq m$.

5.10 Example: product domains. If $D=\mathrm{Mod}(\mathbf{T},2)$ and $D'=\mathrm{Mod}(\mathbf{T}',2)$, then the product $D\times D'$ (which is easily seen to be another Scott domain) is specified by the theory $\mathbf{T}\uplus\mathbf{T}'$ whose atomic propositions are the disjoint union of those in **T** and **T′**, and whose axioms are all those of **T** and **T′**.

5.11 Example: function domains. If D and D' are Scott domains, then the set $(D\rightarrow D')$ of continuous functions from D to D', when partially ordered as a subposet of $[D,D']$, is known to again be a Scott domain. If D and D' are specified by theories **T** and **T′**, then $(D\rightarrow D')$ can be specified by the following theory.

The atomic propositions of the theory are given by pairs of propositions, written $\{\phi\}P\{\psi\}$ (*cf.* 'Hoare triples'), where ϕ is a non-stict Horn proposition in **T** with $\phi\not\leq_{\mathbf{T}}\bot$, and ψ is a non-strict Horn proposition in **T′**.

The axioms of the theory are:

$$(5.1)\qquad \{\phi'\}P\{\psi\}\vdash\{\phi\}P\{\psi'\} \qquad \text{provided } \phi\leq_{\mathbf{T}}\phi' \text{ and } \psi\leq_{\mathbf{T}'}\psi'$$

$$(5.2)\qquad \top\vdash\{\phi\}P\{\top\}$$

$$(5.3)\qquad \{\phi\}P\{\psi\}\wedge\{\phi\}P\{\psi'\}\vdash\{\phi\}P\{\psi\wedge\psi'\}$$

$$(5.4)\qquad \{\phi\}P\{\bot\}\vdash\bot\ .$$

(This description of $(D\rightarrow D')$ is based upon the observation that for $A,B\in\mathbf{M}_\bot$, a continuous function $\mathbf{M}_\bot(A,2)\longrightarrow\mathbf{M}_\bot(B,2)$ corresponds to a monotone function $(\mathbf{M}_\bot(A,2))_{\mathrm{f}}\longrightarrow\mathbf{M}_\bot(B,2)$, which from the proof of 5.8(i) is the same as a montone function $\{a\in A\mid a\neq\bot\}^{\mathrm{op}}\longrightarrow\mathbf{M}_\bot(B,2)$. The latter amounts to giving a function $\{a\in A\mid a\neq\bot\}\times B\longrightarrow 2$ which is separately order-reversing in its first variable and in its second variable preserves (order,) finite meets and bottom; but such a function corresponds to a relation $P\subseteq\{a\in A\mid a\neq\bot\}\times B$ satisfying:

$$(a',b)\in P \text{ implies } (a,b')\in P, \text{ when } a\leq a' \text{ and } b\leq b' \qquad (cf.\ (5.1))$$

$$(a,\top)\in P \qquad (cf.\ (5.2))$$

$$(a,b)\in P \text{ and } (a,b')\in P \text{ imply } (a,b\wedge b')\in P \qquad (cf.\ (5.3))$$

$$(a,\bot)\notin P \qquad (cf.\ (5.4)).$$

Compare this with Scott's notion of 'approximable mapping' between information systems—*cf.* [LW, 2.1].)

5.12 Exercise (for budding domain theorists). If D is a Scott domain, then so are the upper (Smyth) powerdomain, $P_u(D)$, and lower (Hoare) powerdomain, $P_\ell(D)$. If D is specified by the theory **T**, are there nice descriptions in terms of **T** of theories corresponding to $P_u(D)$ and $P_\ell(D)$?

5.13 Remark: interpretations as approximations. Recall from 3.8(viii) the notion of embedding-projection pairs between dcpo's. Thus an embedding between Scott domains $i: D \longrightarrow E$ is a continuos function possessing a continuous right-adjoint-left-inverse $i_*: E \longrightarrow D$. (Such an embedding gives an 'approximation' of the domain E by the domain D. Embeddings are used in the solution of recursive domain equations by calculating initial fixed points of continuous functors on the category of domains and embeddings.)

Embedding-projection pairs have a very simple description in terms of propositional theories. Recall from 5.6 the notion of an interpretation $I: \mathbf{T} \longrightarrow \mathbf{T}'$. An interpretation is *conservative* if whenever an interpreted sequent $\{I(\gamma) \mid \gamma \in \Gamma\} \vdash I(\phi)$ is a theorem of $\mathbf{T}'$, then the original sequent $\Gamma \vdash \phi$ is already a theorem of **T**. Then if $D = \mathrm{Mod}(\mathbf{T}, 2)$ and $E = \mathrm{Mod}(\mathbf{T}', 2)$, one can show that *embeddings* $D \longrightarrow E$ *correspond precisely to conservative interpretations of* **T** *in* $\mathbf{T}'$.

Part B

Predicate Logic

6 Terms and Equations

6.1 Definition. A (*many-sorted, algebraic*) *signature* Σ is specified by

- a set of *sorts* $A, B, \ldots$;
- a set of *operators* (or *function symbols*) $f, g, \ldots$, together with a map assigning to each operator f its *type*, which is a non-empty list of sorts. The notation

  $$f: A_1 \ldots A_n \longrightarrow B$$

  wl be used to indicate that f has type $A_1, \ldots, A_n, B$. (The number n is called the *arity* of f. In the case $n=0$, f is more usually called a *constant* of sort B.)

6.2 Definition. Let Σ be a signature. For each sort A in Σ, fix a countably infinite set of *variables of sort A*. (We assume these sets are disjoint for different sorts.) Then the *terms* over Σ and their sorts are defined recursively as follows, where we write $t{:}A$ to indicate that t is a well-formed term of sort A:

- if x is a variable of sort A, then $x{:}A$
- if $f: A_1 \ldots A_n \longrightarrow B$ is an operator and $t_1{:}A_1, \ldots, t_n{:}A_n$, then $f(t_1, \ldots, t_n){:}B$. (In the case $n=0$, we will write f instead of $f()$.)

The usual, set-theoretic semantics for terms is based upon having a 'structure' for Σ, which consists of

–a set MA for each sort A; and
–a function $Mf{:} MA_1 \times \cdots \times MA_n \longrightarrow MB$ for each operator $f: A_1 \ldots A_n \longrightarrow B$

(The set $MA_1 \times \cdots \times MA_n$ is the cartesian product of the sets MA_i and consists of n-tuples $(a_1, \ldots, a_n)$ with $a_i \in MA_i$. In the case $n=0$, this cartesian product contains just one element, the 0-tuple $()$, and so specifying the function Mf amounts to picking a particular element of MB.)

An *environment* ρ assigns to each variable x of each sort A an element $\rho(x) \in MA$ in the structure M. In the presence of such an environment, each term t, of sort A say, can be assigned a meaning as an element $[\![t]\!]\rho$ in MA:

– $[\![x]\!]\rho = \rho(x)$
– $[\![f(t_1, \ldots, t_n)]\!] = Mf([\![t_1]\!]\rho, \ldots, [\![t_n]\!]\rho)$.

For our purposes, there are two things which are unsatisfactory about this environment-style semantics of terms:

(i) It implicitly assumes that all the sorts are interpreted as non-empty sets; for if any of the MA were empty, then there are no environments at all.

(ii) It is given in terms of *elements of sets*; we would like a formulation in terms of *functions between sets* which we could then generalize by replacing the role of the category of sets and functions by other categories.

One can easily solve the first problem by using partial environments ρ in $[\![t]\!]\rho$, with the domain of definition of ρ a finite set of variables which includes those occurring in t. But then we can also sovle the second problem by noting that for any fixed list of distinct variables $x_1:A_1,\ldots,x_n:A_n$, the partial environments defined just on this set of variables comprise the set $MA_1\times\cdots\times MA_n$; and then $\rho\longmapsto[\![t]\!]\rho$ is a function $MA_1\times\cdots\times MA_n\longrightarrow MB$ (if $t:B$) which captures the meaning of t in the structure. So we can get a semantics in terms of functions rather than elements if we consider terms 'in context':

6.3 Definition. A *context* $\mathbf{x}$ is a finite list $x_1,\ldots,x_n$ of distinct variables. (The case $n=0$ is allowed, yielding the *empty context*, [].) The *sort* of $\mathbf{x}$ is the list $\mathbf{A}=A_1,\ldots,A_n$ of the sorts of each x_i.

A *term-in-context* $t(\mathbf{x})$ consists of a term t together with a context $\mathbf{x}$ containing all the variables occurring in t (and possibly some others as well). In particular, if t contains no variables at all (we call such a term *closed*) we can consider it in the empty context; in this case we will abbreviate $t([\,])$ to just t.

Now suppose that **C** is a category with finite products (see Glossary). A *structure* M in **C** for a signature Σ is specified by

- an object $MA\in\mathrm{ob}\mathbf{C}$ for each sort A in Σ
- a morphism $Mf:\prod M\mathbf{A}\longrightarrow MB$ in **C** for each operator $f:\mathbf{A}\longrightarrow B$ in Σ.

($\prod M\mathbf{A}$ denotes the finite product in **C** of the objects $MA_1,\ldots,MA_n$ when $\mathbf{A}$ is the list of sorts $A_1,\ldots,A_n$. In the case $n=0$, this is the product of no objects, which is the terminal object 1 in **C**—see Glossary. In the case $n=1$, the product of one object is just the object itself, with identity morphism as product projection.)

For each term-in-context $t(\mathbf{x})$, where $\mathbf{x}:\mathbf{A}$ and $t:B$ say, define a morphism

$$[\![t(\mathbf{x})]\!]_M:\prod M\mathbf{A}\longrightarrow MB$$

in **C** by recursion on the structure of t as in Table 6.4. (Recall that $\langle[\![t_1(\mathbf{x})]\!]_M,\ldots,[\![t_m(\mathbf{x})]\!]_M\rangle$ denotes the unique morphism into the product whose composition with each product projection π_j is $[\![t_j(\mathbf{x})]\!]_M$.)

- If t is a variable x_i, then $[\![t(\mathbf{x})]\!]_M$ is π_i, the i^{th} product projection morphism.

- If t is $f(t_1,\ldots,t_m)$, where $f:B_1\ldots B_m\longrightarrow B$ is an operator, then $[\![t(\mathbf{x})]\!]_M$ is the composition

$$\prod M\mathbf{A}\xrightarrow{\langle[\![t_1(\mathbf{x})]\!]_M,\ldots,[\![t_m(\mathbf{x})]\!]_M\rangle}MB_1\times\cdots\times MB_m\xrightarrow{Mf}MB.$$

6.4 Table: categorical semantics of terms-in-context

We will write $[\![t(\mathbf{x})]\!]$ for $[\![t(\mathbf{x})]\!]_M$ when the structure M is understood. Note that for a single variable x, $[\![x(x)]\!]$ is just id_{MA} (when $x:A$). Note also that for a closed term $[\![t]\!]$ is a 'global element' of MB, that is, a morphism $1 \longrightarrow MB$ from the terminal object to MB in $\mathbf{C}$.

6.5 Definition. An *equation-in-context* over some signature Σ

$$s = t \ (\mathbf{x})$$

is specified by a context $\mathbf{x}$ and a pair of terms s and t (over Σ) *of the same sort,* whose variables occur in $\mathbf{x}$. If $\mathbf{x}$ consists only of variables which occur in s or t, we will abbreviate the above to just $s = t$.

A (many-sorted) *algebraic theory* $\mathbf{T}$ consists of a signature Σ together with a set of equations-in-context over Σ, called the *axioms* of $\mathbf{T}$.

A structure M (in some category $\mathbf{C}$ with finite products) for the signature Σ is said to *satisfy* the above equation if $[\![s(\mathbf{x})]\!]_M$ and $[\![t(\mathbf{x})]\!]_M$ are equal morphisms in $\mathbf{C}$. (Note that they already have equal domain and codomain.) M is a $\mathbf{T}$-*algebra in the category* $\mathbf{C}$ if it is a structure for the underlying signature of $\mathbf{T}$ and satisfies all axioms of $\mathbf{T}$.

There is a whole *category* of $\mathbf{T}$-algebras in $\mathbf{C}$, denoted $\mathrm{Mod}(\mathbf{T},\mathbf{C})$. A morphism $h: M \longrightarrow N$ in this category is called a $\mathbf{T}$-*algebra homomorphism* and is specified by a family $h_A: MA \longrightarrow NA$ of morphisms in $\mathbf{C}$ indexed by the sorts, and satisfying for each operator $f: \mathrm{A} \longrightarrow B$ that

$$\begin{array}{ccc} \prod M\mathrm{A} & \xrightarrow{\prod h_{\mathrm{A}}} & \prod N\mathrm{A} \\ {\scriptstyle Mf}\downarrow & & \downarrow{\scriptstyle Nf} \\ MB & \xrightarrow[h_{\mathrm{B}}]{} & NB \end{array}$$

commutes in $\mathbf{C}$ (where $\prod h_{\mathrm{A}}$ denotes $h_{A_1} \times \cdots \times h_{A_n}$). The identity on M in $\mathrm{Mod}(\mathbf{T},\mathbf{C})$ has component id_{MA} at sort A; and the composition of h and k in $\mathrm{Mod}(\mathbf{T},\mathbf{C})$ has component $k_A \circ h_A$ at sort A.

6.6 Examples. (i) Referring to 5.1, the algebraic theory of 'pointed meet-semilattices' has the following specification:

A single sort A.
Three operators $\top: \longrightarrow A$, $\bot: \longrightarrow A$ and $\wedge: A\,A \longrightarrow A$.
Five equations(-in-context)

$$\begin{aligned} \wedge(x,y) &= \wedge(y,x) \\ \wedge(\wedge(x,y),z) &= \wedge(x,\wedge(y,z)) \\ \wedge(x,\top) &= x \\ \wedge(x,x) &= x \\ \wedge(x,\bot) &= \bot. \end{aligned}$$

Unravelling the definition of the semantics of terms in Table 6.4, one finds that an algebra for this theory in a category with finite products amounts to having an object M together with morphisms $t,b: 1 \longrightarrow M$ and $m: M \times M \longrightarrow M$ such that the following diagrams commute:

$$\begin{array}{ccc} M\times M & \xrightarrow{\langle\pi_2,\pi_1\rangle} & M\times M \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & M & \end{array} \qquad \begin{array}{ccc} M\times M\times M & \xrightarrow{\langle m\circ\langle\pi_1,\pi_2\rangle,\pi_3\rangle} & M\times M \\ {\scriptstyle \langle\pi_1,m\circ\langle\pi_2,\pi_3\rangle\rangle}\downarrow & & \downarrow{\scriptstyle m} \\ M\times M & \xrightarrow[m]{} & M \end{array}$$

$$\begin{array}{ccccc} M & \xrightarrow{\langle \mathrm{id},\mathrm{T}\circ\langle\rangle\rangle} & M\times M & \xleftarrow{\langle \mathrm{id},\mathrm{id}\rangle} & M \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \mathrm{id}} & \\ & & M & & \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\langle \mathrm{id},\mathrm{T}\circ\langle\rangle\rangle} & M\times M \\ {\scriptstyle \langle\rangle}\downarrow & & \downarrow{\scriptstyle m} \\ 1 & \xrightarrow[\perp]{} & M\,. \end{array}$$

(Recall that $\langle\rangle$ is our notation for the unique morphism $M \longrightarrow 1$.)

(ii) Ⓣ When **C** is **Set**, the category of sets and functions, the notion of an algebra for an algebraic theory in **C** coincides with the usual notion from Universal Algebra (except that traditionally only single-sorted signatures were considered.) Thus a **T**-algebra in **Set** for the algebraic theory of (i) is precisely a pointed meet-semilattice in the sense of Definition 5.1.

(**Exercise.** The opposite† category $\mathbf{Set}^{\mathrm{op}}$ has binary products given by disjoint union and terminal object given by the empty set. Are there any algebras in $\mathbf{Set}^{\mathrm{op}}$ for the algebraic theory of (i) ?)

(iii) Ⓢ Why study **T**-algebras in categories other than the category of sets and functions? One good reason (the same one which in the development of mathematics stimulated the construction of first the rationals, then the reals, then the complex numbers) is that we may be interested in particular kinds of equation which may not always admit solutions in ordinary, set-valued structures. For example, in programming language semantics one might want, for a given signature, to have solutions for finite sets of mutually recursive equations

$$\begin{aligned} x_1 &= t_1(x_1,\ldots,x_n) \\ &\vdots \\ x_n &= t_n(x_1,\ldots,x_n)\,. \end{aligned}$$

This is not always possible for structures valued in **Set**, but it is possible for structures valued in the category of pointed dcpo's (*i.e.* dcpo's with bottom) and continuous functions. (In this category the binary product of D and E is the cartesian product $D\times E=\{(d,e)\mid d\in D \text{ and } e\in E\}$ of the underlying sets, partially ordered by: $(d,e)\le(d',e')$ iff $d\le d'$ and $e\le e'$. The terminal object is the one element poset. Hence **T**-algebras in this category amount to ordinary, set-valued **T**-algebras in which the sets interpreting the sorts are pointed dcpo's and the functions interpreting the operators are all continuous.)

There are very many different categories with finite products, and algebras for an algebraic theory in one may have very different detailed structure from algebras for the same theory in another category. Nevertheless, we can use a familiar kind of equational logic and still preserve satisfaction of equations, whatever the underlying category. The rules of this logic are given in Table 6.7.

† See Glossary.

(Weakening)	$\dfrac{s=t\ (\mathbf{x})}{s=t\ (\mathbf{y})}$	provided $\mathbf{x}\subseteq\mathbf{y}$
(Reflexivity)	$\dfrac{}{t=t\ (\mathbf{x})}$	
(Symmetry)	$\dfrac{s=t\ (\mathbf{x})}{t=s\ (\mathbf{x})}$	
(Transitivity)	$\dfrac{r=s\ (\mathbf{x}) \qquad s=t\ (\mathbf{x})}{r=t\ (\mathbf{x})}$	
(Substitution)	$\dfrac{s=s'\ (\mathbf{x}) \qquad t=t'\ (\mathbf{x},y)}{t(s/y)=t'(s'/y)\ (\mathbf{x})}$	

6.7 Table: Equational Logic

In the (Substitution) rule in Table 6.7, $t(s/y)$ denotes the result of substituting the term s for the variable y throughout t (and s and y have the same sort).

A set of equations-in-context is *closed* under the rules in Table 6.7 if whenever it contains the hypotheses of one of the rules, it also contains the conclusion. If **T** is an algebraic theory over some signature Σ, the (equational) *theorems* of **T** comprise the least set of equations-in-context over Σ which contains the axioms of **T** and is closed under the rules in Table 6.7.

6.8 Proposition (Soundness). *Let* **C** *be a category with finite products,* **T** *an algebraic theory and M a* **T**-*algebra in* **C**. *Then M satisfies any equation-in-context which is a theorem of* **T**.

Proof. The corresponding properties of actual equality of morphisms in **C** impy that the collection of equations-in-context satisfied by M is closed under the (Reflexivity), (Symmetry) and (Transitivity) rules in Table 6.7. Closure under the (Substitution) and (Weakening) rules is a consequence of the lemma below, which can be proved by induction on the structure of terms. □

6.9 Substitution Lemma. *If $t(\mathbf{x})$ is a term-in-context with $\mathbf{x}=x_1,\ldots,x_n$, and if $s_i(\mathbf{y})$ $(i=1,\ldots,n)$ is a term-in-context of the same sort as x_i, let $t(\mathbf{s}(\mathbf{y}))$ denote the term-in-context resulting from simultaneously substituting each s_i for x_i in t. Then*

$$[\![t(\mathbf{s}(\mathbf{y}))]\!] \;=\; [\![t(\mathbf{x})]\!]\circ\langle [\![s_1(\mathbf{y})]\!],\ldots,[\![s_n(\mathbf{y})]\!]\rangle.$$

6.10 Corollary. (Weakening) *Suppose that $t(\mathbf{x})$ is a term-in-context and that* **y** *is another context containing all the variables in* **x**. *Then ($t(\mathbf{y})$ is also a term-in-context and)*

$$[\![t(\mathbf{y})]\!] \;=\; [\![t(\mathbf{x})]\!]\circ\pi$$

where π is the unique morphism whose composition with the i^{th} product projection π_i is $\pi_{\alpha(i)}$, with $\alpha(i)$ defined so that $y_{\alpha(i)}$ is the (unique) member of **y** *equal to x_i.*

Proof. This is just the special case of the Substitution Lemma with $s_i=y_{\alpha(i)}$. □

6.11 Remark. The labelling of equations with contexts is natural if we are considering the categorical semantics, which assigns meanings to terms-in-context rather than to terms by themselves. However, even if we restrict attention to set-valued structures and use a familiar environment-style semantics, the possibility of interpreting some of the sorts of a signature by the empty set means that equational logic for unlabelled equations is not sound. Here is an example of this (lifted from [GM]). Let **T** be the following algebraic theory:

$$\begin{array}{rl} \text{Sorts:} & B, H \\ \text{Operators:} & t: \longrightarrow B,\ f: \longrightarrow B \\ & n: B \longrightarrow B,\ c: H \longrightarrow B \\ & a: B\ B \longrightarrow B,\ o: B\ B \longrightarrow B \\ \text{Axioms:} & o(n(x), x) = t \\ & a(n(x), x) = f \\ & o(x, x) = x \\ & a(x, x) = x \\ & n(c(y)) = c(y)\,. \end{array}$$

With unlabelled equations we could argue that

$$\begin{array}{ll} t = o(n(c(y)), c(y))) & \text{(using the first axiom)} \\ \quad = o(c(y), c(y)) & \text{(using the fifth axiom)} \\ \quad = c(y) & \text{(using the third axiom)} \\ \quad = a(c(y), c(y)) & \text{(using the second axiom)} \\ \quad = a(n(c(y)), c(y)) & \text{(using the fifth axiom)} \\ \quad = f & \text{(using the second axiom)} \end{array}$$

and hence conclude by transitivity that $t = f$ is a theorem. But that is an unsound conclusion, since that equation is not satisfied by the model of the theory with $M(B) = \{0,1\}$, $M(H) = \emptyset$, $M(t) = 1$, $M(f) = 0$, *etc.* With equations-in-context, the closest we can get to the above argument is to conclude that $t = f\ (y)$ is a theorem of **T**, where y is a variable of sort H. This means that for any model M of **T** in a category, the morphisms

$$[\![t(y)]\!] = \left(M(H) \xrightarrow{\ \langle\rangle\ } 1 \xrightarrow{\ M(t)\ } M(B)\right) \quad \text{and} \quad [\![f(y)]\!] = \left(M(H) \xrightarrow{\ \langle\rangle\ } 1 \xrightarrow{\ M(f)\ } M(B)\right)$$

are always equal—which does not necessarily mean that $M(t)$ and $M(f)$ are equal (as the above model demonstrates).

6.12 Remarks. (i) **(Completeness).** There is an easily proved converse to Proposition 6.8, *viz: an equation-in-context is a theorem of* **T** *if it is satisfied by all* **T***-algebras in* **Set**. This is because for any context $\mathrm{x} = x_1, \ldots, x_n$, with $x_i : A_i$ say, we can form the 'free **T**-algebra on indeterminates $x_1 : A_1, \ldots, x_n : A_n$', $\mathrm{F_T(x)}$, which has the property that an equation in the context x is satisfied by $\mathrm{F_T(x)}$ iff it is a theorem of **T**. In fact $\mathrm{F_T(x)}$ at sort A consists of the set of terms-in-context $s(\mathrm{x})$ with $s : A$, quotiented by the equivalence relation

$$s(\mathrm{x}) \sim t(\mathrm{x}) \quad \text{if and only if} \quad s = t\ (\mathrm{x}) \text{ is a theorem of } \mathbf{T}.$$

(The collection of these quotient sets can be endowed with an obvious **T**-algebra structure.)

(ii) (**Classifying categories**). The construction of these free **T**-algebras can be subsumed in the construction of a single category with finite products Cl(**T**), called the *classifying category* of the algebraic theory **T**. Cl(**T**) is specified uniquely up to equivalence of categories by the requirement that that for each category **C** with finite products there is a (natural) equivalence of categories

$$\mathrm{FP}(\mathrm{Cl}(\mathbf{T}),\mathbf{C}) \simeq \mathrm{Mod}(\mathbf{T},\mathbf{C}) ,$$

where the category on the left-hand side consists of the finite product preserving† functors $\mathrm{Cl}(\mathbf{T})\longrightarrow\mathbf{C}$ and natural transformations. Briefly, one way to construct Cl(**T**) is to take contexts **x** as its objects, and for morphisms $\mathbf{x}\longrightarrow\mathbf{y}$ take equivalence classes of lists $\mathbf{t}(\mathbf{x}) = t_1(\mathbf{x}),\ldots,t_m(\mathbf{x})$ of terms-in-context with m=(length of **y**) and t_j of the same sort as y_j, under the equivalence relation which identifies $\mathbf{s}(\mathbf{x})$ with $\mathbf{t}(\mathbf{x})$ when $s_j = t_j$ (**x**) is a theorem of **T** for each $j = 1,\ldots,m$. The identity morphism on **x** is the equivalence class of $\mathbf{x}(\mathbf{x})$; and the composition of $[\mathbf{s}(\mathbf{x})]:\mathbf{x}\longrightarrow\mathbf{y}$ and $[\mathbf{t}(\mathbf{y})]:\mathbf{y}\longrightarrow\mathbf{z}$ is $[\mathbf{t}(\mathbf{s}(\mathbf{x}))]$ (using notation for substitution as in 6.9). One can prove (**Exercise**) that the empty context [] is a terminal object and that the binary product of **x** and **y** is given by any context $\mathbf{x}'\mathbf{y}'$ where $\mathbf{x}'$ (respectively $\mathbf{y}'$) has the same length and sort as **x** (respectively **y**).

(An alternative construction: it can be shown that Cl(**T**) is equivalent to the opposite of the full subcategory† of Mod(**T**,**C**) whose objects are the free **T**-algebras $\mathrm{F}_{\mathbf{T}}(\mathbf{x})$ described in (i).)

(iii) (**Internal languages**). Up to equivalence, every small† category **C** with finite products is the classifying category of an algebraic theory. (The restriction on the size of **C** is only there because we are assuming theories to be specified by a set of symbols, rather than a proper class of them.) To see this, let $\Sigma_{\mathbf{C}}$ be the signature with

- one sort X for each $X \in \mathrm{ob}\mathbf{C}$
- one operator $f : X_1 \cdots X_n \longrightarrow Y$ for each non-empty list $X_1,\ldots,X_n,Y$ of objects and each morphism $f: X_1 \times\cdots\times X_n \longrightarrow Y$ in **C**.

(Pardon the overloading of symbols.) The terms over this signature constitute the *internal language* of **C**. There is a distinguished structure for $\Sigma_{\mathbf{C}}$ in **C**, namely the one sending each X and each f to itself. Let $\mathbf{T}_{\mathbf{C}}$ be the algebraic theory over $\Sigma_{\mathbf{C}}$ whose axioms are all those equation-in-context which are satisfied by this structure. Then one can prove that $\mathrm{Cl}(\mathbf{T}_{\mathbf{C}}) \simeq \mathbf{C}$.

The internal language of **C** provides a means for describing properties of objects and morphisms in **C**. For example, writing

$$\mathbf{C} \models s = t \ (\mathbf{x})$$

to indicate that the canonical $\mathbf{T}_{\mathbf{C}}$-algebra in **C** satisfies the equation-in-context, we can prove:

(a) *$f: X \longrightarrow X$ is the identity morphism iff* $\mathbf{C} \models f(x) = x \ (x)$.

(b) *$f: X \longrightarrow Z$ is the composition of $g: X \longrightarrow Y$ and $h: Y \longrightarrow Z$ iff* $\mathbf{C} \models f(x) = h(g(x)) \ (x)$.

† See Glossary.

(c) *An object* T *is a terminal object iff there is some morphism* $t: 1 \longrightarrow T$ *with* $\mathbf{C} \models x = t\ (x)$ (*where* x *is a variable of sort* T).

(d) $X \xleftarrow{p} Z \xrightarrow{q} Y$ *is a binary product diagram iff there is some morphism* $r: X \times Y \longrightarrow Z$ *with* $\mathbf{C} \models z = r(p(z), q(z))\ (z)$, $\mathbf{C} \models p(r(x,y)) = x\ (x,y)$ *and* $\mathbf{C} \models q(r(x,y)) = y\ (x,y)$.

Thus the internal language of a category **C** makes it look like a category of sets and functions: give an object X, its 'elements' are the terms $t: X$; and a morphism $f: X \longrightarrow Y$ yields a function sending 'elements' $t: X$ to 'elements' $f(t): Y$.

(iv) Taken together, (ii) and (iii) set up a correspondence (first observed by Lawvere) between algebraic theories and categories with finite products. We can view a particular small category with finite products **C** as specifying a (many-sorted) algebraic theory *independently of any particular presentation* in terms of a signature and axioms: there may be many such presentations whose syntactic details are different, but which nevertheless specify the 'same' theory, in the sense that their classifying categories are equivalent to the given category **C**.

Under the identification of algebraic theories with their classifying categories, the algebras of the theory correspond to functors out of the classifying category which preserve finite products (and homomorphisms correspond to natural transformations between such functors). In particular, a finite product preserving functor $\mathrm{Cl}(\mathbf{T}) \longrightarrow \mathrm{Cl}(\mathbf{T}')$ gives a good notion of an *interpretation* $\mathbf{T} \longrightarrow \mathbf{T}'$ of one theory in another.

Once one has this categorical view of algebraic theories, one sees that there are many 'naturally occurring' algebraic theories which do not arise in terms of a presentation with operators and equations. One example of this of relevance to the theory of computation is the algebraic theory which as a category with finite products has for objects the finite cartesian powers of the natural numbers **N**, and whose morphisms $\mathbf{N}^n \longrightarrow \mathbf{N}^m$ are m-tuples of n-ary primitive recursive functions.

6.13 Summary. We conclude this section by summarizing the important features of the categorical semantics of terms and equations in a category with finite products.

- Sorts are interpreted as object.

- A term is only interpreted in a context (a list of distinct variables containing at least the variables mentioned in the term) and a term-in-context is interpreted as a morphism with
  - the codomain of the morphism determined by the sort of the term;
  - the domain of the morphism determined by the context;
  - variables-in-context interpreted as product projection morphisms (identity morphisms being a special case of these);
  - substitution of terms for variables interpreted via composition and pairing;
  - weakening of contexts interpreted via composition with a product projection morphism.

- An equation is only considered in a context (containing at least the variables mentioned), and an equation-in-context is satisfied if the two morphisms interpreting the equated terms-in-context are actually equal in the category.

7 Indexed Preorders

We saw in sections 4 and 5 the relationship between certain kinds of preordered set and theories in propositional logics. And in the previous section we saw how the sorts and terms of an algebraic theory are modelled by the objects and morphisms of a category with finite products. The next step is to put these two things together and consider logics in which there are predicates asserting properties of terms which can be combined using the propositional connectives and which may contain quantification of variables ranging over the sorts. The precise syntax and categorical semantics of such 'first-order predicate logics' will be given in the next section. In this section we introduce the category-theoretic structure required—a combination of categories with finite products and preordered sets.

7.1 **Definition.** Let **C** be a category. A **C**-*indexed preordered set* **X** is specified by

- a preordered set $\mathbf{X}(I)$ for each $I \in \mathrm{ob}\mathbf{C}$
- a monotone function $\mathbf{X}(\alpha): \mathbf{X}(J) \longrightarrow \mathbf{X}(I)$ for each morphism $\alpha: I \longrightarrow J$ in **C**

satisfying

- for each $I \in \mathrm{ob}\mathbf{C}$, $\mathbf{X}(\mathrm{id}_I) \cong \mathrm{id}_{\mathbf{X}(I)}$
- for $I \xrightarrow{\alpha} J \xrightarrow{\beta} K$ in **C**, $\mathbf{X}(\alpha) \circ \mathbf{X}(\beta) \cong \mathbf{X}(\beta \circ \alpha)$.

$\mathbf{X}(I)$ will be called the '*fibre of* **X** *at* I'; and we will usually write α^* for $\mathbf{X}(\alpha)$ and call it '*pullback along* α'.

Thus a **C**-indexed preordered set is something less than a functor $\mathbf{C}^{\mathrm{op}} \longrightarrow$ **Preord**, since the latter would be specified by similar information, but with equalities rather than isomorphisms in the above two requirements. If all the $\mathbf{X}(I)$ are posets, then the isomorphisms are necessarily equalities and the two notions coincide. Thus a **C**-*indexed poset* is a functor from $\mathbf{C}^{\mathrm{op}}$ to the category of posets and monotone functions. (**Exercise.** Generalize the 'poset reflection' construction from section 1 to give a construction of **C**-indexed posets from **C**-indexed preordered sets.)

7.2 **Examples.** (i) Ⓣ **Set**-indexed poset **P**:
- fibre at I is poset of subsets of I, $\mathbf{P}(I)$ (see 1.3(i))
- $\mathbf{P}(\alpha)$ is inverse image along α, α^{-1} (see 2.2(i)).

(ii) Ⓛ Suppose **C** has pullbacks. Then we have a **C**-indexed poset $\mathrm{Sub}_{\mathbf{C}}$:
- fibre at I is poset of subobjects of I, $\mathrm{Sub}_{\mathbf{C}}(I)$ (see 1.3(ii))
- $\mathrm{Sub}_{\mathbf{C}}(\alpha)$ is pullback of subobjects along α, α^{-1} (see 2.2(ii)).

(iii) Ⓚ **Set**-indexed preordered set **R**:
- fibre at I is $\mathbf{R}(I)$ (see 1.3(iii))
- $\mathbf{R}(\alpha)$ is the monotone function α^* of 2.2(iii).

(iv) Ⓢ Let **Dcpo** be the category of dcpo's (*cf.* 1.7(iii)) and continuous functions (*cf.* 2.2(iv)). Then we have a **Dcpo**-indexed poset **F**:

–fibre at I is the poset $\mathbf{F}(I)$ of Scott closed subsets (see 1.10(iv))

–$\mathbf{F}(\alpha)$ is inverse image along α, α^{-1} (see 2.2(iv)).

7.3 Definition. Let **C** be a category and **X** be a **C**-indexed preordered set. Say that **X** *has finite meets* (respectively *finite joins* or *Heyting implication*) if for each $I \in \mathrm{ob}\mathbf{C}$, the preordered set $\mathbf{X}(I)$ has finite meets (respectively finite joins, or Heyting implication) which are preserved by each monotone function $\alpha^*: \mathbf{X}(J) \longrightarrow \mathbf{X}(I)$ ($\alpha: I \longrightarrow J$ in **C**).

Top and bottom in $\mathbf{X}(I)$ will be denoted by $\top$ and $\bot$ as usual; similarly $\wedge$ and $\vee$ will be used for binary meets and joins and $\Rightarrow$ for Heyting implication. Then preservation of these operations by the α^* means

$$\alpha^*(\top) \cong \top,\ \alpha^*(x \wedge x') \cong \alpha^*(x) \wedge \alpha^*(x'),\ \alpha^*(\bot) \cong \bot,\ \alpha^*(x \vee x') \cong \alpha^*(x) \vee \alpha^*(x')$$
$$\text{and}\quad \alpha^*(x \Rightarrow x') \cong \alpha^*(x) \Rightarrow \alpha^*(x').$$

7.4 Examples. Ⓣ Ⓚ Ⓢ Referring to 7.2, the **Set**-indexed preordered sets **P** and **R** have finite meets, finite joins and Heyting implications. The **Dcpo**-indexed poset **F** has finite meets and finite joins; however, it does not have Heyting implications.

Next we give the **C**-indexed versions of monotone functions and adjoints.

7.5 Definitions. (i) Let **X** and **Y** be **C**-indexed preordered sets. A **C**-*indexed monotone function* $f: \mathbf{X} \longrightarrow \mathbf{Y}$ is specified by

- a monotone function $f_I: \mathbf{X}(I) \longrightarrow \mathbf{Y}(I)$ for each $I \in \mathrm{ob}\mathbf{C}$

satisfying that for each $\alpha: I \longrightarrow J$ in **C**, $f_I \circ \alpha^* \cong \alpha^* \circ f_J$. (Thus f is something weaker than a natural transformation, but if **Y** is actually a **C**-indexed *poset*, then the f_I define a natural transformation from **X** to **Y**.)

Evidently such **C**-indexed monotone functions can be composed: $(g \circ f)_I = g_I \circ f_I$; and there is an identity at **X**, $\mathrm{id}_{\mathbf{X}}$, for this composition: $(\mathrm{id}_{\mathbf{X}})_I = \mathrm{id}_{\mathbf{X}(I)}$. Thus we get a category of **C**-indexed preordered sets and **C**-indexed monotone functions, denoted **C**-**Preord**.

The collection **C**-**Preord**(**X**,**Y**) of all **C**-indexed monotone functions from **X** to **Y** is itself preordered by defining

$$f \leq g \quad \text{if and only if} \quad \text{for all } I \in \mathrm{ob}\mathbf{C},\ f_I \leq g_I \text{ in } [\mathbf{X}(I), \mathbf{Y}(I)].$$

(ii) Given $f: \mathbf{X} \longrightarrow \mathbf{Y}$ and $g: \mathbf{Y} \longrightarrow \mathbf{X}$ in **C**-**Preord**, say that *f is left adjoint to g* (and *g is right adjoint to f*) and write $f \dashv g$, if

$$f \circ g \leq \mathrm{id}_Y \text{ in } \mathbf{C}\text{-}\mathbf{Preord}(\mathbf{Y},\mathbf{Y}) \quad \text{and} \quad \mathrm{id}_X \leq g \circ f \text{ in } \mathbf{C}\text{-}\mathbf{Preord}(\mathbf{X},\mathbf{X}).$$

7.6 Proposition. (Analogue of 3.3) *A* **C**-*indexed monotone function* $g: \mathbf{Y} \longrightarrow \mathbf{X}$ *has a left adjoint if and only if for each* $I \in \mathrm{ob}\mathbf{C}$ *there is a function assigning to each* $x \in \mathbf{X}(I)$ *an element* $f_I(x) \in \mathbf{Y}(I)$ *satisfying*

- *$x \le g_I(f_I(x))$ in $\mathbf{X}(I)$*
- *for all $y \in \mathbf{Y}(I)$, if $x \le g_I(y)$ in $\mathbf{X}(I)$, then $f_I(x) \le y$ in $\mathbf{Y}(I)$*
- *for all $\alpha: I \longrightarrow J$ in $\mathbf{C}$, $\alpha^*(f_J(x)) \cong f_I(\alpha^*(x))$ in $\mathbf{Y}(J)$.*

Proof. By Proposition 3.3, the first two conditions give that f_I is a monotone function which is left adjoint to g_I. But then the last condition says that $I \longmapsto f_I$ determines a $\mathbf{C}$-indexed monotone function $\mathbf{X} \longrightarrow \mathbf{Y}$. Arguing as in Proposition 3.2, we have for each $I \in \mathrm{ob}\mathbf{C}$ that $\mathrm{id}_{X(I)} \le g_I \circ f_I$ and $f_I \circ g_I \le \mathrm{id}_{Y(I)}$; hence $\mathrm{id}_X \le g \circ f$ and $f \circ g \le \mathrm{id}_Y$, so that $f \dashv g$. □

7.7 **Remark.** In 3.8(i) top and bottom elements in a preordered set X were characterized in terms of right and left adjoints to $\langle\rangle: X \longrightarrow 1$; similarly, in 3.8(ii), binary meets and joins in X were characterized in terms of right and left adjoints to the diagonal $\Delta: X \longrightarrow X \times X$. Now that we have a notion of adjoints for $\mathbf{C}$-indexed monotone functions we can examine the analogous properties for a $\mathbf{C}$-indexed preordered set $\mathbf{X}$ of $\langle\rangle: \mathbf{X} \longrightarrow \mathbf{1}$ and $\Delta: \mathbf{X} \longrightarrow \mathbf{X} \times \mathbf{X}$, where

> $\mathbf{1}$ is the $\mathbf{C}$-indexed preordered set with $\mathbf{1}(I) = 1$ (one element poset); and
>
> $\mathbf{X} \times \mathbf{Y}$ is defined by: $(\mathbf{X} \times \mathbf{Y})(I) = \mathbf{X}(I) \times \mathbf{Y}(I)$ (product in **Preord**) and $(\mathbf{X} \times \mathbf{Y})(\alpha) = \mathbf{X}(\alpha) \times \mathbf{Y}(\alpha)$.

Using Proposition 7.6, we find in each case that the existence of such adjoints is equivalent not to just the existence of the appropriate meet or join in each fibre, but also the preservation of those meets or joins under the pullback operations. Thus for example Δ has a right adjoint iff each $\mathbf{X}(I)$ has binary meets and for each $\alpha: I \longrightarrow J$ and $y, y' \in \mathbf{X}(J)$, $\alpha^*(y \wedge y') \cong \alpha^*(y) \wedge \alpha^*(y')$ in $\mathbf{X}(I)$.

7.8 **Construction.** Let $\mathbf{C}$ be a category with binary products and $\mathbf{X}$ a $\mathbf{C}$-indexed preordered set. For each $K \in \mathrm{ob}\mathbf{C}$, we can define a new $\mathbf{C}$-indexed preordered set $\mathbf{X}^K$ as follows:

- For $I \in \mathrm{ob}\mathbf{C}$, let $\mathbf{X}^K(I) = \mathbf{X}(K \times I)$.
- For each $\alpha: I \longrightarrow J$ in $\mathbf{C}$, let $\alpha^*: \mathbf{X}^K(J) \longrightarrow \mathbf{X}^K(I)$ be $\mathbf{X}(\mathrm{id}_K \times \alpha)$.

Recall (see Glossary) that $\mathrm{id}_K \times \alpha: I \times K \longrightarrow J \times K$ denotes the unique morphism with $\pi_1 \circ (\mathrm{id}_K \times \alpha) = \mathrm{id}_K \circ \pi_1$ and $\pi_2 \circ (\mathrm{id}_K \times \alpha) = \alpha \circ \pi_2$. The uniqueness part of this property gives that $\mathrm{id}_K \times \mathrm{id}_I = \mathrm{id}_{K \times I}$ and that $(\mathrm{id}_K \times \beta) \circ (\mathrm{id}_K \times \alpha) = \mathrm{id}_K \times (\beta \circ \alpha)$. Thus these definitions do indeed give a $\mathbf{C}$-indexed preordered set.

There is a $\mathbf{C}$-indexed monotone function $\Delta: \mathbf{X} \longrightarrow \mathbf{X}^K$ whose component at $I \in \mathrm{ob}\mathbf{C}$ is

$$\Delta_I = (\pi_2)^*: \mathbf{X}(I) \longrightarrow \mathbf{X}(K \times I) = \mathbf{X}^K(I).$$

For, if $\alpha: I \longrightarrow J$ in $\mathbf{C}$, then since $\pi_2 \circ (\mathrm{id}_K \times \alpha) = \alpha \circ \pi_2$, we have

$$\Delta_I \circ \alpha^* = (\pi_2)^* \circ \alpha^* \cong (\mathrm{id}_K \times \alpha)^* \circ (\pi_2)^* = \mathbf{X}^K(\alpha) \circ \Delta_J,$$

so that the Δ_I satisfy the condition in Definition 7.5.

7.9 **Example.** Let X be a preordered set. We can construct a **Set**-indexed preordered set $(- \to X)$ out of X as follows:

- For each set I, let $(I \to X)$ be the set functions from I to X, preordered by

$$x \leq y \text{ if and only if for all } i \in I,\ x(i) \leq y(i) \text{ in } X.$$

In this context, one tends to refer to the elements of $(I \to X)$ as *I-indexed collections of elements of X*, and to write a typical element x as $(x_i \mid i \in I)$.

- For each function $\alpha: I \longrightarrow J$, $\alpha^*: (J \to X) \longrightarrow (I \to X)$ is the monotone function defined by

$$\alpha^*(y_j \mid j \in J) = (y_{\alpha(i)} \mid i \in I).$$

For a fixed set K, one can prove (**Exercise**) that the **Set**-indexed monotone function $\Delta: (- \to X) \longrightarrow (- \to X)^K$ constructed in 7.8 has a right adjoint iff the monotone function $X \longrightarrow (K \to X)$ defined by $x \longmapsto (x \mid k \in K)$ has a right adjoint, iff the meet $\bigwedge_{k \in K} x_k$ exists for any K-indexed collection $(x_k \mid k \in K)$ of elements of X. Similarly $\Delta: (- \to X) \longrightarrow (- \to X)^K$ has a left adjoint iff all K-indexed collections of elements of X have a join. This motivates the following

7.10 Definition. Let **C** be a category with binary products and let $K \in$ ob**C**. A **C**-indexed preordered set **X** *has K-indexed meets* if the **C**-indexed monotone function $\Delta: \mathbf{X} \longrightarrow \mathbf{X}^K$ constructed in 7.8 has a right adjoint, denoted $\forall_K: \mathbf{X}^K \longrightarrow \mathbf{X}$. Similarly, **X** *has K-indexed joins* if Δ has a left adjoint, denoted $\exists_K$.

In view of Proposition 7.6, these conditions on **X** amount to the following:

(i) *K-indexed meets.* For $I \in$ ob**C** and $z \in \mathbf{X}(K \times I)$, there is $\forall_{K,I}(z) \in \mathbf{X}(I)$ satisfying

- $(\pi_2)^*(\forall_{K,I}(z)) \leq z$ in $\mathbf{X}(K \times I)$ (where $\pi_2: K \times I \longrightarrow I$) and
- if $x \in \mathbf{X}(I)$ and $(\pi_2)^*(x) \leq z$ in $\mathbf{X}(K \times I)$, then $x \leq \forall_{K,I}(z)$ in $\mathbf{X}(I)$.

Moreover

- for each $\alpha: I \longrightarrow J$ in **C**, $\alpha^*(\forall_{K,J}(z)) \cong \forall_{K,I}\big((\mathrm{id}_K \times \alpha)^*(z)\big)$, all $z \in \mathbf{X}(K \times J)$.

(The first two conditions say that $\forall_{K,I}$ is right adjoint to π_2; the third condition says that the right adjoint is 'stable'.)

(ii) *K-indexed joins.* For $I \in$ ob**C** and $z \in \mathbf{X}(K \times I)$, there is $\exists_{K,I}(z) \in \mathbf{X}(I)$ satisfying

- $z \leq (\pi_2)^*(\exists_{K,I}(z))$ in $\mathbf{X}(K \times I)$ (where $\pi_2: K \times I \longrightarrow I$) and
- if $x \in \mathbf{X}(I)$ and $z \leq (\pi_2)^*(x)$ in $\mathbf{X}(K \times I)$, then $\exists_{K,I}(z) \leq x$ in $\mathbf{X}(I)$.

Moreover

- for each $\alpha: I \longrightarrow J$ in **C**, $\alpha^*(\exists_{K,J}(z)) \cong \exists_{K,I}\big((\mathrm{id}_K \times \alpha)^*(z)\big)$, all $z \in \mathbf{X}(K \times J)$.

(The first two conditions say that $\exists_{K,I}$ is left adjoint to π_2 ; the third condition says that the left adjoint is 'stable'.)

7.11 Examples. (i) Ⓣ We saw in 3.8(iii) that for each $\alpha: I \longrightarrow J$ in **Set**, $\alpha^{-1}: \mathbf{P}(J) \longrightarrow \mathbf{P}(I)$ has both left $(\exists\alpha)$ and right $(\forall\alpha)$ adjoints. In particular for $\pi_2: K \times I \longrightarrow I$, and $A \subseteq K \times I$

$$\exists_{K,I}(A) = \exists\pi_2(A) = \{i \in I \mid \exists k \in K\ (k, i) \in A)\}$$
$$\forall_{K,I}(A) = \forall\pi_2(A) = \{i \in I \mid \forall k \in K\ (k, i) \in A)\}$$

Moreover, if $\alpha: I \longrightarrow J$ and $S \subseteq K \times J$, then

$$\begin{aligned}\alpha^{-1}(\exists_{K,J}(S)) &= \{i \in I \mid \exists k \in K\ (k, \alpha(i)) \in S\} \\ &= \{i \in I \mid \exists k \in K\ (k, i) \in (\mathrm{id}_K \times \alpha)^{-1} S\} \\ &= \exists_{K,I}\big((\mathrm{id}_K \times \alpha)^{-1}(S)\big),\end{aligned}$$

and similarly for $\forall$. Thus **P** has K-indexed joins and meets for all $K \in \mathrm{ob}\mathbf{Set}$.

(ii) Ⓚ The **Set**-indexed preordered set **R** has K-indexed meets and joins for all sets K. From 3.8(vi) we know that the adjoints $\exists_{K,I}$ and $\forall_{K,I}$ exist for all sets K and I; indeed for $p \in (K \times I \to \mathbf{PN})$ we have

$$\exists_{K,I}(p) = \exists\pi_2 : i \in I \longmapsto \bigcup_{k \in K} p(k, i)$$

and

$$\forall\pi_2(p) : i \in I \longmapsto \bigcap_{(k,i') \in K \times I} \delta_I(i', i) \Rightarrow p(k, i'),$$

where the second expression uses the result at the end of 3.15(iv) for $\forall\pi_2$. In fact since we are dealing with a projection π_2 rather than a general function α, the expression for $\forall\pi_2$ can be replaced by a simpler, isomorphic one—namely

$$\forall_{K,I}(p) : i \in I \longmapsto \bigcap_{k \in K} p(k, i).$$

With these particular expressions for the left and right adjoints to $\pi_2 : K \times I \longrightarrow I$, it is easy to calculate (same calculation as in (i), in fact) that they are actually stable up to equality, *i.e.* for all functions $\alpha: I \longrightarrow J$ we have

$$\alpha^* \circ \exists_{K,J} = \exists_{K,I} \circ (\mathrm{id}_K \times \alpha)^* \quad \text{and} \quad \alpha^* \circ \forall_{K,J} = \forall_{K,I} \circ (\mathrm{id}_K \times \alpha)^*.$$

(iii) Ⓢ Consider the **Dcpo**-indexed poset **F** of Example 7.2(iv). In 3.8(v) we saw for each morphism $\alpha: I \longrightarrow J$ in **Dcpo** that $\alpha^{-1} : \mathbf{F}(J) \longrightarrow \mathbf{F}(I)$ has a left adjoint $\alpha_!$, sending a Scott closed subset S of I to the closure of the direct image $\alpha(S) = \{\alpha(i) \mid i \in S\}$. One might therefore expect that **F** has all K-indexed joins ($K \in \mathrm{ob}\mathbf{Dcpo}$). This is not the case, because in general *the left adjoints* $(\pi_2)_!$ *to projections fail the third, stability condition in* 7.10(ii).

(**Proof.** Suppose $\exists_{K,J}$ were stable. Then for each $j \in J$ and $S \in \mathbf{F}(K \times J)$, the stability condition for $\alpha: 1 \longrightarrow J$ where α sends the unique element of 1 to j (such an α is continuous) implies that $j \in (\pi_2)_!(S)$ iff for some $k \in K$, $(k, j) \in S$. In other words we would have that $\exists\pi_2 : \mathbf{P}(K \times J) \longrightarrow \mathbf{P}(J)$ takes Scott closed subsets to Scott closed subsets. But this is by no means the case for all choices of dcpo's K and J. For example, let $K = \mathrm{N} \cup \{\infty\}$ made into a ('discrete') dcpo by taking equality as the partial order; and let $J = \mathrm{N} \cup \{\infty\}$ made into a dcpo by taking the usual total ordering on N and adding ∞ as a top element. Then $S = \{(n, m) \in \mathrm{N} \times \mathrm{N} \mid n \geq m\}$ is a Scott closed subset of $K \times J$ (**Exercise**!), but $\pi_2(S) = \{j \in J \mid j \neq \infty\}$ is not a Scott closed subset of J. □)

Thus **F** does not have K-indexed joins for all $K \in \mathrm{ob}\mathbf{Dcpo}$ even though left adjoints to $\mathbf{F}(\alpha)$ exist for all continuous α. In contrast, **F** *does have K-indexed meets for all* $K \in \mathrm{ob}\mathbf{Dcpo}$, even though right adjoints to $\mathbf{F}(\alpha)$ do not exist for all α. (**Exercise.** With J and K as in the previous paragraph, show that the identity function on $\mathrm{N} \cup \{\infty\}$ gives a morphism $\alpha: K \longrightarrow J$ in **Dcpo** for which $\mathbf{F}(\alpha)$ does not have a right adjoint.) For one can prove (**Exercise**) that for all dcpo's K and I, the dual image function $\forall\pi_2 : \mathbf{P}(K \times I) \longrightarrow \mathbf{P}(I)$ takes Scott closed subsets of $K \times I$ to Scott closed subsets of I and hence restricts to give a right adjoint to $(\pi_2)^{-1} : \mathbf{F}(I) \longrightarrow \mathbf{F}(K \times I)$ which inherits the third, stability condition in 7.10(i) from **P**.

To conclude this section we draw together the structure we will use to interpret first-order predicate logic (without equality) and give it a name—a rather ugly one, unfortunately.

7.12 Definition. A *first-order hyperdoctrine* (**C**,**H**) is specified by a category **C** with finite products (called the *base category* of the hyperdoctrine) together with a **C**-indexed preordered set **H** having finite meets, finite joins, implications and K-indexed meets and joins for all $K \in \mathrm{ob}\mathbf{C}$.

Thus of the examples in 7.11, (**Set**,**P**) and (**Set**,**R**) are first-order hyperdoctrines, but (**Dcpo**,**F**) is not.

8 First-Order Logic

In this section we will use the structures of the previous section to give the 'hyperdoctrine' style semantics of first order logic (without equality—the semantics of equality is discussed in section 10).

8.1 Definitions. Let us augment the notion of *signature* given in 6.1 by allowing not just sorts and function symbols, but also *relation symbols*. Each such relation symbol in a signature is to come supplied with a *type*, which in this case is a finite list of sorts. The notation

$$R \subseteq A_1 \dots A_n$$

will be used to indicate that R has type $A_1 \dots A_n$. (**Remark.** The case $n=0$ is allowed, in which case R amounts to an atomic proposition in the sense of section 4.)

The terms of various sorts for such a signature Σ are just as before, in 6.2. Now however, we can also form formulas. The *(first-order) formulas* ϕ over Σ are defined recursively by the following clauses; at the same time we define the finite set of *free variables* of a formula ϕ:

- *Atomic formulas.* $R(t_1, \dots, t_n)$ is a formula for each relation symbol $R \subseteq A_1 \dots A_n$ and terms $t_1 : A_1, \dots, t_n : A_n$; its free variables are all variables occurring in the t_i.
- *Truth and falsity.* $\top$ and $\bot$ are formulas, with no free variables.
- *Negation.* $\neg\phi$ is a formula if ϕ is, with the same free variables.
- *Conjunction, disjunction and implication.* $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \Rightarrow \psi$ are formulas if ϕ and ψ are; their free variables are those of ϕ or ψ.
- *Universal and existential quantification.* $\forall x : A . \phi$ and $\exists x : A . \phi$ are formulas if ϕ is a formula and x is a variable of sort A; their free variables are all those of ϕ *except* x.

With the notion of *context* as in 6.3, we will say that $\phi(\mathbf{x})$ is a *formula-in-context* if ϕ is a formula whose free variables occur in $\mathbf{x}$. Thus for example, if $\phi(x, \mathbf{x})$ is a formula-in-context (with $x : A$ say), then so is $(\exists x : A . \phi)(\mathbf{x})$.

Now suppose that $(\mathbf{C}, \mathbf{H})$ is a first-order hyperdoctrine (see Definition 7.12). A *structure* in the hyperdoctrine for a signature Σ is specified by

- a structure M (in the sense of section 6) in $\mathbf{C}$ for the sorts and function symbols of Σ
- an element $MR \in \mathbf{H}(\prod M\mathrm{A})$ in the fibre of $\mathbf{H}$ at $\prod M\mathrm{A} (= MA_1 \times \dots \times MA_n$, when A is $A_1 \dots A_n)$ for each relation symbol $R \subseteq \mathrm{A}$ in Σ.

Then for each formula-in-context $\phi(\mathrm{x})$ over Σ, where $\mathrm{x}:\mathrm{A}$ say, we can define an element

$$[\![\phi(\mathrm{x})]\!]_M \in \mathbf{H}\big(\textstyle\prod M\mathrm{A}\big)$$

by recursion on the structure of ϕ as in Table 8.2 (using the semantics of terms-in-context given in Table 6.4). Note that the semantics of first-order propositions given in section 4 is a particular case of the semantics of formulas—the case where Σ contains no sorts or function symbols and hence amounts to a set of atomic propositions (*cf.* the **Remark** above).

- If ϕ is $R(t_1,\dots,t_m)$, then $[\![\phi(\mathrm{x})]\!]_M$ is $\langle[\![t_1(\mathrm{x})]\!]_M,\dots,[\![t_m(\mathrm{x})]\!]_M\rangle^*(MR)$, the pullback of $MR\in\mathbf{H}\big(\prod M\mathbf{B}\big)$ along $\langle[\![t_1(\mathrm{x})]\!]_M,\dots,[\![t_m(\mathrm{x})]\!]_M\rangle$: $\prod M\mathrm{A}\longrightarrow\prod M\mathbf{B}$ (where $t_j:B_j$ and $\mathbf{B}=B_1,\dots,B_m$).
- If ϕ is $\top$, then $[\![\phi(\mathrm{x})]\!]_M$ is $\top$, the top element in $\mathbf{H}\big(\prod M\mathrm{A}\big)$.
- If ϕ is $\bot$, then $[\![\phi(\mathrm{x})]\!]_M$ is $\bot$, the bottom element in $\mathbf{H}\big(\prod M\mathrm{A}\big)$.
- If ϕ is $\neg\psi$, then $[\![\phi(\mathrm{x})]\!]_M$ is the pseudocomplement $\neg[\![\psi(\mathrm{x})]\!]_M$.
- If ϕ is $\psi\wedge\theta$, then $[\![\phi(\mathrm{x})]\!]_M$ is the binary meet $[\![\psi(\mathrm{x})]\!]_M\wedge[\![\theta(\mathrm{x})]\!]_M$.
- If ϕ is $\psi\vee\theta$, then $[\![\phi(\mathrm{x})]\!]_M$ is the binary join $[\![\psi(\mathrm{x})]\!]_M\vee[\![\theta(\mathrm{x})]\!]_M$.
- If ϕ is $\psi\Rightarrow\theta$, then $[\![\phi(\mathrm{x})]\!]_M$ is the Heyting implication $[\![\psi(\mathrm{x})]\!]_M\Rightarrow[\![\theta(\mathrm{x})]\!]_M$.
- If ϕ is $\exists x:A.\psi$, then $[\![\phi(\mathrm{x})]\!]_M$ is the MA-indexed join $\exists_{MA,I}\big([\![\psi(x,\mathrm{x})]\!]_M\big)$ (where I stands for the product $\prod M\mathrm{A}$).
- If ϕ is $\forall x:A.\psi$, then $[\![\phi(\mathrm{x})]\!]_M$ is the MA-indexed meet $\forall_{MA,I}\big([\![\psi(x,\mathrm{x})]\!]_M\big)$ (where I stands for the product $\prod M\mathrm{A}$).

8.2 Table: categorical semantics of first-order formulas-in-context

8.3 Examples. (i) Ⓣ The **Set**-indexed poset **P** of 7.2(i) is a first-order hyperdoctrine. The notion of a structure in (**Set**, **P**) amounts to the usual notion from classical model theory of a (many-sorted, set-valued) structure. If M is such a structure, then each $[\![\phi(\mathrm{x})]\!]_M$ is a subset of the cartesian product $\prod M\mathrm{A}$: it is easy to prove that a tuple $\mathbf{a}\in\prod M\mathrm{A}$ is in the subset if and only if the structure M *satisfies* the sentence $\phi(\mathbf{a})$ in the classical, Tarskian sense (see [CK, 1.3]).

(ii) Ⓚ The **Set**-indexed preordered set **R** of 7.2(iii) is a first-order hyperdoctrine. We remarked in 4.2(ii) that the semantics of the propositional connectives in **R** amounts to Kleene's '1945-realizability'. Now we also have a semantics for quantification over the elements of a set. Each $[\![\phi(\mathrm{x})]\!]_M$ is a function $\prod M\mathrm{A}\longrightarrow\mathbf{PN}$.

Writing ' $n \mathbf{r}\ \phi(\mathbf{a})$ ' for ' $n \in [\![\phi(\mathbf{x})]\!]_M(\mathbf{a})$ ' we find

- $n \mathbf{r}\ \exists x{:}A.\phi(\mathbf{a})$ iff for some $a \in MA$, $n \mathbf{r}\ \phi(a,\mathbf{a})$
- $n \mathbf{r}\ \forall x{:}A.\phi(\mathbf{a})$ iff for all $a \in MA$, $n \mathbf{r}\ \phi(a,\mathbf{a})$.

Warning. If MA is countable, this kind of quantification is not the same as 'quantification over the natural numbers' in the world Ⓚ, for which the clauses are

- $n \mathbf{r}\ \exists m.\phi(\mathbf{a})$ iff for some m, $\langle n,m\rangle \mathbf{r}\ \phi(m,\mathbf{a})$
- $n \mathbf{r}\ \forall m.\phi(\mathbf{a})$ iff for all m, $n{\cdot}m\downarrow$ and $n{\cdot}m \mathbf{r}\ \phi(m,\mathbf{a})$.

(iii) Ⓢ The **Dcpo**-indexed poset **F** of 7.2(iv) is not a first-order hyperdoctrine. However, it does support the interpretation of the fragment of first-order logic without $\Rightarrow$ or $\exists$.

8.4 Definition. Let us extend the notion of 'sequent' from section 4 to first-order formulas-in-context. A first-order *sequent-in-context* over a signature Σ is of the form

$$\Gamma \vdash \phi \ (\mathbf{x})$$

where Γ is a finite set of formulas over Σ, ϕ is a single formula over Σ and $\mathbf{x}$ is a context containing (at least) all the free variables in Γ or ϕ. If Γ,ϕ contain no free variables then the sequent in the empty context, $\Gamma \vdash \phi$ ([]), will be written just as $\Gamma \vdash \phi$.

A structure M for Σ in a first-order hyperdoctrine $(\mathbf{C},\mathbf{H})$ *satisfies* such a sequent-in-context if

$$\textstyle\bigwedge_{\gamma\in\Gamma} [\![\gamma(\mathbf{x})]\!]_M \le [\![\phi(\mathbf{x})]\!]_M$$

in $\mathbf{H}(\prod M\mathbf{A})$ (where $\mathbf{x}{:}\mathbf{A}$, say).

Our immediate aim is to extend the results of sections 4 and 6 to show that this notion of satisfaction is sound for *Intuitionistic Predicate Logic* **IPC**. At the moment we are not considering first-order logic with *equality* (*predicates*)—for which one would have distinguished relation symbols $=_A \subseteq A\ A$ for each sort in the clauses for formula formation. (Logics without equality are of relevance to Ⓢ.) This means that the judgement

$$s = t \ (\mathbf{x})$$

that two terms (of the same sort, A say) are equal cannot be replaced by the sequent

$$\emptyset \vdash =_A(s,t) \ (\mathbf{x}).$$

So we will consider systems for deriving both equations-in-context and sequents-in-context simultaneously. 'Adjoint-style' rules for **IPC** are given in in Table 8.5, extending both the propositional rules of Table 4.8 and the equational logic of Table 6.7. (An equivalent, Gentzen-style calculus extending Table 4.3 could be given.)

$$(\mathrm{Wk})\begin{cases} \dfrac{\Gamma\vdash\phi\ (\mathbf{x})}{\Gamma\vdash\phi\ (\mathbf{y})}(\mathbf{x}\subseteq\mathbf{y}) \\[2ex] \dfrac{s=t\ (\mathbf{x})}{s=t\ (\mathbf{y})}(\mathbf{x}\subseteq\mathbf{y}) \end{cases} \qquad (\mathrm{Sub})\begin{cases} \dfrac{s=s'\ (\mathbf{x}) \qquad \Gamma\vdash\phi\ (y,\mathbf{x})}{\Gamma(s/y)\vdash\phi(s'/y)\ (\mathbf{x})} \\[2ex] \dfrac{s=s'\ (\mathbf{x}) \qquad t=t'\ (y,\mathbf{x})}{t(s/y)=t'(s'/y)\ (\mathbf{x})} \end{cases}$$

$$(\mathrm{Id})\begin{cases} \dfrac{}{\Gamma,\phi\vdash\phi\ (\mathbf{x})} \\[2ex] \dfrac{}{t=t\ (\mathbf{x})} \end{cases} \qquad (\mathrm{Cut})\begin{cases} \dfrac{\Gamma\vdash\phi\ (\mathbf{x}) \qquad \Delta,\phi\vdash\psi\ (\mathbf{x})}{\Delta,\Gamma\vdash\psi\ (\mathbf{x})} \\[2ex] \dfrac{r=s\ (\mathbf{x}) \qquad s=t\ (\mathbf{x})}{r=t\ (\mathbf{x})} \end{cases}$$

$$(\mathrm{Sym})\dfrac{s=t\ (\mathbf{x})}{t=s\ (\mathbf{x})}$$

$$(\top)\begin{array}{c} \\ \hline\hline \Gamma\vdash\top \end{array} \qquad (\wedge)\begin{array}{c} \Gamma\vdash\phi \quad \Gamma\vdash\psi \\ \hline\hline \Gamma\vdash\phi\wedge\psi \end{array}$$

$$(\bot)\begin{array}{c} \\ \hline\hline \Gamma,\bot\vdash\phi \end{array} \qquad (\vee)\begin{array}{c} \Gamma,\phi\vdash\theta \quad \Gamma,\psi\vdash\theta \\ \hline\hline \Gamma,\phi\vee\psi\vdash\theta \end{array}$$

$$(\Rightarrow)\begin{array}{c} \Gamma,\phi\vdash\psi \\ \hline\hline \Gamma\vdash\phi\Rightarrow\psi \end{array} \qquad (\neg)\begin{array}{c} \Gamma,\phi\vdash\bot \\ \hline\hline \Gamma\vdash\neg\phi \end{array}$$

$$(\exists)\begin{array}{c} \Gamma,\phi\vdash\psi\ (y,\mathbf{x}) \\ \hline\hline \Gamma,\exists y:A.\phi\vdash\psi\ (\mathbf{x}) \end{array} \qquad (\forall)\begin{array}{c} \Gamma\vdash\phi\ (y,\mathbf{x}) \\ \hline\hline \Gamma\vdash\forall y:A.\phi\ (\mathbf{x}) \end{array}$$

8.5 Table: Adjoint Calculus for IPC

The word *judgement* will be used to refer to something which is either a sequent-in-context or an equation-in-context. The rules in Table 8.5 are of two forms:

$$\frac{\text{judgements}}{\text{judgement}} \quad \text{and} \quad \begin{array}{c}\text{judgements}\\ \hline\hline \text{judgement}\end{array}.$$

A collection of judgements is said to be *closed* under a rule of the first kind if whenever it contains the judgements above the line, it also contains the judgement below the line. The collection of judgements is *closed* under a rule of the second kind if the collection contains the judgements above the double line if and only if it contains the judgement below the double line.

8.5 Remarks. (i) **Side conditions.** It should be noted that in closing under a set of rules, only *well-formed* judgements are considered (that is, ones conforming to Definitions 6.5 and 8.5). This means that some side conditions are implicit in some of the rules. For example in the ($\exists$) and ($\forall$) rules we must have that $y\notin\mathbf{x}$ (for $y,\mathbf{x}$ to be a well-formed context) and that y is not a free variable of Γ or ψ (for the sequent-in-context below the line to be well-formed). And in the (Sub) rules $y\notin\mathbf{x}$ and (hence) y is not a variable occurring in s or s'.

(ii) **Substitution.** The first (Sub) rule in Table 8.5 uses the substitution of terms for variables in formulas. The usual complications enter in through formulas containing variables which are bound by quantifiers. We are only interested in formulas up to 'α-conversion'—changing the names of bound variables. In making a substitution of s for y in ϕ, resulting in $\phi(s/y)$, we assume that the bound variables of ϕ are all distinct and different from the variables in s, so that 'y is free for s in ϕ' and no variable of σ is 'captured' by the scope of a quantifier in ϕ. (We also assume, of course, that s and y are of the same sort).

(iii) **Quantification.** The rules ($\exists$) and ($\forall$) give a formulation of the rules for quantifiers which directly embodies the nature of quantification as adjoint to weakening (*cf.* 3.8(iii)). Equivalent and possibly more familiar looking rules are

$$(\exists')\left\{\begin{array}{c}\dfrac{\Gamma,\phi\vdash\psi\ (y,\mathrm{x})}{\Gamma,\exists y{:}A.\phi\vdash\psi\ (\mathrm{x})}\\[2ex]\dfrac{\Gamma\vdash\phi(s/y)\ (\mathrm{x})}{\Gamma\vdash\exists y{:}A.\phi\ (\mathrm{x})}\end{array}\right.\qquad(\forall')\left\{\begin{array}{c}\dfrac{\Gamma\vdash\phi\ (y,\mathrm{x})}{\Gamma\vdash\forall y{:}A.\phi\ (\mathrm{x})}\\[2ex]\dfrac{\Gamma,\phi(s/y)\vdash\psi\ (\mathrm{x})}{\Gamma,\forall y{:}A.\phi\vdash\psi\ (\mathrm{x})}\end{array}\right.$$

where $s(\mathrm{x})$ is a term-in-context of the same sort as y. (**Exercise.** Show that modulo the other rules in Table 8.5, ($\exists$) can be derived from ($\exists'$) and *vice versa*; show the same for ($\forall$) and ($\forall'$).)

In order to prove that the rules in Table 8.5 are sound for satisfaction of equations-in-context (*cf.* 6.5) in the base category of a hyperdoctrine and satisfaction of sequents-in-context (*cf.* 8.4) in the fibres, we need the following extension of the first clause in Table 8.2.

8.6 Substitution Lemma. *Suppose that $\phi(\mathrm{x})$ is a formula-in-context, with $\mathrm{x}=x_1,\dots,x_n$ and $x_i{:}A_i$ say. Suppose also that $s_i(\mathrm{y})$ is a term-in-context of sort A_i with $\mathrm{y}=y_1,\dots,y_m$ and $y_j{:}B_j$ say. Let $\phi(\mathrm{s}(\mathrm{y}))$ denote the formula (in context y) resulting from simultaneously substituting s_i for x_i in ϕ. If M is a structure in a first-order hyperdoctrine $(\mathbf{C},\mathbf{H})$, then*

$$[\![\phi(\mathrm{s}(\mathrm{y}))]\!]_M\cong\langle[\![s_1(\mathrm{y})]\!]_M,\dots,[\![s_n(\mathrm{y})]\!]_M\rangle^*\big([\![\phi(\mathrm{x})]\!]_M\big)$$

in $\mathbf{H}(MB_1\times\cdots\times MB_m)$.

Proof. The proof is by induction on the structure of ϕ.

If ϕ is atomic, say $R(t_1,\dots,t_r)$, then by the substitution lemma for terms (Lemma 6.9) we have $[\![t_k(\mathrm{s}(\mathrm{x}))]\!]=[\![t_k(\mathrm{y})]\!]\circ\langle[\![s_1(\mathrm{y})]\!]_M,\dots,[\![s_n(\mathrm{y})]\!]_M\rangle$, so that

$$\begin{aligned}[\![\phi(\mathrm{s}(\mathrm{x}))]\!]_M=[\![R(\mathrm{t}(\mathrm{s}(\mathrm{x})))]\!]&=\langle[\![t_1(\mathrm{s}(\mathrm{y}))]\!],\dots,[\![t_r(\mathrm{s}(\mathrm{y}))]\!]\rangle^*(MR)\\&=\big(\langle[\![t_1(\mathrm{x})]\!],\dots,[\![t_r(\mathrm{x})]\!]\rangle\circ\langle[\![s_1(\mathrm{y})]\!],\dots,[\![s_n(\mathrm{y})]\!]\rangle\big)^*(MR)\\&\cong\langle[\![s_1(\mathrm{y})]\!],\dots,[\![s_n(\mathrm{y})]\!]\rangle^*\big(\langle[\![t_1(\mathrm{x})]\!],\dots,[\![t_r(\mathrm{x})]\!]\rangle^*(MR)\big)\\&=\langle[\![s_1(\mathrm{y})]\!],\dots,[\![s_n(\mathrm{y})]\!]\rangle^*\big([\![R(\mathrm{t}(\mathrm{x}))]\!]\big)\\&=\langle[\![s_1(\mathrm{y})]\!],\dots,[\![s_n(\mathrm{y})]\!]\rangle^*\big([\![\phi(\mathrm{x})]\!]\big).\end{aligned}$$

If ϕ is $\top,\bot,\neg\psi,\psi\wedge\theta,\psi\vee\theta$, or $\psi\Rightarrow\theta$, then the result follows from the induction hypothesis plus the fact that the corresponding order-theoretic operations in the fibres of $\mathbf{H}$ are preserved by the pullback functions α^*.

If ϕ is $\exists x{:}A.\psi$ or $\forall x{:}A.\psi$, then the result follows from the induction hypothesis plus the stability property of $\exists_{K,I}$ and $\forall_{K,I}$ with respect to the pullback functions α^* (see the third clauses of (i) and (ii) in Definition 7.10). □

8.7 Corollary. **(Weakening)** *Suppose that* $\phi(\mathbf{x})$ *is a formula-in-context and that* $\mathbf{y}$ *is another context containing all the variables in* $\mathbf{x}$. *Then*

$$[\![\phi(\mathbf{y})]\!] \cong \pi^*\big([\![\phi(\mathbf{x})]\!]\big)$$

where π *is the unique morphism in* **C** *whose composition with the* i^{th} *product projection morphism* π_i *is* $\pi_{\alpha(i)}$, *with* $\alpha(i)$ *defined so that* $y_{\alpha(i)}$ *is the (unique) member of* $\mathbf{y}$ *equal to* x_i.

Proof. This is the special case of the Substitution Lemma with $s_i = y_{\alpha(i)}$. □

8.8 Proposition. **(Soundness)** *If* Σ *is a signature and* M *is a structure for* Σ *in a first-order hyperdoctrine, then the collection of judgements satisfied by* M *is closed under the rules for Intuitionistic Predicate Logic in Table 8.5.*

Proof. Closure under the rules coming from propositional and equational logics are covered by the proofs of Propositions 4.7 and 6.8 respectively. That leaves the rules for quantifiers and the first rules in (Wk) and (Sub). The first (Sub) rule is a consequence of Lemma 8.6 and the first (Wk) rule follows from Corollary 8.7.

For closure under ($\forall$), note that by Corollary 8.7 when $y \notin \mathbf{x}$ and is not a free variable of γ, then $[\![\gamma(y,\mathbf{x})]\!] \cong (\pi_2)^*[\![\gamma(\mathbf{x})]\!]$. Thus a structure satisfies $\Gamma \vdash \phi\ (y,\mathbf{x})$ iff $(\pi_2)^*\big(\bigwedge_{\gamma\in\Gamma}[\![\gamma(\mathbf{x})]\!]\big) \cong \bigwedge_{\gamma\in\Gamma}(\pi_2)^*[\![\gamma(\mathbf{x})]\!] \leq [\![\phi(y,\mathbf{x})]\!]$ in $\mathbf{H}\big(MA\times\prod M\mathbf{A}\big)$ (where $y{:}A$, $\mathbf{x}{:}\mathbf{A}$ say, and $I=\prod M\mathbf{A}$), iff $\bigwedge_{\gamma\in\Gamma}[\![\gamma(\mathbf{x})]\!] \leq \forall_{MA,I}[\![\phi(y,\mathbf{x})]\!]$ in $\mathbf{H}\big(\prod M\mathbf{A}\big)$ (since $\forall_{MA,I}$ is right adjoint to π_2), iff the structure satisfies $\Gamma \vdash \forall y{:}A.\phi\ (\mathbf{x})$.

The argument for closure under ($\exists$) is similar to that for ($\forall$) except that we need to use not only the fact that $\exists_{MA,I}$ is left adjoint to $(\pi_2)^*$, but also the stability property of the adjoint called 'Frobenius Reciprocity', which we saw in 3.10 is a consequence of $(\pi_2)^*$ preserving $\Rightarrow$. □

8.9 Remarks. (i) **Classifying hyperdoctrines.** The constructions in 4.9 of the classifying preorder of a propositional theory and in 6.12(ii) of the classifying category of an algebraic theory can be synthesised into the construction of a *classifying hyperdoctrine* $(\mathbf{C_T},\mathbf{H_T})$ of a *first-order theory* **T**. Such a theory is specified by a signature and a set of axioms, which are judgements (equations-in-context and sequents-in-context) over the signature; the *theorems* of **T** comprise the least set of judgements containing the axioms and closed under the rules in Table 8.5.

Then $\mathbf{C_T}$ is the classifying category (with finite products) for the 'algebraic part' of **T**, constructed as indicated in 6.12(ii): objects are contexts and morphisms are equivalence classes of lists of terms-in-context under the equivalence relation given by the equations-in-context which are theorems of **T**.

To get the fibre of $\mathbf{H_T}$ at an object (*i.e.* a context) $\mathbf{x}$, take the formulas in context $\mathbf{x}$ preordered by the relation given by the sequents in context $\mathbf{x}$ which are theorems of **T**; then let $\mathbf{H_T}(\mathbf{x})$ be the poset reflection of this preordered set.

The reason why we have to take poset reflections becomes apparent when we next define the pullback functions $\alpha^*: \mathbf{H_T}(y) \longrightarrow \mathbf{H_T}(x)$ associated to the morphisms $\alpha: x \longrightarrow y$ in $\mathbf{C_T}$. These are induced by the operation of substituting a term for a free variable in a formula. Since we take equivalence classes of terms under T-provable equality to get the morphisms of $\mathbf{C_T}$, the first (Sub) rule in Table 8.5 gives that α^* is well-defined on formulas (via substitution of a representative term) only up to T-provable isomorphism: $\phi(x) \cong \psi(x)$ iff $\phi \vdash \psi\ (x)$ and $\psi \vdash \phi\ (x)$ are theorems of **T**).

The $\mathbf{C_T}$-indexed poset $\mathbf{H_T}$ which results from this construction does indeed make $(\mathbf{C_T}, \mathbf{H_T})$ a first-order hyperdoctrine. It contains a structure for the signature of **T** which satisfies exactly those judgements which are theorems of **T**. And $(\mathbf{C_T}, \mathbf{H_T})$ enjoys a universal property with respect to other first-order hyperdoctrines whereby models of **T** in a hyperdoctrine $(\mathbf{C}, \mathbf{H})$ correspond to morphisms $(\mathbf{C_T}, \mathbf{H_T}) \longrightarrow (\mathbf{C}, \mathbf{H})$. (We leave the definition of a 'morphism of first-order hyperdoctrines' to the reader's imagination.)

(ii) (**Internal languages**). By combining the constructions in 4.13 and 6.12(iii), one can prove that, up to equivalence, every first-order hyperdoctrine $(\mathbf{C}, \mathbf{H})$ (with **C** a small† category) is the classifier of a first-order theory **T**. Roughly speaking, one takes as signature for **T** sorts naming the objects of **C**, function symbols naming the morphisms of **C** and relation symbols naming the elements of the fibres of **H**. There is an evident structure for this signature in $(\mathbf{C}, \mathbf{H})$ and taking the axioms of **T** to be those judgements which are satisfied by the structure, it is the case that $(\mathbf{C_T}, \mathbf{H_T})$ is equivalent to the original hyperdoctrine $(\mathbf{C}, \mathbf{H})$.

† See Glossary.

9 Example: disjunction and explicit definability properties

First-order hyperdoctrines do more than just provide a semantics for first-order intuitionistic logic. The constructions sketched in 8.9 are the basis of a corresponence between first-order theories in **IPC** and first-order hyperdoctines which enables us to view hyperdoctrines as specifying theories independently of any particular presentation in terms of signature and axioms. Is this viewpoint useful? Although the notion of 'first-order hyperdoctrine' may seem a rather complicated one, it is essentially an *algebraic* notion, whereas the traditional notion of theory, via syntax and proof rules, is not. So we can construct new theories out of old by performing algebraic constructions on hyperdoctrines. Better still, since there is a whole *category* of theories (theory morphisms = interpretations between theories = models of theories in classifying hyperdoctrines = hyperdoctrine morphisms), constructions of theories can be specified by the typical 'universal' constructions of category theory. This enables powerful methods to be brought to bear to prove properties of *theories in general* (rather than of a particular theory, such as first-order arithmetic, or whatever). Here, in outline, is an example of this. (Other, more involved examples can be found in [Pi].)

Let **T** be a first-order theory. If ϕ is a first-order formula (over the signature of T) which contains no free variables, write

$$\mathbf{T} \vdash \phi$$

to indicate that the sequent $\emptyset \vdash \phi$ (in the empty context) is a theorem of **T**. (A formula with no free variables is usually called a *sentence*; a term with no variables is called a *closed term.*) One of the justifications for calling **IPC** a *constructive* logic (*cf.* [Dum, 6.1]) is that it has the following two properties:

9.1 Theorem. (Disjunction and Explicit Definability in IPC) *Let* **T** *be a first-order theory whose only axioms are equations-in-context. Then*

(i) **T** *has the* disjunction property: *for all sentences* ϕ *and* ψ, *if* $\mathbf{T} \vdash \phi \vee \psi$ *then* $\mathbf{T} \vdash \phi$ *or* $\mathbf{T} \vdash \psi$.

(ii) **T** *has the* explicit definability property: *if* $\exists x{:}A.\phi$ *is a sentence and* $\mathbf{T} \vdash \exists x{:}A.\phi$, *then there is some closed term* $t{:}A$ *with* $\mathbf{T} \vdash \phi(t/x)$.

Proof. We will sketch a proof of (i) and (ii) using hyperdoctrines and a particular instance of a categorical technique called 'glueing' (see [Johl, 4.2]), which is also able to prove similar results for more complicated logics (see [LS, §22] for example).

First note that under the correspondence of first-order theories with first-order hyperdoctrines indicated in 8.9, properties (i) and (ii) of a theory correspond to the following properties of a first-order hyperdoctrine $(\mathbf{C}, \mathbf{H})$:

(i)′ *For all* $\phi,\psi\in\mathbf{H}(1)$, *if* $\phi\vee\psi\cong\top$ *then* $\phi\cong\top$ *or* $\psi\cong\top$.

(ii)′ *For all* $K\in\mathrm{ob}\mathbf{C}$ *and* $\phi\in\mathbf{H}(K\times 1)$, *if* $\exists_{K,1}(\phi)\cong\top$ *then for some* $k:1\longrightarrow K$ *in* $\mathbf{C}$ $\langle k,\mathrm{id}_1\rangle^*(\phi)\cong\top$.

So we have to prove that the classifying hyperdoctrine $(\mathbf{C_T},\mathbf{H_T})$ has these properties when **T** satisfies the condition

(iii) *The only axioms of* **T** *are equations-in-context.*

This condition on **T** can be expessed equivalently as a condition on the classifying hyperdoctrine $(\mathbf{C_T},\mathbf{H_T})$, in terms of the universal property enjoyed by the latter. Indeed, (iii) means that $\mathbf{H_T}$ is 'freely generated' over $\mathbf{C_T}$ by the relation symbols in the signature of **T**:

9.2 Definition. If $(\mathbf{C},\mathbf{H})$ is a first-order hyperdoctrine and for each $I\in\mathrm{ob}\mathbf{C}$ we are given a subset $X(I)\subseteq\mathbf{H}(I)$, then **H** is *freely generated over* **C** *by the elements* $x\in X(I)$ ($I\in\mathrm{ob}\mathbf{C}$) if given any first-order hyperdoctrine **K** over **C** equipped with functions $\delta_I:X(I)\longrightarrow\mathbf{K}(I)$ ($I\in\mathrm{ob}\mathbf{C}$), there is a **C**-indexed monotone function $\bar\delta:\mathbf{H}\longrightarrow\mathbf{K}$ defined uniquely up to isomorphism by the properties

- $\bar\delta_I$ preserves finite meets, finite joins, Heyting implications and K-indexed meets and joins (all $K\in\mathrm{ob}\mathbf{C}$); and
- $\bar\delta_I(\ulcorner x\urcorner)=\delta_I(x)$ for all $I\in\mathrm{ob}\mathbf{C}$ and $x\in X(I)$.

When **T** satisfies (iii), the universal property of the classifying hyperdoctrine construction mentioned in 8.9(i) implies that $\mathbf{H_T}$ is freely generated over $\mathbf{C_T}$ by (**T**-provable equivalence classes of) all the atomic formulas $R(\mathrm{x})$, where $\mathrm{x}\in\mathrm{ob}\mathbf{C_T}$ is a context and R is a relation symbol of the appropriate sort.

So we have now translated the original problem in Theorem 9.1 into a purely category-theoretic one:

Given a first-order hyperdoctrine $(\mathbf{C},\mathbf{H})$ *with* **H** *freely generated over* **C** *by some sets of elements* $X(I)\subseteq\mathbf{H}(I)$ ($I\in\mathrm{ob}\mathbf{C}$), *prove that it satisfies* (i)′ *and* (ii)′.

(We need only prove this for the case when **C** is a small† category, since we are tacitly assuming that theories **T** contain only a set (rather than a proper class) of symbols—which means that the classifiers $\mathbf{C_T}$ are small.)

The proof goes by constructing a new first-order hyperdoctrine over **C**, $\hat{\mathbf{H}}$, which satisfies (i)′ and (ii)′ and which *contains* **H** *as a retract.* This means that there are **C**-indexed monotone functions $\iota:\mathbf{H}\longrightarrow\hat{\mathbf{H}},\rho:\hat{\mathbf{H}}\longrightarrow\mathbf{H}$ preserving finite and indexed meets and joins and Heyting implication, and satisfying $\rho\circ\iota\cong\mathrm{id}_{\mathbf{H}}$. Under these circumstances it is easy to see that **H** satisfies (i)′ and (ii)′ when $\hat{\mathbf{H}}$ does.

9.3 Construction. (Glueing) $(\mathbf{C},\hat{\mathbf{H}})$ is obtained from $(\mathbf{C},\mathbf{H})$ by 'glueing' it to $(\mathbf{Set},\mathbf{P})$ (the hyperdoctrine of 8.3(i)) along the global sections functor. We first give the construction at the level of Heyting preordered sets in (a), and then apply it fibrewise to hyperdoctrines in (b).

(a) Let $\gamma:H\longrightarrow K$ be a monotone function between Heyting preordered sets which preserves finite meets. Define

† See Glossary.

$$\mathrm{Gl}(\gamma) = \{(k,h)\in K\times H \mid k\leq\gamma(h)\}.$$

Then the relation

$$(k,h)\leq(k',h') \quad \text{iff} \quad k\leq k' \text{ in } K \text{ and } h\leq h' \text{ in } H$$

makes $\mathrm{Gl}(\gamma)$ into another Heyting preordered set with

- top element $(\top,\top)$
- binary meet $(k,h)\wedge(k',h') = (k\wedge k', h\wedge h')$
- bottom element $(\bot,\bot)$
- binary join $(k,h)\vee(k',h') = (k\vee k', h\vee h')$
- Heyting implication $(k,h)\Rightarrow(k',h') = ((k\Rightarrow k')\wedge\gamma(h\Rightarrow h'),\, h\Rightarrow h')$.

and the second projection function $\pi_2 : K\times H \longrightarrow H$ restricts to give a morphism in **Heyt**, $\rho:\mathrm{Gl}(\gamma)\longrightarrow H$.

(b) Each first-order hyperdoctrine $(\mathbf{C},\mathbf{H})$ with $\mathbf{C}$ a locally small† category is related to the particular first-order hyperdoctrine $(\mathbf{Set},\mathbf{P})$ of 8.3(i) by taking 'global sections': there is a finite product preserving functor $\Gamma:\mathbf{C}\longrightarrow\mathbf{Set}$ defined by

$$\Gamma(I\xrightarrow{\ \alpha\ }J) = \mathbf{C}(1,I)\xrightarrow{\ \alpha_*\ }\mathbf{C}(1,J)$$
$$(1\xrightarrow{\ i\ }I)\longmapsto(1\xrightarrow{\ \alpha\circ i\ }J),$$

and then there is a $\mathbf{C}$-indexed monotone function $\gamma_- : \mathbf{H}(-)\longrightarrow\mathbf{P}(\Gamma(-))$ whose component at $I\in\mathrm{ob}\mathbf{C}$, $\gamma_I:\mathbf{H}(I)\longrightarrow\mathbf{P}(\Gamma(I))$, is defined by

$$\gamma_I(\phi) = \{i\in\mathbf{C}(1,I) \mid i^*(\phi)\cong\top \text{ in } \mathbf{H}(1)\}.$$

It is easy to see that each γ_I preserves finite meets; so we may apply the construction in (a) to obtain

$$\hat{\mathbf{H}}(I) = \mathrm{Gl}(\gamma_I).$$

Thus a typical element of $\mathrm{Gl}(\gamma_I)$ is a pair (A,ϕ) where $A\subseteq\mathbf{C}(1,I)$, $\phi\in\mathbf{H}(I)$ and for each $i\in A$, $i^*(\phi)\cong\top$. Furthermore, for each $\alpha:I\longrightarrow J$ in $\mathbf{C}$ we get a monotone function

$$\alpha^*:\hat{\mathbf{H}}(J)\longrightarrow\hat{\mathbf{H}}(I)$$

by defining $\alpha^*(B,\psi) = (\alpha^{-1}(B),\alpha^*(\psi))$. This gives a new $\mathbf{C}$-indexed preordered set $\hat{\mathbf{H}}$ and in fact $(\mathbf{C},\hat{\mathbf{H}})$ is again a first-order hyperdoctrine, with finite meets, finite joins and Heyting implication in the fibres given as in (a) and for each $K\in\mathrm{ob}\mathbf{C}$, K-indexed joins and meets given by

- $\exists_{K,I}(B,\psi) = \big(\{i\in\Gamma(I) \mid \langle k,i\rangle\in B, \text{ some } k\in\Gamma(K)\}, \exists_{K,I}(\psi)\big)$
- $\forall_{K,I}(B,\psi) = \big(\{i\in\Gamma(I) \mid \langle \mathrm{id}_K, i\rangle^*(\psi)\cong\top \text{ and } \langle k,i\rangle\in B, \text{ all } k\in\Gamma(K)\},\forall_{K,I}(\psi)\big)$.

Note also that second projection on each fibre determines a $\mathbf{C}$-indexed monotone function $\rho:\hat{\mathbf{H}}\longrightarrow\mathbf{H}$ preserving finite meets, finite joins, Heyting implication and K-indexed meets and joins (all $K\in\mathrm{ob}\mathbf{C}$).

It is easy to see that $(\mathbf{Set},\mathbf{P})$ satisfies (i)′ and (ii)′ and the crucial point is that these properties are inherited by the 'glued' hyperdoctrine:

† See Glossary.

9.4 Lemma. $\hat{\mathbf{H}}$ *satisfies the hyperdoctrine versions of the disjunction and explicit definability properties* (i)′ *and* (ii)′.

Proof. For (i)′, given $(A,\phi),(B,\psi)\in\hat{\mathbf{H}}(1)$, if

$$\top \cong (A,\phi)\vee(B,\psi) = (A\cup B, \phi\vee\psi),$$

then $A\cup B = \{\mathrm{id}_1\}$, so

either $\mathrm{id}_1\in A$ and then $\top\cong(\mathrm{id}_1)^*(\phi)\cong\phi$ (since $(A,\phi)\in\hat{\mathbf{H}}(1)$),
or $\mathrm{id}_1\in B$ and then $\top\cong(\mathrm{id}_1)^*(\psi)\cong\psi$ (since $(B,\psi)\in\hat{\mathbf{H}}(1)$).

Similarly for (ii)′, given $(A,\phi)\in\hat{\mathbf{H}}(K\times 1)$, if

$$\top\cong\exists_{K,1}(A,\phi) = \big(\{\mathrm{id}_1 \mid \langle k,\mathrm{id}_1\rangle\in A,\ \text{some } k\in\Gamma(K)\}, \exists_{K,1}(\phi)\big)$$

then the first component is not empty, *i.e.* there is some $k{:}1\longrightarrow K$ in $\mathbf{C}$ with $\langle k,\mathrm{id}_1\rangle\in A$, which means that $\top\cong\langle k,\mathrm{id}_1\rangle^*(\phi)$ since $(A,\phi)\in\hat{\mathbf{H}}(K\times 1)$. □

9.5 Lemma. *If* $\mathbf{H}$ *is freely generated over* $\mathbf{C}$ *by sets of elements* $X(I)\subseteq\mathbf{H}(I)$ *(as in Definition 9.2), then* $\rho{:}\hat{\mathbf{H}}\longrightarrow\mathbf{H}$ *has a right inverse, i.e. there is* $\iota{:}\mathbf{H}\longrightarrow\hat{\mathbf{H}}$ *preserving the hyperdoctrine structure and satisfying* $\rho\circ\iota\cong\mathrm{id}_{\mathbf{H}}$.

Proof. For each $I\in\mathrm{ob}\mathbf{C}$, define $\delta_I{:}X(I)\longrightarrow\hat{\mathbf{H}}(I)$ by $\delta_I(x) = (\gamma_I(x),x)$. Then by Definition 9.2, δ induces $\bar{\delta}{:}\mathbf{H}\longrightarrow\hat{\mathbf{H}}$ preserving the hyperdoctrine structure and with the property that for all $I\in\mathrm{ob}\mathbf{C}$ and $x\in X(I)$, $\bar{\delta}_I(\ulcorner x\urcorner) = \delta_I(x)$. Then

$$\rho_I(\bar{\delta}_I(\ulcorner x\urcorner)) = \rho_I(\delta_I(x)) = x$$

so that $\rho\circ\bar{\delta}$ and $\mathrm{id}_{\mathbf{H}}$ are both $\mathbf{C}$-indexed monotone functions $\mathbf{H}\longrightarrow\mathbf{H}$ preserving the hyperdoctrine structure which have the same effect on the elements $x\in X(I)$ $(I\in\mathrm{ob}\mathbf{C})$; so by the uniqueness part of Definition 9.2, $\rho\circ\bar{\delta}\cong\mathrm{id}_{\mathbf{H}}$. So we can take $\iota=\bar{\delta}$ □

We can now complete the proof of Theorem 9.1 by showing that $\mathbf{H}$ satisfies (i)′ and (ii)′. Given $\phi,\psi\in\mathbf{H}(1)$, if $\top\cong\phi\vee\psi$, then using ι from Lemma 9.5,

$$\top\cong\iota_1(\top)\cong\iota_1(\phi\vee\psi)\cong\iota_1(\phi)\vee\iota_1(\psi) \quad\text{in}\quad \hat{\mathbf{H}}(1).$$

So by Lemma 9.5 either $\top\cong\iota_1(\phi)$ in which case $\top\cong\rho_1(\iota_1(\phi))\cong\phi$, or else $\top\cong\iota_1(\psi)$ in which case $\top\cong\rho_1(\iota_1(\psi))\cong\psi$. Thus $\mathbf{H}$ satisfies (i)′. And given $\phi\in\mathbf{H}(K\times 1)$ (some $K\in\mathrm{ob}\mathbf{C}$), if $\top\cong\exists_{K,1}(\phi)$, then

$$\top\cong\iota_1(\top)\cong\iota_1\big(\exists_{K,1}(\phi)\big)\cong\exists_{K,1}(\iota_1(\phi)) \quad\text{in}\quad \hat{\mathbf{H}}(1).$$

So by Lemma 9.5 there is some $k{:}1\longrightarrow K$ in $\mathbf{C}$ with $\top\cong\langle k,\mathrm{id}_1\rangle^*(\iota_1(\phi))$ and hence

$$\top\cong\rho_{K\times 1}\big(\langle k,\mathrm{id}_1\rangle^*(\iota_1(\phi))\big)\cong\langle k,\mathrm{id}_1\rangle^*\rho_1(\iota_1(\phi))\cong\langle k,\mathrm{id}_1\rangle^*(\phi)$$

so that $\mathbf{H}$ also satisfies (ii)′. □

10 Equality

We have seen that the categorical semantics of the propositional connectives and first-order quantifiers provides a characterization of these logical operations in terms of various categorical adjunctions. We are now going to see that, within the context of first-order logic, the same is true of *equality predicates.* (As with much of the material in these notes, this observation originated with Lawvere [La2].) This is perhaps surprising if one thinks of equality as some fairly arbitrary equivalence relation on terms; but the assumption that all functions and relations respect the relation is enough to give the equality predicates a canonical nature with, once again, a neat adjoint characterization.

Let Σ be a signature of sorts, function and relation symbols. Add to the clauses in 8.1 defining the first-order formulas over Σ the following clause:

- *Equality.* $t = t'$ is a formula if t and t' are terms over Σ of the same sort.

We wish to extending the rules of **IPC** to those of First-Order Intuitionistic Predicate Calculus with Equality, $\mathbf{IPC}^{=}$. The formulation of **IPC** we used had both sequents and equations. Since we now have equality predicates, it is natural to give the rules just using sequents-in-context, replacing an equation-in-context $t = t'$ (x) by the sequent-in-context $\emptyset \vdash t = t'$ (x). So we discard all the rules in Table 8.5 which mention equations-in-context and add to the remainder the usual *equality axioms*

(*10.1*) $\quad \top \vdash x = x \ \ (x)$

(*10.2*) $\quad x = x' \vdash x' = x \ \ (x, x')$

(*10.3*) $\quad x = x' \wedge x' = x'' \vdash x = x'' \ \ (x, x', x'')$

(*10.4*) $\quad x = x' \vdash t(x'/x) = t \ \ (x, x', \mathbf{y})$ $\quad$ (where $t(x, \mathbf{y})$ is a term-in-context)

(*10.5*) $\quad x = x' \wedge \phi \vdash \phi(x'/x) \ \ (x, x', \mathbf{y})$ $\quad$ (where $\phi(x, \mathbf{y})$ is a formula-in-context)

together with the following simplification of the (Sub) rule in Table 8.5

$$(\mathrm{Sub})\ \frac{\Gamma \vdash \phi \ \ (y, \mathbf{x})}{\Gamma(s/y) \vdash \phi(s/y) \ \ (\mathbf{x})} \qquad (s(\mathbf{x}) \text{ a term-in-context of the same sort as } y)$$

So the rules for $\mathbf{IPC}^{=}$ consist of (*10.1*)–(*10.5*) (regarded as rules with empty hypotheses), (Sub), and from Table 8.5 the first (Wk) rule, the first (Id) rule, the first (Cut) rule and the bi-rules ($\top$), ($\wedge$), ($\bot$), ($\vee$), ($\Rightarrow$), ($\neg$), ($\exists$) and ($\forall$).

10.1 Proposition. *Modulo the other rules of* $\mathbf{IPC}^{=}$, (*10.1*)–(*10.5*) *are equivalent to the bi-rule*

$$(\mathrm{Equ})\ \frac{\Gamma \vdash \phi(x/x') \ \ (x, \mathbf{y})}{\overline{\Gamma, x = x' \vdash \phi \ \ (x, x', \mathbf{y})}}\,.$$

Proof is left as an (instructive?) **Exercise** in first-order logic. $\square$

$$(\mathrm{Wk})\frac{\Gamma\vdash\phi\ (\mathbf{x})}{\Gamma\vdash\phi\ (\mathbf{y})}(\mathbf{x}\subseteq\mathbf{y}) \qquad (\mathrm{Sub})\frac{\Gamma\vdash\phi\ (y,\mathbf{x})}{\Gamma(s/y)\vdash\phi(s/y)\ (\mathbf{x})}$$

$$(\mathrm{Id})\frac{}{\Gamma,\phi\vdash\phi\ (\mathbf{x})} \qquad (\mathrm{Cut})\frac{\Gamma\vdash\phi\ (\mathbf{x})\quad \Delta,\phi\vdash\psi\ (\mathbf{x})}{\Delta,\Gamma\vdash\psi\ (\mathbf{x})}$$

$$(\mathrm{Equ})\frac{\Gamma\vdash\phi(x/x')\ (x,\mathbf{y})}{\overline{\Gamma,x=x'\vdash\phi\ (x,x',\mathbf{y})}}$$

$$(\top)\frac{}{\overline{\Gamma\vdash\top}} \qquad (\wedge)\frac{\Gamma\vdash\phi\quad\Gamma\vdash\psi}{\overline{\Gamma\vdash\phi\wedge\psi}}$$

$$(\bot)\frac{}{\overline{\Gamma,\bot\vdash\phi}} \qquad (\vee)\frac{\Gamma,\phi\vdash\theta\quad\Gamma,\psi\vdash\theta}{\overline{\Gamma,\phi\vee\psi\vdash\theta}}$$

$$(\Rightarrow)\frac{\Gamma,\phi\vdash\psi}{\overline{\Gamma\vdash\phi\Rightarrow\psi}} \qquad (\neg)\frac{\Gamma,\phi\vdash\bot}{\overline{\Gamma\vdash\neg\phi}}$$

$$(\exists)\frac{\Gamma,\phi\vdash\psi\ (y,\mathbf{x})}{\overline{\Gamma,\exists y:A.\phi\vdash\psi\ (\mathbf{x})}} \qquad (\forall)\frac{\Gamma\vdash\phi\ (y,\mathbf{x})}{\overline{\Gamma\vdash\forall y:A.\phi\ (\mathbf{x})}}$$

10.2 Table: Adjoint Calculus for IPC⁼

Using the simplification afforded by this proposition, the rules for **IPC⁼** are collected in Table 10.2: the first four rules embody basic properties of entailment and each of the other 'bi-rules' embody an adjoint characterization of a particular logical operator.

In what sense is the rule (Equ) capturing the meaning of equality in terms of an adjunction? To answer this question, let us consider how equality is modelled in a fisrt-order hyperdoctrine $(\mathbf{C},\mathbf{H})$. To interpret a formulas-in-context $\phi(\mathbf{x})$ when ϕ is $t=t'$ we need for each $I\in\mathrm{ob}\mathbf{C}$ some element

$$\delta_I\in\mathbf{H}(I\times I)$$

so that we can define for y,y': B,

$$[\![y=y'\ (y,y')]\!]_M = \delta_{MB}\,,$$

and hence (substitution being modelled by the pullback functions) in general

$$(10.6)\quad [\![t=t'\ (\mathbf{x})]\!]_M = \langle[\![t(\mathbf{x})]\!]_M,[\![t'(\mathbf{x})]\!]_M\rangle^*(\delta_{MB})\in\mathbf{H}(\textstyle\prod M\mathbf{A})$$

where $\mathbf{x}:\mathbf{A}$ and t,t': B, say.

10.3 Proposition. *Let* (**C**,**H**) *be a first-order hyperdoctrine equipped with elements* $(\delta_I \in \mathbf{H}(I\times I) \mid I \in \mathrm{ob}\mathbf{C})$. *Extend the semantics of formulas-in-context to formulas involving equality by adding the clause (10.6) to Table 8.2. Then the collection of sequents-in-context satisfied by any structure in* (**C**,**H**) *is closed under the rules in Table 10.2 if and only if the elements* $(\delta_I \in \mathbf{H}(I\times I) \mid I \in \mathrm{ob}\mathbf{C})$ *satisfy*

(*10.7*) δ_I *is a value of the left adjoint to* $\Delta^*: \mathbf{H}(I\times I) \longrightarrow \mathbf{H}(I)$ *at* $\top$ *(see 3.3), where* $\Delta = \langle \mathrm{id}_I, \mathrm{id}_I \rangle : I \longrightarrow I\times I$ *is the diagonal morphism.*

Proof. Suppose first that (*10.7*) holds. Then the proof of the soundness of all the rules except the new (Equ) is just as in the proof of 8.8—except that for (Wk), (Sub), (∃) and (∀) we must extend the substitution lemma 8.6 (and thereby also the weakening lemma 8.7), since we are now dealing with a larger class of formulas. But we do indeed have the substitution property

$$[\![\phi(\mathbf{s}(\mathbf{y}))]\!] \cong \langle [\![s_1(\mathbf{y})]\!], \ldots, [\![s_n(\mathbf{y})]\!] \rangle^*\big([\![\phi(\mathbf{x})]\!]\big),$$

the proof being as in 8.6 by induction on the structure of ϕ, with a new clause for the case ϕ is $t = t'$ (which is just like the case for atomic formulas because of the way we defined $[\![t = t'(\mathbf{x})]\!]$). For (Equ), first note that modulo (⇒) and (∀) it is equivalent to a simpler rule without 'parameters':

$$(10.8)\qquad \frac{\emptyset \vdash \phi(x/x')\ (x)}{x = x' \vdash \phi\ (x, x')} \;.$$

By definition, a structure satisfies $x = x' \vdash \phi\ (x, x')$ iff $\delta_{MA} \leq [\![\phi(x,x')]\!]$. But by (*10.7*) $\delta_{MA} = \exists\Delta(\top)$ is the left adjoint of Δ^* at $\top$, so the sequent-in-context is satisfied iff $\top \leq \Delta^*\big([\![\phi(x,x')]\!]\big) = \langle [\![x(x)]\!], [\![x(x)]\!] \rangle^*\big([\![\phi(x,x')]\!]\big) \cong [\![\phi(x, x/x')]\!]$ (by the substitution property), which is to say that the structure satisfies $\top \vdash \phi(x/x')\ (x)$. So the above rule is sound, and hence so is (Equ) as well.

Conversely if the sequents satisfied by any structure are closed under the rules in Table 10.2, given $I \in \mathrm{ob}\mathbf{C}$ and $R \in \mathbf{H}(I\times I)$ we can take a signature with one sort $\ulcorner I \urcorner$ and one binary relation symbol $\ulcorner R \urcorner \subseteq \ulcorner I \urcorner \ulcorner I \urcorner$ and consider the structure sending $\ulcorner I \urcorner$ to I and $\ulcorner R \urcorner$ to R. Then as above $[\![\ulcorner R \urcorner(x, x/x')]\!] \cong \Delta^*\big([\![\ulcorner R \urcorner(x,x')]\!]\big) = \Delta^*(R)$ and closure under (Equ) gives: $\top \leq \Delta^*(R)$ iff $\delta_I \leq R$. Since this holds for all $R \in \mathbf{H}(I\times I)$, δ_I is necessarily a value of the left adjoint to Δ^* at $\top$. □

Note that property (*10.7*) determines the δ_I ($I \in \mathrm{ob}\mathbf{C}$) uniquely up to isomorphism and gives equality a status similar to the other logical operators.

10.4 Definition. We will call (**C**,**H**) a *first-order hyperdoctrine with equality* if it is a first-order hyperdoctrine for which each $\Delta^*: \mathbf{H}(I\times I) \longrightarrow \mathbf{H}(I)$ ($I \in \mathrm{ob}\mathbf{C}$) has a left adjoint at $\top$.

10.5 Examples. (i) Ⓣ Ⓚ (**Set**,**P**) and (**Set**,**R**) are first-order hyperdoctrines with equality, since we saw in 3.8 that the **Set**-indexed posets **P** and **R** have both left and right adjoints for *all* pullback functions α^*. (This is no accident—see Remark 10.6 below.)

(ii) Ⓢ We saw in section 7 that (**Dcpo**,**F**) is not a first-order hyperdoctrine, but that it does support the interpretation of a fragment of first-order logic (that without

$\Rightarrow$ or $\exists$). Since we know from 3.8(v) that **F** has left adjoints for all pullback functions, it is the case that it satisfies condition (*10.7*). However, it would be *wrong* to say that **F** supports the sound interpretation of a fragment of first-order logic with equality, since without $\Rightarrow$ in the logic, we cannot reduce (*10.7*) to (*10.8*) as we did in the proof of 10.3. As with other 'left adjoint' rules, (*10.7*) has a 'stability' condition on the adjoint built into its formulation: to satisfy it we need not only that $\exists\Delta(\top) \in \mathbf{H}(I\times I)$ exists, but also that for all $\gamma \in \mathbf{H}(I\times I)$, $\gamma \wedge \exists\Delta(\top)$ is a value of the left adjoint to Δ^* at $\Delta^*(\gamma)$. (This is in fact a particular instance of 'Frobenius Reciprocity'—see 3.10.) In **F**, $\exists\Delta(\top)$ fails to satisfy this 'stability' condition in general.

10.6 Remarks. (i) **(Generalized quantifiers)** $\mathbf{IPC}^{=}$ has much greater expressive power than **IPC**. This is reflected by the consequences for a first-order hyperdoctrine of satisfying the apparently limited extra property (*10.7*). For if $(\mathbf{C},\mathbf{H})$ is a first-order hyperdoctrine with equality, then for every $\alpha : I \longrightarrow J$ in **C**, $\alpha^* : \mathbf{H}(J) \longrightarrow \mathbf{H}(I)$ *has both left and right adjoints*, which will be denoted $\exists\alpha$ and $\forall\alpha$ respectively. Indeed, for $\phi \in \mathbf{H}(I)$ we can define

$$\exists\alpha(\phi) = \exists_{I,J}\big((\alpha\times \mathrm{id}_J)^*\delta_J \wedge (\pi_1)^*\phi\big)$$

$$\forall\alpha(\phi) = \forall_{I,J}\big((\alpha\times \mathrm{id}_J)^*\delta_J \Rightarrow (\pi_1)^*\phi\big).$$

It is easier to see what these expressions mean and to prove that they have the required properties if we use Proposition 10.3 and work in $\mathbf{IPC}^{=}$, where $\exists\alpha(\phi)$ and $\forall\alpha(\phi)$ correspond to *generalized quantifiers which are adjoint to substitution.* Thus if $t(x)$ is a term-in-context, $\phi(x)$ is a formula-in-context, $x{:}A$ and y is a variable of the same sort as t, define formulas-in-context $(\exists_t x{:}A.\phi)(y)$ and $(\forall_t x{:}A.\phi)(y)$ by

$$\exists_t x{:}A.\phi = \exists x{:}A.(t = y \wedge \phi)$$

$$\forall_t x{:}A.\phi = \forall x{:}A.(t = y \Rightarrow \phi).$$

(*Cf.* the similar expressions in 3.8(iii) and 3.15(iv).) Then

$$(\exists_t)\frac{\phi \vdash \psi(t/y)\ (x)}{\exists_t x{:}A.\phi \vdash \psi\ (y)} \qquad \text{and} \qquad (\forall_t)\frac{\psi(t/y) \vdash \phi\ (x)}{\psi \vdash \forall_t x{:}A.\phi\ (y)}$$

are derived rules of $\mathbf{IPC}^{=}$ **(Exercise)**.

(ii) **('Beck-Chevalley Conditions')** We required the adjoints $\exists_{K,I}\,(=\exists\pi_2)$ and $\forall_{K,I}\,(=\forall\pi_2)$ to have certain stability properties with respect to the pullback functions in order to model the interaction of quantification with substitution correctly. Similarly the general adjoints $\exists\alpha$ and $\forall\alpha$ have stability properties reflecting the interaction of the generalized quantifiers with substitution on their parameters. These stability conditions are all instances of 'Beck-Chevalley conditions': in general if

$$\begin{array}{ccc} L & \xrightarrow{\ \delta\ } & K \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ J & \xrightarrow[\ \alpha\ ]{} & I \end{array}$$

is a commutative square in **C**, then we say that the left adjoints to the pullback functions satisfy a Beck-Chevalley Condition for the above square if $\beta^*\circ\exists\alpha\cong\exists\delta\circ\gamma^*$. Note that $\gamma^*\circ\alpha^*\cong\delta^*\circ\beta^*$ (since $\alpha\circ\gamma=\beta\circ\delta$), $\mathrm{id}\leq\alpha^*\circ\exists\alpha$ (since $\exists\alpha\dashv\alpha^*$) and $\exists\delta\circ\delta^*\leq\mathrm{id}$ (since $\exists\delta\dashv\delta^*$), so that

$$\exists\delta\circ\gamma^*\leq\exists\delta\circ\gamma^*\circ\alpha^*\circ\exists\alpha\cong\exists\delta\circ\delta^*\circ\beta^*\circ\exists\alpha\leq\beta^*\circ\exists\alpha\,.$$

So the Beck-Chevalley Condition amounts to requiring the other inequality: $\beta^*\circ\exists\alpha\leq\exists\delta\circ\gamma^*$. Dually, $\beta^*\circ\forall\alpha\leq\forall\delta\circ\gamma^*$ holds automatically and the right adjoints are said to satisfy a Beck-Chevalley condition for the above square if the reverse inequality holds, so that $\forall\delta\circ\gamma^*\cong\beta^*\circ\forall\alpha$.

With this terminology, it is the case that in a first-order hyperdoctrine with equality the left and right adjoints satisfy the Beck-Chevalley Condition for certain commutative squares which (by virtue of the finite products in **C**) are *pullback squares*. These are the squares of the form

$$\begin{array}{ccc} K\times I & \xrightarrow{\pi_2} & I \\ {\scriptstyle \mathrm{id}\times\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ K\times J & \xrightarrow[\pi_2]{} & J \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I & \xrightarrow{\langle \mathrm{id},\alpha\rangle} & I\times J \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\times\mathrm{id}} \\ J & \xrightarrow[\Delta]{} & J\times J \end{array}\,.$$

If **C** has all pullbacks, obviously a sufficient condition on **H** to ensure the adjoints to the pullback functions satisfy these stability conditions is that the Beck-Chevalley Condition holds for all pullback squares. This is the case for **(Set,P)** and **(Set,R)**. (**Exercise.** Apply 3.4 and 3.5 to deduce that the Beck-Chevalley Condition for all pullback squares holds for $\exists$ iff it holds for $\forall$.)

(iii) **Beyond hyperdoctrines.** Following the pattern set in sections 4, 6 and 8, you are perhaps expecting me to say something about and a correspondence between theories in **IPC**$^=$ and first-order hyperdoctrines with equality. However, the situation is not as satisfactory one one might hope. We have formulated **IPC**$^=$ just using judgements which are sequents-in-context (as is usual). So in a theory, if we wish to assert that two terms (of the same sort) are equal, we stipulate $\emptyset\vdash s=t\ (\mathbf{x})$ as an axiom. This sequent-in-context is satisfied in a hyperdoctrine if $\top\cong\langle[\![s(\mathbf{x})]\!],[\![t(\mathbf{x})]\!]\rangle^*(\exists\Delta(\top))$. For this it is sufficient, but unfortunately not always necessary, that $[\![s(\mathbf{x})]\!]$ and $[\![t(\mathbf{x})]\!]$ be equal morphisms in the base category (which is to say that the equation-in-context $s=t\ (\mathbf{x})$ is satisfied). So in trying to reconstruct a first-order hyperdoctrine with equality as the classifier of some theory over **IPC**$^=$, we will not always be able to capture in sequents some of the identities which hold between morphisms in the base category.

The best way of getting round this problem is to restrict attention to first-order hyperdoctrines as in 7.2(ii), *i.e.* of the form $\mathrm{Sub}_{\mathbf{C}}$ where **C** is a category with finite limits (and sufficient extra structure to ensure $\mathrm{Sub}_{\mathbf{C}}$ is a first-order hyperdoctrine). Such a hyperdoctrine has equality at $I\in\mathrm{ob}\mathbf{C}$ given by the diagonal subobject $\Delta: I\rightarrowtail I\times I$, and we have $\alpha=\beta$ iff $\langle\alpha,\beta\rangle$ factors as $\Delta\circ\gamma$ for some γ, iff $\langle\alpha,\beta\rangle^*(\Delta)\cong\top$ in $\mathrm{Sub}_{\mathbf{C}}$. There is a correspondence between such **C** and theories in **IPC**$^=$. (Abnd there is a simple construction reflecting an arbitrary first-order hyperdoctrines with equality into such special ones.) The categorical semantics of first-order logic in this (special) style interprets

sorts as objects of **C**,
terms(-in-contex) as morphisms of **C** and
formulas(-in-context) as subobjects of **C**.

See [MR] for an exposition. Even for first-order logic *without* equality, one can give a version of the categorical semantics in this style, except that formulas are interpreted as subobjects lying in a *distinguished class* of subobjects which has to be given as an extra part of the structure of **C** (and which in general will be a proper subclass of all subobjects).

Glossary

CATEGORY. A category **C** is specified by

- A collection ob**C** of *objects* of **C**: $X,Y,Z,\ldots$
- A collection mor**C** of *morphisms* of **C**: $f,g,h,\ldots$
- Operations assigning to each $f \in$ mor**C** its *domain* dom(f)$\in$ob**C** and *codomain* cod(f)$\in$ob**C**. We write $f: X \longrightarrow Y$ or $X \xrightarrow{f} Y$ to indicate that dom(f)$=X$ and cod(f)$=Y$.
- An operation assigning to each $X \in$ ob**C** the corresponding *identity morphism* $\mathrm{id}_X: X \longrightarrow X$.
- An operation assigning to each composable pair of morphisms $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ (*composable* means cod(f) = dom(g)) their *composition* $g \circ f: X \longrightarrow Y$. This operation of composition is
  unitary: $(\mathrm{id}_Y \circ f) = f = (f \circ \mathrm{id}_X)$ and
  associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

CODOMAIN. Dual of DOMAIN.

COMMUTATIVE DIAGRAM. See DIAGRAM.

COMPOSITION. See CATEGORY.

CONTRAVARIANT FUNCTOR. See FUNCTOR.

COVER. A MORPHISM $f: X \longrightarrow Y$ in a CATEGORY is a cover if whenever $f = m \circ e$ with m a MONOMORPHISM, it is the case that m is an ISOMORPHISM. In a category with PULLBACKS, a cover f is said to be *stable* if the result of pulling it back along any morphism is again a cover.

DIAGRAM. A diagram D in a CATEGORY **C** is specified by a GRAPH $s,t: E \rightrightarrows V$ and two functions $D: V \longrightarrow$ ob**C**, $D: E \longrightarrow$ mor**C** satisfying that for each $e \in E$ dom($D(e)$) = $D(s(e))$ and cod($D(e)$) = $D(t(e))$. The diagram *commutes* if for every pair of paths (= lists of edges with $t(e_i) = s(e_{i+1})$) $e_0,\ldots,e_n$ and $e'_0,\ldots,e'_m$ with $s(e_0) = s(e'_0)$ and $t(e_n) = t(e'_m)$, it is the case that the compositions $D(e_n) \circ \cdots \circ D(e_0)$ and $D(e'_m) \circ \cdots \circ D(e'_0)$ are equal morphisms of **C**.

DOMAIN. See CATEGORY.

DUAL CONCEPT. Concept applied to the OPPOSITE CATEGORY.

EPIMORPHISM. Dual of MONOMORPHISM.

EQUIVALENT CATEGORIES. Two CATEGORIES **C** and **D** are equivalent if there are FUNCTORS $F:\mathbf{C}\longrightarrow\mathbf{D}$ and $G:\mathbf{D}\longrightarrow\mathbf{C}$, and NATURAL ISOMORPHISMS $F\circ G\cong \mathrm{Id}_{\mathbf{D}}$ and $\mathrm{Id}_{\mathbf{C}}\cong G\circ F$. In this case we say F is an *equivalence* with *essential inverse* G, and write $\mathbf{C}\simeq\mathbf{D}$ to indicate that **C** and **D** are equivalent.

FAITHFUL FUNCTOR. A FUNCTOR $F:\mathbf{C}\longrightarrow\mathbf{D}$ is faithful if for all $f,g:X\rightrightarrows Y$ in **C**, $F(f)=F(g)$ implies $f=g$.

FULL FUNCTOR. A FUNCTOR $F:\mathbf{C}\longrightarrow\mathbf{D}$ is full if for all $X,X'\in\mathrm{ob}\mathbf{C}$ and all $g:F(X)\longrightarrow F(X')$ in **D**, there is some $f:X\longrightarrow X'$ in **C** with $F(f)=g$.

FULL SUBCATEGORY. A SUBCATEGORY **D** of a CATEGORY **C** is full if for all $X,Y\in\mathrm{ob}\mathbf{D}$ every MORPHISM $X\longrightarrow Y$ in **C** is a morphism in **D** (so that $\mathbf{D}(X,Y)=\mathbf{C}(X,Y)$). Thus each full subcategory of **C** is uniquely determined by specifying the subset of ob**C** comprising its OBJECTS.

FUNCTOR. A (covariant) functor $F:\mathbf{C}\longrightarrow\mathbf{D}$ between CATEGORIES is specified by

- An operation sending OBJECTS X in **C** to objects $F(X)$ in **D**.
- An operation sending MORPHISMS $f:X\longrightarrow X'$ in **C** to morphisms $F(f)$: $F(X)\longrightarrow F(X')$ in **D**. This operation should satisfy

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$$
$$F(g\circ f) = F(g)\circ F(f).$$

A *contravariant functor* from **C** to **D** is a functor $\mathbf{C}^{\mathrm{op}}\longrightarrow\mathbf{D}$.

The *composition of functors* $F:\mathbf{C}\longrightarrow\mathbf{D}$ and $G:\mathbf{D}\longrightarrow\mathbf{E}$ is the functor $G\circ F:\mathbf{C}\longrightarrow\mathbf{E}$ with $(G\circ F)(X)=G(F(X))$ and $(G\circ F)(f)=G(F(f))$.

The *identity functor* $\mathrm{Id}:\mathbf{C}\longrightarrow\mathbf{C}$ has $\mathrm{Id}(X)=X$ and $\mathrm{Id}(f)=f$.

FUNCTOR CATEGORY. Given CATEGORIES **C** and **D**, the functor category $[\mathbf{C},\mathbf{D}]$ (also written $\mathbf{D}^{\mathbf{C}}$) has as OBJECTS all FUNCTORS $\mathbf{C}\longrightarrow\mathbf{D}$ and NATURAL TRANSFORMATIONS as MORPHISMS, with IDENTITIES and COMPOSITION given by *identity natural transformations* and by *vertical composition* of natural transformations.

GRAPH. A (directed) graph is specified by a set V of *vertices*, a set E of *edges* and two functions $s,t:E\rightrightarrows V$ assigning to each edge $e\in E$ its *source* $s(e)\in V$ and *target* $t(e)\in V$.

HOM-FUNCTOR. If **C** is a CATEGORY and $X,Y\in\mathrm{ob}\mathbf{C}$, then $\mathbf{C}(X,Y)$ denotes the collection of MORPHISMS $X\longrightarrow Y$ in **C**. **C** is called *locally small* if each $\mathbf{C}(X,Y)$ is a set, *i.e.* an OBJECT in the category **Set** of sets and functions. Then $(X,Y)\longmapsto\mathbf{C}(X,Y)$ is the object part of the hom-functor

$$\mathbf{C}(-,+):\mathbf{C}^{\mathrm{op}}\times\mathbf{C}\longrightarrow\mathbf{Set}$$

which assigns to a pair $X'\xrightarrow{f}X$, $Y\xrightarrow{g}Y'$ of morphisms the function

$$\begin{aligned} C(f,g)\colon C(X,Y) &\longrightarrow C(X',Y') \\ h &\longmapsto g\circ h\circ f\,. \end{aligned}$$

$C(f,\mathrm{id})$ is usually written f^* and called *precomposition with f*; $C(\mathrm{id},g)$ is usually called g_* and called *postcomposition with g*. Thus $C(f,g)(h) = f^*(g_*(h)) = g_*(f^*(h))$.

IDENTITY. See CATEGORY.

IMAGE FACTORIZATION. A CATEGORY **C** is said to have image factorizations if every MORPHISM f factors as $f = m\circ e$ with m a MONOMORPHISM and e a COVER.

INITIAL OBJECT. Dual of TERMINAL OBJECT.

ISOMORPHISM. A MORPHISM $f\colon X\longrightarrow Y$ in a CATEGORY is an isomorphism (*iso*) if there is a $g\colon Y\longrightarrow X$ with $g\circ f = \mathrm{id}_X$ and $f\circ g = \mathrm{id}_Y$. Such a g is necessarily unique, is called the *inverse* of f and denoted f^{-1}. We write $f\colon X\cong Y$ to indicate that f is an isomorphism. We write $X\cong Y$ if such an f exists and say that X and Y are *isomorphic* objects in **C**.

LOCALLY SMALL. See HOM-FUNCTOR.

MONOMORPHISM. A MORPHISM $f\colon X\longrightarrow Y$ in a CATEGORY is a monomorphism (*mono, monic*) if for all $g,h\colon X'\rightrightarrows X$, $f\circ g = f\circ h$ implies that $g = h$. The notation $f\colon X\rightarrowtail Y$ is used to indicate that f is a monomorphism.

MORPHISM. See CATEGORY.

NATURAL ISOMORPHISM. A natural isomorphism $\theta: F\cong G$ between FUNCTORS $F,G\colon \mathbf{C}\rightrightarrows\mathbf{D}$ is an ISOMORPHISM between F and G regarded as OBJECTS of the FUNCTOR CATEGORY $[\mathbf{C},\mathbf{D}]$.

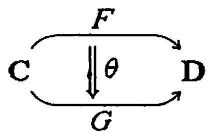
NATURAL TRANSFORMATION. If **C**,**D** are CATEGORIES and $F,G\colon \mathbf{C}\rightrightarrows\mathbf{D}$ are FUNCTORS, a natural transformation $\theta\colon F\longrightarrow G$ is specified by

- An operation assigning to each object X in **C** a morphism $\theta_X\colon F(X)\longrightarrow G(X)$ in **D** (called the *component* of θ at X). This operation should satisfy the condition for each $f\colon X\longrightarrow Y$ in **C**, $G(f)\circ\theta_X = \theta_Y\circ F(f)$, that is, the square

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\theta_X}\downarrow & & \downarrow{\scriptstyle\theta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

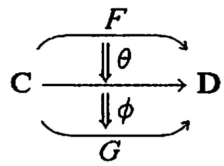
commutes. (One says that the family $(\theta_X \mid X\in \mathrm{ob}\mathbf{C})$ is 'natural in X'.)

We often write

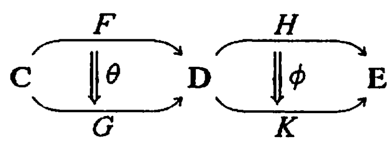


to indicate that $F:\mathbf{C}\longrightarrow\mathbf{D}$, $G:\mathbf{C}\longrightarrow\mathbf{D}$ and $\theta:F\longrightarrow G$.

The *vertical composition* $\phi\circ\theta: F\longrightarrow H$ of natural transformations



has components $(\phi\circ\theta)_X = \phi_X\circ\theta_X$. The *horizontal composition* $\phi{*}\theta: H\circ F\longrightarrow K\circ G$ of natural transformations



has components $(\phi{*}\theta)_X = \phi_{G(X)}\circ H(\theta_X)\bigl(= K(\theta_X)\circ\phi_{F(X)}$, by naturality of $\phi\bigr)$.

OBJECT. See CATEGORY.

OPPOSITE CATEGORY. The opposite $\mathbf{C}^{\mathrm{op}}$ of a CATEGORY $\mathbf{C}$ is the category with

- $\mathrm{ob}\mathbf{C}^{\mathrm{op}} = \mathrm{ob}\mathbf{C}$;
- $\mathrm{mor}\mathbf{C}^{\mathrm{op}} = \{f^{\mathrm{op}} \mid f\in\mathrm{mor}\mathbf{C}\}$, where f^{op} is just a formal copy of f (in other words take $f\longmapsto f^{\mathrm{op}}$ to be some bijection whose codomain is disjoint from its domain);
- $\mathrm{dom}(f^{\mathrm{op}})$ in $\mathbf{C}^{\mathrm{op}}$ is $\mathrm{cod}(f)$ in $\mathbf{C}$; $\mathrm{cod}(f^{\mathrm{op}})$ in $\mathbf{C}^{\mathrm{op}}$ is $\mathrm{dom}(f)$ in $\mathbf{C}$ (in other words, $f^{\mathrm{op}}: X\longrightarrow Y$ in $\mathbf{C}^{\mathrm{op}}$ iff $f: Y\longrightarrow X$ in $\mathbf{C}$);
- IDENTITY on X in $\mathbf{C}^{\mathrm{op}}$ is $(\mathrm{id}_X)^{\mathrm{op}}$;
- $(f^{\mathrm{op}}\circ g^{\mathrm{op}})$ in $\mathbf{C}^{\mathrm{op}}$ is $(g\circ f)^{\mathrm{op}}$.

PRODUCTS. Given a set I and an I-indexed collection $(X_i \mid i\in I)$ of OBJECTS in a CATEGORY, the product of these objects is specified by

- an object $\prod_{j\in I}X_j$
- an I-indexed collection of MORPHISMS $\bigl(\pi_i:\prod_{j\in I}X_j\longrightarrow X_i \mid i\in I\bigr)$

having the following universal property:

> for any object Y and I-indexed family $(f_i: Y\longrightarrow X_i \mid i\in I)$ of morphisms, there is a unique morphism $\langle f_i \mid i\in I\rangle: Y\longrightarrow\prod_{j\in I}X_j$ satisfying for all $i\in I$ that $\pi_i\circ\langle f_i \mid i\in I\rangle = f_i$.

The π_i are called the *product projection morphisms*. We say that the category *has all I-indexed products* if we are given the above structure for all families $(X_i \mid i \in I)$. We say that it *has finite products* if we are given the above structure for all families indexed by a finite sets. By induction on the number of elements in a finite set, this is equivalent to having I-indexed products for $I = \emptyset$ and $I = 2$.

In the case that $I = \emptyset$, the product of no objects is an object 1, called the TERMINAL OBJECT, with the universal property

for any object Y there is a unique morphism $\langle\rangle : Y \longrightarrow 1$.

In the case $I = 2$, we obtain the notion of *binary* product and use the notation

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

for the product of X and Y, and denote by

$$\langle f, g\rangle : Z \longrightarrow X \times Y$$

the unique morphism satisfying $\pi_1 \circ \langle f, g\rangle = f : Z \longrightarrow X$ and $\pi_2 \circ \langle f, g\rangle = g : Z \longrightarrow Y$. If $f : X' \longrightarrow X$ and $g : Y' \longrightarrow Y$, then $\langle f \circ \pi_1, g \circ \pi_2\rangle$ is denoted

$$f \times g : X' \times Y' \longrightarrow X \times Y.$$

A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ *preserves* the product of $(X_i \mid i \in I)$ in $\mathbf{C}$ if $F\left(\prod_{i \in I} X_i\right)$ together with the morphisms $F(\pi_i)$ has the above universal property for the product of $(F(X_i) \mid i \in I)$ in $\mathbf{D}$. This is the same as requiring the morphism

$$\langle F(\pi_i) \mid i \in I\rangle : F\left(\textstyle\prod_{i \in I} X_i\right) \longrightarrow \textstyle\prod_{i \in I} F(X_i)$$

in $\mathbf{D}$ to be an ISOMORPHISM.

(Dual concept is called *Coproduct*, or *Sum.*)

PULLBACKS. Given MORPHISMS $Y \xrightarrow{f} X \xleftarrow{g} Z$ in a CATEGORY, the pullback of g along f is a morphism $f^*(g) : Y \times_X Z \longrightarrow Y$ equipped with a morphism $\varepsilon_f(g) : Y \times_X Z \longrightarrow Z$ making the square

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{\varepsilon_f(g)} & Z \\ {\scriptstyle f^*(g)}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

commute ($f \circ f^*(g) = g \circ \varepsilon_f(g)$) and with the universal property

for any pair of morphisms $Y \xleftarrow{a} W \xrightarrow{b} Z$ with $f \circ a = g \circ b$, there is a unique morphism $c : W \longrightarrow Y \times_X Z$ satisfying $f^*(g) \circ c = a$ and $\varepsilon_f(g) \circ c = b$.

(We say that a COMMUTATIVE square in a category *is a pullback square* if it has a similar universal property to the one above.) We say that the category *has pullbacks* if we are given the above structure for each pair of morphisms with common codomain.

(Alternative name (Fr.): *Fibred Product.*)

PUSHOUT. Dual of PULLBACK. (Alternative name (Fr.): *Fibred Sum.*)

SMALL CATEGORY. A CATEGORY **C** is small if mor**C** and (hence) ob**C** are sets, *i.e.* objects in the category **Set** of sets and functions.

SUBCATEGORY. A CATEGORY **D** is a subcategory of another category **C** if ob**D** is a subset of ob**C**, mor**D** is a subset of mor**C** and the operations in **D** for DOMAIN, CODOMAIN, IDENTITY and COMPOSITION are the restrictions of those in **C**. The *inclusion* functor **D**⟶**C** obtained by restricting the identity on **C**, is FAITHFUL.

TERMINAL OBJECT. See PRODUCTS.

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