Annals of Pure and Applied Logic 41 (1989) 33–81
North-Holland

CONCEPTUAL COMPLETENESS FOR FIRST-ORDER INTUITIONISTIC LOGIC: AN APPLICATION OF CATEGORICAL LOGIC

Andrew M. PITTS
Mathematics Division, University of Sussex, Falmer, Brighton BN1 9QH, UK

Communicated by D. van Dalen
Received 28 February 1986; revised 29 June 1987

Introduction

This paper concerns properties of interpretations between first-order theories in intuitionistic logic, and in particular how certain *syntactic* properties of such interpretations can be characterised by their *model-theoretic* properties.

We allow theories written in possibly many-sorted languages. Given two such theories $\mathcal{T}$ and $\mathcal{T}'$, an *interpretation* of $\mathcal{T}$ in $\mathcal{T}'$ will here mean a *model* of $\mathcal{T}$ in the 'first-order intuitionistically definable sets of $\mathcal{T}'$': these can be built up from the basic sorts of $\mathcal{T}'$ using the operations of finite cartesian product, finite disjoint union, separating out a $\mathcal{T}'$-definable subset and quotienting by a $\mathcal{T}'$-definable equivalence relation. (Allowing the formation of quotients (i.e. allowing equality relations in $\mathcal{T}$ to be interpreted by equivalence relation in $\mathcal{T}'$) and more especially the formation of disjoint unions, makes this a more general notion of 'interpretation' or 'translation' than is usually encountered in the literature.) Such an interpretation $I:\mathcal{T}\to\mathcal{T}'$ gives one a way of producing models of $\mathcal{T}$ from models of $\mathcal{T}'$: given a model M of $\mathcal{T}'$ in some semantics for first order intuitionistic logic, restricting along I yields a model $I^*(M)$ of $\mathcal{T}$. Similarly if $h:M_1\cong M_2$ is an isomorphism of $\mathcal{T}'$-models, one gets by restriction along I, an isomorphism $I^*(h):I^*(M_1)\cong I^*(M_2)$ of $\mathcal{T}$-models. Fixing a particular semantics, let us say that I induces an *equivalence between the models of $\mathcal{T}'$ and the models of $\mathcal{T}$* in the semantics if both of the following statements hold:

(a) every $\mathcal{T}$-model is isomorphic to one of the form $I^*(M)$ with M a $\mathcal{T}'$-model;

(b) if M_1, M_2 are $\mathcal{T}'$-models and $k:I^*(M_1)\cong I^*(M_2)$ is a $\mathcal{T}$-model isomorphism, then there is a unique $\mathcal{T}'$-model isomorphism $h:M_1\cong M_2$ with $k=I^*(h)$.

(In category-theoretic language: I induces an equivalence between models if the functor I^* gives an equivalence of categories between the category of $\mathcal{T}'$-models and isomorphisms and the cagegory of $\mathcal{T}$-models and isomorphisms.)

Conditions (a) and (b) are not much of a constraint on I if there are not many models of $\mathcal{T}$ or $\mathcal{T}'$ in the given semantics. So let us now assume that the semantics is *complete*, in the sense that a sentence is intuitionistically derivable from a theory $\mathcal{T}$ iff it is satisfied by every $\mathcal{T}$-model in the semantics. (For example, one could consider sheaf models over complete Heyting algebras, as in [5].)

A main result of this paper (Theorem 2.10(ii)) is then:

Conceptual Completeness for First-Order Intuitionistic Logic. *If an interpretation $I:\mathcal{T}\to\mathcal{T}'$ induces an equivalence between the models of $\mathcal{T}'$ and the models of $\mathcal{T}$ (in some fixed, complete semantics), then I is already a 'syntactic equivalence', i.e. there is an interpretation $J:\mathcal{T}'\to\mathcal{T}$ with the compositions $J\circ I$ and $I\circ J$ isomorphic to the identity interpretations on $\mathcal{T}$ and $\mathcal{T}'$ respectively.*

This theorem is in fact a consequence of the following, stronger result (Theorem 2.10(i)):

If an interpretation $I:\mathcal{T}\to\mathcal{T}'$ satisfies just condition (b) *above (for models in the fixed, complete semantics), then $\mathcal{T}'$ is syntactically equivalent to a quotient theory of $\mathcal{T}$, i.e. one that can be obtained by adding additional axioms to $\mathcal{T}$ without changing its underlying language.*

A version of this second result for (countable theories in) *classical* first-order logic and set-valued models, was announced by Haim Gaifman in 1975. A proof has been given by Michael Makkai in [12], using a mixture of category-theoretic and classical model-theoretic techniques (and the Omitting Types theorem in particular). The latter methods are of little help in proving the intuitionistic versions stated above. Instead, we exploit the full power of the category-theoretic ones, as we now indicate:

Firstly, the category-theoretic approach to logic developed by Freyd, Joyal, Makkai, Reyes and others, allows the two theorems above to be reformulated as statements about certain categories and functors. The way in which one can use categories and functors in place of theories and models is recalled in Section 1. At least one advantage of this approach is to allow a very smooth treatment of the particular notion of interpretation that we have to consider: this is carried out in Section 2.

More crucially, the category-theoretic reformulation enables the powerful *functorial* techniques of category theory to be directly applied. Thus at the heart of our proof is a certain functorial construction which from a conceptual point of view produces (generalised) spaces from theories and (generalised) continuous maps from interpretations. The word 'generalised' in the previous sentence refers to the fact that the construction in fact yields *Grothendieck toposes* and *geometric morphisms* rather than topological spaces and continuous functions. A Grothendieck topos is a generalisation of the notion of topological space via its category of set-valued sheaves. They were originally introduced by Grothendieck and his school of algebraic geometry [1] in order to develop notions of sheaf and cohomology adequate for algebraic geometry. Later, an intimate connection between Grothendieck toposes and the so-called *geometric fragment* ($=$, $\wedge$, $\exists$, $\bigvee$) of infinitary first-order logic emerged. (See [10], [6] and the references therein.) It is really this aspect that it is to the fore here. The properties of geometric morphisms between Grothendieck toposes that we need are set out in Section 3: the only ones that appear to be new are the (rather easy) characterisation of *localic* morphisms in terms of the associated diagonal morphism being an inclusion (Proposition 3.5(ii)) and as a corollary, a sufficient condition for being localic in terms of the existence of *descent data* for sheaves (Proposition 3.7).

The method by which we utilise the sheaf-theoretic results of Section 3 is the 'topos of filters' construction introduced in [14]. There it was used to prove an interpolation theorem for interpretations between first-order intuitionistic theories and indeed that result plays a key role here. The relevant properties of the topos of filters construction are summarized at the beginning of Section 4 and then used to prove the conceptual completeness theorem. In an appendix, the details of the construction are given in a somewhat different form from that in [14], in terms of sites and sheaves.

The original instance of this kind of result about interpretations was the *Pretopos Conceptual Completeness Theorem* of Makkai and Reyes [10, Theorem 7.1.8]. This deals with the coherent fragment ($=$, $\wedge$, $\vee$, $\exists$) of first-order logic, where it is natural to consider categories of models and *homomorphisms* rather than isomorphisms. A version of this theorem for pretoposes has been proved by the author in [15] using similar methods to those outlined above: it is a *constructive* version, in the sense that the arguments can be carried out in the category theory of an arbitrary elementary topos with natural number object. Although we have not emphasised this aspect it remains true here, *provided* the conclusion of the conceptual completeness theorem stated above is suitably modified by taking 'syntactic equivalence' to mean that the functor induced between classifying Heyting pretoposes by an interpretation I (as in 2.3) is full, faithful and essentially surjective. (Constructively this is weaker than asserting the existence of J with $J \circ I \cong \mathrm{Id}$, $\mathrm{Id} \cong I \circ J$, but just as useful in practice.)

I gratefully acknowledge many helpful conversations on the subject matter of this paper with P. Freyd, A. Joyal, M. Makkai and M. Zawadowski. I also acknowledge the helpful comments of the referee.

1. Theories and categories

In this section we survey those parts of the category-theoretic approach to first-order logic that enable the properties of theories and interpretations in which we are interested to be reformulated as properties of certain kinds of category and functor. The reader is referred to the account of Makkai and Reyes [10] for fuller details.

Let $\mathscr{L}$ be a many-sorted language: thus $\mathscr{L}$ consists of a collection of *sort, relation* and *function* symbols. Moreover each relation or function symbol has a designated *type*; formally these are just finite lists of sort symbols (non-empty lists in the latter case), but to suggest the intended interpretation we will denote the type of a relation symbol R by

$$R \rightarrowtail S_0 \times \cdots \times S_{n-1}$$

and the type of a function symbol f by

$$f : S_0 \times \cdots \times S_{n-1} \to S,$$

where $S_0, \ldots, S_{n-1}$ and S are sort symbols. (Since we allow the case $n = 0$, constant symbols are special instances of function symbols.)

Introducing variables for each sort, the *terms* of $\mathscr{L}$ are built up from these and the function symbols in the usual way, each term being assigned its particular sort. Then the *first-order formulae* of $\mathscr{L}$ are constructed using $\wedge$, $\vee$, $\to$, $\exists$, $\forall$ from the *atomic* formulae: $\top$ (truth), $\bot$ (falsity), $R(t_0, \ldots, t_{n-1})$ and $t = t'$ (where R is a relation symbol and t_i, t, t' are terms of appropriate sorts).

For such a language $\mathscr{L}$, the usual notions of *structure* and *satisfaction* of a formula by a structure can be generalised by first seeing how to define these notions in terms of *category-theoretic* properties of the category of sets and functions, and then by replacing the latter by an arbitrary category with these properties. The basic idea is that an $\mathscr{L}$-structure, M, in a category $\mathscr{C}$ assigns to each sort symbol S an *object MS* of $\mathscr{C}$, to each function symbol f a *morphism Mf* with appropriate domain and codomain, and to each relation symbols R a *subobject MR* of an appropriate object of $\mathscr{C}$. To preserve the intended interpretation of the typings of the function and relation symbols, we assume that $\mathscr{C}$ has finite products: thus if $R \rightarrowtail S_0 \times \cdots \times S_{n-1}$ and $f : S_0 \times \cdots \times S_{n-1} \to S$, then MR is to be a subobject of the finite product $MS_0 \times \cdots \times MS_{n-1}$ in $\mathscr{C}$ and Mf a morphism in $\mathscr{C}$ from $MS_0 \times \cdots \times MS_{n-1}$ to MS.

One can now define the value of terms in the $\mathscr{L}$-structure M. Given a term t of sort S and a finite list $\boldsymbol{x} = x_0 \cdots x_{n-1}$ of distinct variables amongst which lie the variables mentioned in t, we define a morphism

$$M(t; \boldsymbol{x}) : MS_0 \times \cdots \times MS_{n-1} \to MS$$

(where x_i is of sort S_i) by structural induction;

· $M(x_i; \boldsymbol{x})$ is the ith projection morphism, π_i;

· $M(f(t_0, \ldots, t_{m-1}); \boldsymbol{x})$ is $Mf \circ \langle M(t_j; \boldsymbol{x}) \mid j < m \rangle$

(where f has type $S'_0 \times \cdots \times S'_{m-1} \to S$ and $\langle M(t_j; \boldsymbol{x}) \mid j < m \rangle$ denotes the unique morphism whose composition with each π_j is $M(t_j; \boldsymbol{x})$.

But to define the value of first-order formulae in the $\mathscr{L}$-structure, we have to make further assumptions about the category $\mathscr{C}$. Thus given a formula ϕ and a finite list of distinct variables $\boldsymbol{x}$ amongst which lie the free variables of ϕ, we define by structural induction a subobject

$$M(\phi;\boldsymbol{x}) \rightarrowtail MS_0 \times \cdots \times MS_{n-1}$$

(where S_i is the sort of x_i). The clauses of this definition (which will be given below) give the category-theoretic explanations of the logical symbols $=$, $\top$, $\wedge$, $\perp$, $\vee$, $\rightarrow$, $\exists$ and $\forall$. Equality is interpreted by means of equalizers of parallel pairs of maps; and once we assume that $\mathscr{C}$ has equalizers as well as finite products then of course it has all finite limits, including pullbacks. The propositional connectives are interpreted by the corresponding lattice-theoretic operations on the partially ordered sets $\mathrm{Sub}_{\mathscr{C}}(X)$ of subobjects of an object X in $\mathscr{C}$: $\top$ and $\wedge$ require finite meets (which exist since $\mathscr{C}$ has finite limits), $\perp$ and $\vee$ require finite joins (and this is an added assumption on $\mathscr{C}$) and finally $\rightarrow$ requires Heyting implication (again, an added assumption on $\mathscr{C}$), so that $\mathrm{Sub}_{\mathscr{C}}(X)$ is a Heyting algebra. For quantification, it was a key observation of Lawvere that $\exists$ and $\forall$ can be interpreted in terms of left and right adjoints to the order-preserving operations of pulling back subobjects along morphisms; the existence of such adjoints is a further requirement on $\mathscr{C}$. This does not quite finish the list of necessary assumptions on $\mathscr{C}$. Under this explanation of the value of terms and formulae of $\mathscr{L}$ in $\mathscr{C}$, the operation of substituting a term for a variable in a formula becomes that of pulling back the subobject which is the value of the formula along the morphism which is the value of the term. Then, to ensure that substitution has the correct properties with respect to $=$, $\top$, $\wedge$, $\perp$, $\vee$, $\rightarrow$, $\exists$ and $\forall$, we have to assume that the category-theoretic notions mentioned above have suitable stability properties with respect to pulling back along a morphism.

Collecting together the various requirements on $\mathscr{C}$ mentioned in the previous paragraph and eliminating some redundancies, one arrives at the notion of a *logos* (a terminology popularised by P. Freyd):

1.1. Definition. A category $\mathscr{C}$ is called a *logos* if it has the following properties:

(i) $\mathscr{C}$ has finite limits.

(ii) For each object X of $\mathscr{C}$, the partially ordered set $\mathrm{Sub}_{\mathscr{C}}(X)$ of subobjects of X has finite (including empty) joins.

(iii) For each morphism $f: Y \rightarrow X$ in $\mathscr{C}$, the order-preserving operation $f^{-1}: \mathrm{Sub}_{\mathscr{C}}(X) \rightarrow \mathrm{Sub}_{\mathscr{C}}(Y)$ of pulling back a subobject along f (which exists by virtue of (i)) has both left and right adjoints, denoted $\exists f$ and $\forall f$ respectively. These adjoints are called the operations of *existential* and *universal quantification* of subobjects along f.

(iv) For each pullback square

$$\begin{array}{ccc} W & \xrightarrow{h} & Z \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

in $\mathscr{C}$, it is the case that $g^{-1} \circ \exists f = \exists h \circ k^{-1}$.

1.2. Remarks and notation. (i) As usual, $m : A \rightarrowtail X$ will denote that a morphism m is a monomorphism. We will (harmlessly) confuse such an m with the subobject of X which it determines. Clause (i) of Definition 1.1 guarantees that $\mathrm{Sub}_{\mathscr{C}}(X)$ has a top element, denoted $\top$, and binary meet, denoted $A \wedge B$.

(ii) The bottom element and binary join in $\mathrm{Sub}_{\mathscr{C}}(X)$ that are guaranteed by 1.1(ii) will be denoted $\perp$ and $A \vee B$ respectively. Note that pulling back preserves joins, since by 1.1(iii) each f^{-1} has a right adjoint.

(iii) Call a morphism $f : X \to Y$ a *cover* if $\exists f(\top) = \top$ in $\mathrm{Sub}_{\mathscr{C}}(Y)$. We will denote this by

$$f : X \twoheadrightarrow Y.$$

The existence of the left adjoint $\exists f$ satisfying 1.1(iv) is equivalent to asking that every morphism factor as a cover composed with a monomorphism and that covers be stable under pullback.

(iv) Clause (iv) of 1.1 has come to be called a *Beck–Chevalley condition*; it implies the same condition for $\forall$.

(v) Note that each $\mathrm{Sub}_{\mathscr{C}}(X)$ is a Heyting algebra: finite meets and joins have been mentioned above and for $A, B \in \mathrm{Sub}_{\mathscr{C}}(X)$, the Heyting implication $A \to B$ is given by $\forall a\,(a^{-1}B)$ where a is a monomorphism representing A. From this observation and the Beck–Chevalley condition for $\forall$, it follows that the operations f^{-1} preserve $\to$.

(vi) Note that being a logos is a *category-theoretic property*: if $\mathscr{C}$ is a logos and $\mathscr{D}$ is a category equivalent to $\mathscr{C}$, then $\mathscr{D}$ is also a logos.

Suppose now that $\mathscr{C}$ is a logos and $\mathscr{L}$ a many-sorted language. As indicated above, an *$\mathscr{L}$-structure in $\mathscr{C}$* assigns to each sort symbol S an object MS of $\mathscr{C}$, to each function symbol $f : S_0 \times \cdots \times S_{n-1} \to S$ a morphism

$$Mf : MS_0 \times \cdots \times MS_{n-1} \to MS$$

in $\mathscr{C}$, and to each relation symbol $R \rightarrowtail S_0 \times \cdots \times S_{n-1}$ a subobject

$$MR \rightarrowtail MS_0 \times \cdots \times MS_{n-1}.$$

Now if ϕ is a formula of $\mathscr{L}$ whose free variables occur in the finite list $\boldsymbol{x}$, define

$$M(\phi; \boldsymbol{x}) \rightarrowtail \boldsymbol{MS}$$

(where x_i is of sort S_i and $\boldsymbol{MS}$ abbreviates $MS_0 \times \cdots \times MS_{n-1}$) by structural induction:

(a)
$$\begin{array}{ccc} M(R(\boldsymbol{t}); \boldsymbol{x}) & \rightarrowtail & \boldsymbol{MS} \\ \downarrow & & \downarrow {\scriptstyle \langle M(t_i; \boldsymbol{x}) \mid i<n \rangle} \\ MR & \rightarrowtail & \boldsymbol{MS} \end{array}$$
is a pullback square;

(b)
$$M(t = t'; \boldsymbol{x}) \rightarrowtail \boldsymbol{MS} \underset{M(t'; \boldsymbol{x})}{\overset{M(t; \boldsymbol{x})}{\rightrightarrows}} MS$$
is an equalizer

(where $M(t; \boldsymbol{x})$ is defined as above);

(c) $M(\top; \boldsymbol{x}) = \top, \qquad M(\bot; \boldsymbol{x}) = \bot;$

(d) $M(\phi \# \psi; \boldsymbol{x}) = M(\phi; \boldsymbol{x}) \# M(\psi; \boldsymbol{x})$ (where $\#$ is $\wedge$, $\vee$ or $\rightarrow$);

(e) $M(Qx\, \phi; \boldsymbol{x}) = Q\pi_2 M(\phi; \boldsymbol{x}x)$

(where Q is $\exists$ or $\forall$, x has sort S and $\pi_2 : \boldsymbol{MS} \times \boldsymbol{MS} \rightarrow \boldsymbol{MS}$ is the second projection).

If σ is a sentence of $\mathscr{L}$ (i.e. a formula with no free variables), then $M(\sigma;\ )$ is a subobject of the empty product, i.e. of the terminal object 1 of $\mathscr{C}$. Say that M *satisfies* σ and write

$$M \models \sigma$$

if $M(\sigma;\ ) \rightarrowtail 1$ is the top subobject $\top$. This notion of satisfaction is sound for the *intuitionistic* predicate calculus (IPC), in the sense that a sentence σ of $\mathscr{L}$ is always satisfied by any $\mathscr{L}$-structure in any logos if it is provable in IPC. However, one must exercise a little care over the precise meaning of 'provable in IPC'. For classical logic and the model theory of set-valued structures it is customary to make the somewhat unnatural assumption that sorts are interpreted by *non-empty* sets. When we move to intuitionistic logic this assumption would translate to the *very* unnatural one that sorts be interpreted by *inhabited* objects. (An object X of a logos is inhabited if the unique morphism from X to the terminal object 1 is a cover.) We make no such assumption and so require a formulation of the axioms and rules of IPC that reflect this fact. One good way of doing this is via a calculus of *labelled* sequents, where the usual notion of sequent is modified by including a list of variables, amongst which should be the free variables of any formula mentioned in the sequent. The reader can see this carried out in detail for higher order logic in [4]. (An alternative approach is taken by Makkai and Reyes in [10, Chapter 5].)

With those provisos, and writing

$$\text{IPC} \vdash \sigma$$

to mean that the sentence σ is derivable in such a suitable system of deduction for first order intuitionistic predicate logic, let us record the result mentioned above:

1.3. Soundness Theorem. *If $\mathscr{C}$ is a logos, $\mathscr{L}$ a many-sorted language and M an $\mathscr{L}$-structure in $\mathscr{C}$, then for any $\mathscr{L}$-sentence σ, if* IPC $\vdash \sigma$, *then $M \models \sigma$.*

1.4. Definition. Let $\mathscr{T}$ be a *theory* in IPC, i.e. a many-sorted language $\mathscr{L}$ plus a collection of $\mathscr{L}$-sentences closed under deduction in IPC; write $\mathscr{T} \vdash \sigma$ to denote that a sentence σ is in this collection. By a *model* of $\mathscr{T}$ in a logos $\mathscr{C}$ is meant an $\mathscr{L}$-structure M in $\mathscr{C}$ such that $M \models \sigma$ whenever $T \vdash \sigma$.

This notion of model encompasses the possibly more familiar ones. Thus when $\mathscr{C}$ is the category of sets, one regains the classical notion of model. More generally, when $\mathscr{C}$ is the category of set-valued sheaves on a topological space, one regains the topological semantics for IPC (including Kripke and Beth semantics).

We wish to organise the models of a theory $\mathscr{T}$ in a logos $\mathscr{C}$ into a category and hence must consider homomorphisms of models. More generally, given $\mathscr{L}$-structures M and N in $\mathscr{C}$, a *homomorphism* $h: M \to N$ is given by a collection of morphisms

$$h_s : MS \to NS$$

indexed by the sort symbols of $\mathscr{L}$, which satisfies that for each function symbol $f: S_0 \times \cdots \times S_{n-1} \to S$ the diagram in $\mathscr{C}$

$$\begin{tikzcd} MS_0 \times \cdots \times MS_{n-1} \arrow[r, "Mf"] \arrow[d, "h_{s_0} \times \cdots \times h_{s_{n-1}}"'] & MS \arrow[d, "h_s"] \\ NS_0 \times \cdots \times NS_{n-1} \arrow[r, "Nf"'] & NS \end{tikzcd}$$

commutes, and that for each relation symbol $R \rightarrowtail S_0 \times \cdots \times S_{n-1}$ there is a commutative square

$$\begin{tikzcd} MR \arrow[r, rightarrowtail] \arrow[d] & MS_0 \times \cdots \times MS_{n-1} \arrow[d, "h_{s_0} \times \cdots \times h_{s_{n-1}}"] \\ NS \arrow[r, rightarrowtail] & NS_0 \times \cdots \times NS_{n-1} \end{tikzcd}$$

in $\mathscr{C}$ (i.e. $\exists(h_{s_0} \times \cdots \times h_{s_{n-1}})MR \leq NR$ in $\mathrm{Sub}_{\mathscr{C}}(NS_0 \times \cdots \times NS_{n-1})$).

Homomorphisms compose in the obvious manner and each $\mathscr{L}$-structure has an identity homomorphism: in this way, one gets a category of $\mathscr{L}$-structures and homomorphisms in the logos $\mathscr{C}$ which will be denoted

$$\mathrm{Mod}(\mathscr{L}, \mathscr{C}).$$

Then, if $\mathscr{T}$ is a theory in the language $\mathscr{L}$,

$$\mathrm{Mod}(\mathscr{T}, \mathscr{C})$$

will denote the full subcategory of $\mathrm{Mod}(\mathscr{L}, \mathscr{C})$ whose objects are the models of $\mathscr{T}$ in $\mathscr{C}$.

Next we consider the relationship between models of a theory $\mathscr{T}$ in *different* logoses. To do this we must introduce the notion of a morphism of logoses. Since a logos is a category with certain properties, it is clear that a logos morphism should be a functor which preserves these properties:

1.5. Definition. A *morphism of logoses* is a functor $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ between logoses which preserves finite limits, finite joins of subobjects and both existential and universal quantification of subobjects along maps. (The latter condition means that $\mathscr{F}(\exists f A) = \exists(\mathscr{F}f)(\mathscr{F}A)$ and $\mathscr{F}(\forall f A) = \forall(\mathscr{F}f)(\mathscr{F}A)$.)

Now if $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is such a morphism and M is an $\mathscr{L}$-structure in $\mathscr{C}$, since $\mathscr{F}$ preserves finite products and monomorphisms evidently one gets an $\mathscr{L}$-structure $\mathscr{F}_*M$ in $\mathscr{D}$ by letting

$$(\mathscr{F}_*M)S = \mathscr{F}(MS)$$

and for $f:S_0\times\cdots\times S_{n-1}\to S$ and $R\rightarrowtail S_0\times\cdots\times S_{n-1}$ letting $(\mathscr{F}_*M)f$ be

$$(\mathscr{F}_*M)S_0\times\cdots\times(\mathscr{F}_*M)S_{n-1}\cong\mathscr{F}(MS_0\times\cdots\times MS_{n-1})\xrightarrow{\mathscr{F}(Mf)}(\mathscr{F}_*M)S$$

and $(\mathscr{F}_*M)R$ be

$$\mathscr{F}(MR)\rightarrowtail\mathscr{F}(MS_0\times\cdots\times MS_{n-1})\cong(\mathscr{F}_*M)S_0\times\cdots\times(\mathscr{F}_*M)S_{n-1}.$$

Similarly, for a homomorphism $h:M\to N$ of $\mathscr{L}$-structures in $\mathscr{C}$, one gets a homomorphism $\mathscr{F}_*h:\mathscr{F}_*M\to\mathscr{F}_*N$ of $\mathscr{L}$-structures in $\mathscr{D}$, where

$$(\mathscr{F}_*h)_S = \mathscr{F}(h_S).$$

In this way we get a functor

$$\mathscr{F}_*:\mathrm{Mod}(\mathscr{L},\mathscr{C})\to\mathrm{Mod}(\mathscr{L},\mathscr{D}).$$

Moreover, for any theory $\mathscr{T}$ in the language $\mathscr{L}$, this functor restricts to one between categories of models:

$$\mathscr{F}_*:\mathrm{Mod}(\mathscr{T},\mathscr{C})\to\mathrm{Mod}(\mathscr{T},\mathscr{D}).$$

This is because the operation of applying $\mathscr{F}$ to subobjects preserves the interpretation of first order logic in $\mathscr{C}$, i.e. we have that the subobjects $\mathscr{F}(M(\phi;\boldsymbol{x}))$ and $\mathscr{F}_*M(\phi;\boldsymbol{x})$ correspond under the isomorphism $\mathscr{F}(M\boldsymbol{S})\cong(\mathscr{F}_*M)\boldsymbol{S}$; and hence in particular $M\vDash\sigma$ implies $\mathscr{F}_*M\vDash\sigma$ for any sentence σ of $\mathscr{L}$.

The assignment of $\mathrm{Mod}(\mathscr{T},\mathscr{C})$ to $\mathscr{C}$ and $\mathscr{F}_*:\mathrm{Mod}(\mathscr{T},\mathscr{C})\to\mathrm{Mod}(\mathscr{T},\mathscr{D})$ to $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ extends to natural transformations between logos morphisms. Given two such morphisms $\mathscr{F},\mathscr{F}':\mathscr{C}\rightrightarrows\mathscr{D}$ and a natural transformation $\alpha:\mathscr{F}\to\mathscr{F}'$, one gets

$$\alpha_*:\mathscr{F}_*\to\mathscr{F}'_*$$

by defining α_* to be the natural transformation whose component at an object M of $\mathrm{Mod}(\mathscr{T},\mathscr{C})$ is the homomorphism of $\mathscr{T}$-models

$$\alpha_*M:\mathscr{F}_*M\to\mathscr{F}'_*M$$

whose value at a sort symbol S is

$$(\alpha_*M)_S=\alpha_{MS}:\mathscr{F}(MS)\to\mathscr{F}'(MS).$$

We now come to a crucial point: the concept of a logos as a place for interpreting intuitionistic first-order logic is sufficiently flexible to allow the construction, for a given theory, of a logos containing a *generic* (or *universal*) model of the theory. The following theorem makes precise what is meant by this:

1.6. Theorem. *Let $\mathcal{T}$ be a theory in* IPC. *There is a logos $\mathcal{C}(\mathcal{T})$ and a model G of $\mathcal{T}$ in $\mathcal{C}(\mathcal{T})$ with the following properties*:

(a) *For any other logos $\mathcal{D}$ and model M of $\mathcal{T}$ in $\mathcal{D}$, there is a logos morphism $\mathcal{F}:\mathcal{C}(\mathcal{T})\to\mathcal{D}$ and an isomorphism $\mathcal{F}_*(G)\cong M$ of $\mathcal{T}$-models in $\mathcal{D}$.*

(b) *If $\mathcal{F}$, $\mathcal{F}':\mathcal{C}(\mathcal{T})\to\mathcal{D}$ are both logos morphisms and $a:\mathcal{F}_*(G)\cong\mathcal{F}'_*(G)$ is a $\mathcal{T}$-model isomorphism in $\mathcal{D}$, then there is a unique natural isomorphism $\alpha:\mathcal{F}\cong\mathcal{F}'$ with $a=\alpha_*(G)$.*

Proof. We refer the reader to Chapter 8 of [10] for a detailed proof of this theorem, cofining ourselves here to a brief description of the logos $\mathcal{C}(\mathcal{T})$ and the *generic model* G.

Let $\mathcal{L}$ be the underlying language of the theory $\mathcal{T}$. Roughly speaking, the objects of $\mathcal{C}(\mathcal{T})$ are pairs $\phi;\boldsymbol{x}$ where ϕ is an $\mathcal{L}$-formula and $\boldsymbol{x}$ a finite list of *distinct* variables containing the free variables of ϕ. However, one does not wish to distinguish a pair $\phi;\boldsymbol{x}$ from a pair $\psi;\boldsymbol{y}$ if $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same length and sorts and ψ is obtained from ϕ by substituting $\boldsymbol{y}$ for $\boldsymbol{x}$ (and changing bound variables, if necessary). So quotient by the evident equivalence relation and let $[\phi;\boldsymbol{x}]$ denote the equivalence class of the pair $\phi;\boldsymbol{x}$: this is the typical object of $\mathcal{C}(\mathcal{T})$.

A morphism $[\phi;\boldsymbol{x}]\to[\psi;\boldsymbol{y}]$ in $\mathcal{C}(\mathcal{T})$ is determined by a formula θ whose free variables lie amongst the list $\boldsymbol{x}$, $\boldsymbol{y}$ (which we take to consist of distinct variables by choosing a suitable representative from the equivalence class $[\psi;\boldsymbol{y}]$), and which satisfies

$$\mathcal{T}\vdash\forall\boldsymbol{x},\boldsymbol{y}\ \theta(\boldsymbol{x},\boldsymbol{y})\to\phi(\boldsymbol{x})\wedge\psi(\boldsymbol{y}), \tag{1}$$

$$\mathcal{T}\vdash\forall\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}'\ \theta(\boldsymbol{x},\boldsymbol{y})\wedge\theta(\boldsymbol{x},\boldsymbol{y}')\to\boldsymbol{y}=\boldsymbol{y}' \quad\text{and} \tag{2}$$

$$\mathcal{T}\vdash\forall\boldsymbol{x}\ (\phi(\boldsymbol{x})\to\exists\boldsymbol{y}\ \theta(\boldsymbol{x},\boldsymbol{y})). \tag{3}$$

However, if θ' is another such formula it should determine the same morphism provided

$$\mathcal{T}\vdash\forall\boldsymbol{x},\boldsymbol{y}\ (\theta\leftrightarrow\theta'). \tag{4}$$

Thus a typical morphism from $[\phi;\boldsymbol{x}]$ to $[\psi;\boldsymbol{y}]$ is an equivalence class of formulae θ satisfying (1), (2) and (3) under the equivalence relation determined by (4).

The generic model G of $\mathcal{T}$ in $\mathcal{C}(\mathcal{T})$ sends a sort symbol S to $GS=[x=x;x]$, where x is of sort S; more generally if $\boldsymbol{x}$ is of sort $\boldsymbol{S}$, then $[x_1=x_1\wedge\cdots\wedge x_{n-1}=x_{n-1};\boldsymbol{x}]$ is the product of the objects GS_i in $\mathcal{C}(\mathcal{T})$. G sends a function symbol $f:\boldsymbol{S}\to S$ to the equivalence class of $f(\boldsymbol{x})=y$; and it sends a relation symbol $R\rightarrowtail\boldsymbol{S}$ to the subobject represented by the monomorphism from $[R(\boldsymbol{x});\boldsymbol{x}]$ to $[\boldsymbol{x}=\boldsymbol{x};\boldsymbol{x}]$ determined by $R(\boldsymbol{x})\wedge\boldsymbol{x}=\boldsymbol{x}'$. Arguing by structural induction, one finds that more generally

$$G(\phi; \boldsymbol{x}) \rightarrowtail GS_0 \times \cdots \times GS_{n-1}$$

is given by the monomorphism

$$[\phi; \boldsymbol{x}] \rightarrowtail [\boldsymbol{x} = \boldsymbol{x}; \boldsymbol{x}]$$

determined by $\phi(\boldsymbol{x}) \wedge \boldsymbol{x} = \boldsymbol{x}'$. $\square$

1.7. Remarks. (i) For reasons that are evident from the above description, $\mathscr{C}(\mathscr{T})$ is often called the *syntactic category* of the theory $\mathscr{T}$. It is a generalisation of the Lindenbaum–Tarski algebra of $\mathscr{T}$ (which is here a Heyting algebra rather than a Boolean algebra since we are dealing with theories in intutionistic rather than classical logic). Indeed, $\mathscr{C}(\mathscr{T})$ contains the Lindenbaum–Tarksi algebra of $\mathscr{T}$ as the lattice of subobjects of the terminal object.

(ii) For logoses $\mathscr{C}$, $\mathscr{D}$, let $\mathrm{LOG}(\mathscr{C}, \mathscr{D})$ denote the category whose objects are logos morphisms from $\mathscr{C}$ to $\mathscr{D}$ and whose morphisms are natural transformations. Let $\mathrm{LOG}_{\cong}(\mathscr{C}, \mathscr{D})$ denote the non-full subcategory of $\mathrm{LOG}(\mathscr{C}, \mathscr{D})$ with the same objects but with only natural isomorphisms for morphisms. Similarly, for a theory $\mathscr{T}$, let $\mathrm{Mod}_{\cong}(T, D)$ denote the non-full subcategory of $\mathrm{Mod}(\mathscr{T}, \mathscr{D})$ consisting of $\mathscr{T}$-models and $\mathscr{T}$-model isomorphisms in $\mathscr{D}$.

In more category-theoretic language, Theorem 1.6 says that the functor

$$\mathrm{LOG}(\mathscr{C}(\mathscr{T}), \mathscr{D}) \rightarrow \mathrm{Mod}(\mathscr{T}, \mathscr{D}),$$

$$(\mathscr{F} \xrightarrow{\alpha} \mathscr{F}') \mapsto (\mathscr{F}_*(G) \xrightarrow{\alpha_*(G)} \mathscr{F}'_*(G))$$

is (a) essentially surjective and (b) full and faithful for isomorphisms.

Thus on restricting it to a functor

$$\mathrm{LOG}_{\cong}(\mathscr{C}(\mathscr{T}), \mathscr{D}) \rightarrow \mathrm{Mod}_{\cong}(\mathscr{T}, \mathscr{D})$$

one obtains an *equivalence* of categories (that is pseudo-natural in $\mathscr{D}$). For the usual category-theoretic reasons, $\mathscr{C}(\mathscr{T})$ is determined by this property up to equivalence and G is determined up to isomorphism.

(iii) Each logos $\mathscr{C}$ is equivalent to a syntactic category $\mathscr{C}(\mathscr{T})$ for some theory $\mathscr{T}$ in IPC. For example, let $\mathscr{L}_{\mathscr{C}}$ be the language which has sort symbols for each object of $\mathscr{C}$, function symbols for the morphisms in $\mathscr{C}$ and relation symbols for the subobjects in $\mathscr{C}$. There is an evident $\mathscr{L}_{\mathscr{C}}$-structure in $\mathscr{C}$ which takes a symbol to the object, morphism or suboject it names; and the collection of sentences that are satisfied by this structure form a theory $\mathscr{T}_{\mathscr{C}}$ for which one has $\mathscr{C}(\mathscr{T}_{\mathscr{C}}) \simeq \mathscr{C}$.

In particular, the properties of being a monomorphism, a finite limit, a finite union of subobjects or the existential or universal quantification of a subobject along a morphism are all expressible by first-order formulae in $\mathscr{L}_{\mathscr{C}}$: cf. Chapter 2, Section 4 of [10]. There are however further categorical concepts which are similarly expressible in $\mathscr{L}_{\mathscr{C}}$, namely the category-theoretic formulations of the notions of disjoint union and quotient by an equivalence relation. We recall their definitions:

1.8. Definitions. Let $\mathscr{C}$ be a logos.

(i) The *disjoint coproduct* of finitely many objects $(X_i \mid i<n)$ of $\mathscr{C}$ is given by an object X together with monomorphisms $(m_i; X_i \rightarrowtail X \mid i<n)$ such that in $\mathrm{Sub}_{\mathscr{C}}(X)$ one has

$$X_i \wedge X_j = \bot \quad \text{when } i \neq j, \quad \text{and} \quad \bigvee_{i<n} X_i = \top\,.$$

(ii) The *effective coequalizer* of an equivalence relation $\langle a, b\rangle : R \rightarrowtail X \times X$ in $\mathscr{C}$ is given by a morphism $q: X \to Q$ which is a cover (i.e. $\exists q(\top) = \top$) and which makes

$$\begin{array}{ccc} R & \xrightarrow{b} & X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[q]{} & Q \end{array}$$

a pullback square in $\mathscr{C}$.

In a logos, finite disjoint coproducts *are* coproducts and effective coequalizers *are* coequalizers. That a diagram in $\mathscr{C}$ is one of these special kinds of colimit is expressible by first-order (indeed, by 'coherent') formulae of $\mathscr{L}_{\mathscr{C}}$. However, a logos in general will not have all finite disjoint coproducts or effective coequalizers for all equivalence relations. Thus, thinking of the syntactic category $\mathscr{C}(\mathscr{T})$ as a category of '$\mathscr{T}$-definable sets and functions', it is in general lacking some of these first order definable concepts.

1.9. Definition. A logos which has all finite disjoint coproducts and effective coequalizers of equivalence relations will be called a *Heyting pretopos*.

1.10. Remarks. (i) The reason for the above terminology is that a Heyting pretopos is in particular a *pretopos*, i.e. a category with finite limits, finite coproducts that are disjoint and stable under pullback and coequalizers of equivalence relations that are effective and stable under pullback (cf. Chapter 3, Section 4 of [10]). Heyting pretoposes are precisely those pretoposes which possess right adjoints to the operations of pulling back subobjects along morphisms.

(ii) Note that a logos morphism necessarily preserves any disjoint coproducts or effective coequalizers of equivalence relations that happen to exist. Thus by a 'morphism of Heyting pretoposes' we shall mean just a morphism of logoses.

It is possible to 'complete' a given logos $\mathscr{C}$ to a Heyting pretopos by freely adjoining finite disjoint coproducts and effective coequalizers for equivalence relations. To be precise, given $\mathscr{C}$ one can construct a Heyting pretopos $\hat{\mathscr{C}}$ and a logos morphism

$$\mathscr{I}_{\mathscr{C}} : \mathscr{C} \to \hat{\mathscr{C}}$$

with the property that for any Heyting pretopos $\mathscr{H}$, the functor induced by $\mathscr{I}_{\mathscr{C}}$

$$\mathscr{I}^*_{\mathscr{C}} : \mathrm{LOG}(\hat{\mathscr{C}}, \mathscr{H}) \to \mathrm{LOG}(\mathscr{C}, \mathscr{H}),$$

$$\left(\mathscr{F} \xrightarrow{\alpha} \mathscr{F}'\right) \mapsto \left(\mathscr{F}\mathscr{I}_{\mathscr{C}} \xrightarrow{\alpha\mathscr{I}_{\mathscr{C}}} \mathscr{F}'\mathscr{I}_{\mathscr{C}}\right)$$

is an equivalence of categories. The construction of $\hat{\mathscr{C}}$ from $\mathscr{C}$ was described by Makkai and Reyes [12, Part II] (for coherent categories, but the construction for logoses is the same; see also [10]). Briefly, it can be described as follows:

First note that if $\mathscr{I} : \mathscr{C} \to \mathscr{H}$ is a logos morphism to a Heyting pretopos, we obtain a full subcategory of $\mathscr{H}$ *which is also a Heyting pretopos* by considering those objects Q of $\mathscr{H}$ for which there is a diagram of the form

$$q : \coprod_{i<n} \mathscr{I}(X_i) \twoheadrightarrow Q \tag{5}$$

where q is a cover whose domain is the disjoint coproduct of finitely many objects in the image of $\mathscr{I}$. Consequently when $\mathscr{I} = \mathscr{I}_{\mathscr{C}}$, we expect this full subcategory to be the whole of $\hat{\mathscr{C}}$. Now in (5), if we take the pullback of q against itself we obtain an equivalence relation on $\coprod_{i<n} \mathscr{I}(X_i)$ whose coequalizer is Q:

$$B \rightarrowtail \left(\coprod_{i<n} \mathscr{I}X_i\right) \times \left(\coprod_{j<n} \mathscr{I}X_j\right) \cong \coprod_{i,j<n} \mathscr{I}(X_i \times X_j). \tag{6}$$

Since the coproduct in (6) is disjoint and stable under pullback, we can express B as

$$\coprod_{i,j<n} B_{ij}$$

for subobjects

$$B_{ij} \rightarrowtail \mathscr{I}(X_i \times X_j), \quad i, j < n \tag{7}$$

Initially, the only subobjects we know about are those that come from $\mathscr{C}$; so let us assume that in (7) $B_{ij} \cong \mathscr{I}(A_{ij})$ where

$$A_{ij} \rightarrowtail X_i \times X_j, \quad i, j > n \tag{8}$$

in $\mathscr{C}$. (It does indeed turn out that $\mathscr{I}_{\mathscr{C}} : \mathscr{C} \to \hat{\mathscr{C}}$ is *full on subobjects* in the sense that for any object X of $\mathscr{C}$, any subobject of $\mathscr{I}_{\mathscr{C}}(X)$ in $\hat{\mathscr{C}}$ is in the image of $\mathscr{I}_{\mathscr{C}}$.)

Putting all this together, one defines a typical object of $\hat{\mathscr{C}}$ to consist of a finite sequence $(X_i \mid i < n)$ of objects of $\mathscr{C}$ plus a matrix (8) of subobjects in $\mathscr{C}$ satisfying the following conditions, which for readability are written in the language $\mathscr{L}_{\mathscr{C}}$

associated to $\mathscr{C}$ as in Remark 1.7(iii):

$$\mathscr{C} \models \bigwedge_{i<n} \forall x_i \, A_{ii}(x_i, x_i),$$

$$\mathscr{C} \models \bigwedge_{i,j<n} \forall x_i, x_j \, (A_{ij}(x_i, x_j) \to A_{ji}(x_j, x_i)) \quad \text{and}$$

$$\mathscr{C} \models \bigwedge_{i,j,k<n} \forall x_i, x_j, x_k \, (A_{ij}(x_i, x_j) \wedge A_{jk}(x_j, x_k) \to A_{ik}(x_i, x_k)).$$

In particular, $\mathscr{I}_{\mathscr{C}}(X)$ will be the object of $\hat{\mathscr{C}}$ with $n=1$, $X_{11}=X$ and A_{11} the diagonal subobject of $X \times X$.

The morphisms of $\hat{\mathscr{C}}$ are specified via their *graphs*. Thus given objects

$$A_{ij} \rightarrowtail X_i \times X_j, \quad i, j < m,$$
$$B_{kl} \rightarrowtail Y_k \times Y_l, \quad k, l < n$$

and $\hat{\mathscr{C}}$, a morphism from the first to the second is given by a matrix of subobjects

$$F_{ik} \rightarrowtail X_i \times Y_k, \quad i < m, k < n$$

satisfying:

$$\mathscr{C} \models \bigwedge_{i<m} \bigwedge_{k<n} \forall x_i, y_k \, (F_{ik}(x_i, y_k) \to A_{ii}(x_i, x_i) \wedge B_{kk}(y_k, y_k)),$$

$$\mathscr{C} \models \bigwedge_{i,j<m} \bigwedge_{k<n} \forall x_i, x_j, y_k \, (A_{ij}(x_i, x_j) \wedge F_{jk}(x_j, x_k) \to F_{ik}(x_i, y_k)),$$

$$\mathscr{C} \models \bigwedge_{i<m} \bigwedge_{k,l<n} \forall x_i, y_k, y_l \, (F_{ik}(x_i, y_k) \wedge B_{kl}(y_k, y_l) \to F_{il}(x_i, y_l)),$$

$$\mathscr{C} \models \bigwedge_{i<m} \bigwedge_{k,l<n} \forall x_i, y_k, y_l \, (F_{ik}(x_i y_k) \wedge F_{il}(x_i, y_l) \to B_{kl}(y_k, y_l)) \quad \text{and}$$

$$\mathscr{C} \models \bigwedge_{i<m} \forall x_i \, (A_{ii}(x_i, x_i) \to \bigvee_{k<n} \exists y_k \, F_{ik}(x_i, y_k)).$$

In particular for $f: X \to Y$ in $\mathscr{C}$, $\mathscr{I}_{\mathscr{C}}(f): \mathscr{I}_{\mathscr{C}}(X) \to \mathscr{I}_{\mathscr{C}}(Y)$ is given simply by the graph of f:

$$\langle \mathrm{id}, f \rangle : X \rightarrowtail X \times Y.$$

These are the essential details of the construction of $\mathscr{I}_{\mathscr{C}}: \mathscr{C} \to \hat{\mathscr{C}}$. We leave to the reader's imagination how composition and identity morphisms are defined for $\hat{\mathscr{C}}$. The proof that $\hat{\mathscr{C}}$ is a Heyting pretopos, that $\mathscr{I}_{\mathscr{C}}$ is a logos morphism and that it has the requisite universal property are then routine calculations, if one uses the characterisation (mentioned in Remark 1.7(iii)) of the relevant category-theoretic concepts by first order formulae of $\mathscr{L}_{\mathscr{C}}$ and the Soundness Theorem 1.3. We record some properties of $\mathscr{I}_{\mathscr{C}}: \mathscr{C} \to \hat{\mathscr{C}}$ that come out of the description just given:

1.11. Lemma. (i) *$\mathscr{I}_{\mathscr{C}}$ is 'full on subobjects', i.e. for each object X of $\mathscr{C}$ the map*

$$\mathrm{Sub}_{\mathscr{C}}(X) \to \mathrm{Sub}_{\hat{\mathscr{C}}}(\mathscr{I}_{\mathscr{C}} X),$$
$$A \mapsto \mathscr{I}_{\mathscr{C}}(A)$$

is surjective.

(ii) *$\mathcal{I}_{\mathscr{C}}$ is 'conservative', i.e. the maps mentioned in* (i) *are also injective* (*and hence are bijections*).

(iii) *Every object Q of $\hat{\mathscr{C}}$ is 'finitely covered via $\mathcal{I}_{\mathscr{C}}$', i.e. there are finitely many objects $(X_i \mid i < n)$ of $\mathscr{C}$ and a cover*

$$\coprod_{i<n} \mathcal{I}_{\mathscr{C}}(X_i) \twoheadrightarrow Q \qquad \text{in } \hat{\mathscr{C}}.$$

1.12. Notation. If $\mathcal{T}$ is a theory in IPC, then as in Theorem 1.6 one has a logos $\mathscr{C}(\mathcal{T})$ and a generic model $G \in \mathrm{Mod}(\mathcal{T}, \mathscr{C}(\mathcal{T}))$. Let $\mathscr{H}(\mathcal{T})$ denote the Heyting pretopos completion $\mathscr{C}(\mathcal{T})\hat{}$, and let $\mathrm{Id}_{\mathcal{T}}$ denote the model of $\mathcal{T}$ in $\mathscr{H}(\mathcal{T})$ obtained from G by appling $\mathcal{I}_{\mathscr{C}(\mathcal{T})} : \mathscr{C}(\mathcal{T}) \to \mathscr{H}(\mathcal{T})$. (The reason for the notation $\mathrm{Id}_{\mathcal{T}}$ will become apparent in Section 2.)

We shall call $\mathscr{H}(\mathcal{T})$ the *classifying Heyting pretopos* of the theory $\mathcal{T}$. Combining Remark 1.7 with the universal property of $\mathcal{I}_{\mathscr{C}(\mathcal{T})} : \mathscr{C}(\mathcal{T}) \to \mathscr{C}(\mathcal{T})\hat{} = \mathscr{H}(\mathcal{T})$, one finds that $\mathrm{Id}_{\mathcal{T}}$ *is universal amongst model of $\mathcal{T}$ in Heyting pretoposes* in the sense that for all Heyting pretoposes $\mathscr{K}$ the functor

$$\mathrm{LOG}_{\cong}(\mathscr{H}(\mathcal{T}), \mathscr{K}) \to \mathrm{Mod}_{\cong}(\mathcal{T}, \mathscr{K}),$$
$$\left(\mathcal{F} \xrightarrow{\alpha} \mathcal{F}'\right) \mapsto \left(\mathcal{F}_*(\mathrm{Id}_{\mathcal{T}}) \xrightarrow{\alpha_*(\mathrm{Id}_{\mathcal{T}})} \mathcal{F}'_*(\mathrm{Id}_{\mathcal{T}})\right) \tag{9}$$

is an equivalence of categories.

Combining the construction of the syntactic category $\mathscr{C}(\mathcal{T})$ from $\mathcal{T}$ and the construction of $\hat{\mathscr{C}}$ from $\mathscr{C}$, one obtains an explicit description of $\mathscr{H}(\mathcal{T})$ in terms of the syntax of the theory $\mathcal{T}$. From this view-point $\mathscr{H}(\mathcal{T})$ is the category of '$\mathcal{T}$-definable first-order sets and functions': on the one hand it contains objects for the basic sorts of $\mathcal{T}$ and is closed under the operations of finite cartesian product, separating out a $\mathcal{T}$-definable subset, finite disjoint union and quotienting by a $\mathcal{T}$-definable equivalence relation; on the other hand, it is not too 'big', in the following sense (cf. Lemma 1.11).

1.13. Lemma. *Let $\mathscr{L}$ be the underlying language of a theory $\mathcal{T}$.*

(i) *Suppose $\boldsymbol{x}$ is a finite list of variables and $\boldsymbol{S}$ the corresponding list of their sorts. Consider the collection of first-order formulae $\phi(\boldsymbol{x})$ of L whose free variables are amongst the list $\boldsymbol{x}$, pre-ordered by $\mathcal{T}$-provability*:

$$\phi(\boldsymbol{x}) \leq \psi(\boldsymbol{x}) \quad \textit{iff} \quad \mathcal{T} \vdash \forall \boldsymbol{x}\, (\phi \to \psi).$$

The resulting pre-ordered set is equivalent to the lattice of subobjects of $\prod_{i<n} \mathrm{Id}_{\mathcal{T}}(S_i)$ in $\mathscr{H}(T)$.

(ii) *For each object Q of $\mathscr{H}(\mathcal{T})$ there is a diagram of the form*

$$\begin{array}{ccc} \cdot & \longrightarrow & Q \\ \big\downarrow & & \\ \coprod_{i<m} \prod_{j<n(i)} \mathrm{Id}_{\mathcal{T}}(S_{ij}) & & \end{array} \qquad \textit{in } \mathscr{H}(\mathcal{T}).$$

2. Interpretations and functors

Equipped with the concepts of a model of a theory in a logos and of the classifying Heyting pretopos of a theory, we can now give the definition of the particular notion of interpretation between theories that underlies the results presented in this paper:

2.1. Definition. Let $\mathcal{T}$ and $\mathcal{T}'$ be two theories in IPC. By an *interpretation* of $\mathcal{T}$ in $\mathcal{T}'$ we shall mean a model of $\mathcal{T}$ in the classifying Heyting pretopos of $\mathcal{T}'$, $\mathcal{H}(\mathcal{T}')$. An *isomorphism* between two such interpretations is just an isomorphism of $\mathcal{T}$-models in $\mathcal{H}(\mathcal{T}')$.

2.2. Remarks. In view of the syntactic nature of the objects and morphisms in the description we gave of $\mathcal{H}(\mathcal{T}')$ in Section 1, the above definition of interpretation can be reformulated in terms of the syntax of $\mathcal{T}$ and $\mathcal{T}'$. In this form it is the same as or includes the various notions of 'interpretation' or 'translation' of one first-order theory into another that have been considered in the literature. It is particularly important to note that since interpretations of $\mathcal{T}$ in $\mathcal{T}'$ are models in $\mathcal{H}(\mathcal{T}')$ rather than $\mathscr{C}(\mathcal{T}')$, we are allowing the basic sorts of $\mathcal{T}$ to be interpreted by *quotients* by definable equivalence relations of *finite disjoint unions* of definable subsets of finite products of the basic sorts of $\mathcal{T}'$. It is this level of generality which permits the Conceptual Completness Theorem (2.10), to be proved below; but at the same time, the features of a notion of 'interpretation' that one might wish are not lost by casting our net this wide. For example, the existence of this kind of interpretation certainly implies relative consistency. For if $\mathcal{T}$ is inconsistent, i.e. $\mathcal{T} \vdash \perp$, and I is an interpretation of $\mathcal{T}$ in $\mathcal{T}'$, then because I is a model of $\mathcal{T}$, $I(\perp;\ ) = \top$ in $\mathrm{Sub}_{\mathcal{H}(\mathcal{T}')}(1)$; but by definition of the semantics, $I(\perp;\ ) = \perp$ and hence by Lemma 1.13(i), $\mathcal{T}' \vdash \perp$, i.e. $\mathcal{T}'$ is also inconsistent.

2.3. Restrictions of models along interpretations. If $I : \mathcal{T} \to \mathcal{T}'$ is an interpretation between two theories and M is a model of $\mathcal{T}'$ in some Heyting pretopos $\mathcal{K}$, then we can *restrict* M along I to obtain a model of $\mathcal{T}$ in $\mathcal{K}$, denoted $I^*(M)$ and defined as follows:

By definition I is a model of $\mathcal{T}$ in $\mathcal{H}(\mathcal{T}')$. Now as in (9) there is an equivalence

$$\mathrm{Mod}_{\cong}(\mathcal{T}', \mathcal{K}) \simeq \mathrm{LOG}_{\cong}(\mathcal{H}(\mathcal{T}'), \mathcal{K})$$

and $M \cong \bar{M}_*(\mathrm{Id}_{\mathcal{T}'})$ for some morphism $\bar{M} : \mathcal{H}(\mathcal{T}') \to \mathcal{K}$ which is uniquely determined by M up to unique isomorphism. Then $I^*(M)$ is defined to be $\bar{M}_*(I)$.

Similarly, if $a : M \cong M'$ is an isomorphism of $\mathcal{T}'$ models in $\mathcal{K}$ we can restrict it along I to obtain $I^*(a) : I^*(M) \cong I^*(M')$ an isomorphism of $\mathcal{T}$ models in $\mathcal{K}$. $I^*(a)$ is defined to be $\bar{a}_*(I)$ where $\bar{a} : \bar{M} \cong \bar{M}'$ is a natural isomorphism such that $\bar{a}_*(\mathrm{Id}_{\mathcal{T}'})$ is equal to

$$\bar{M}_*(\mathrm{Id}_{\mathcal{T}'}) \cong M \overset{a}{\cong} M' \cong \bar{M}'_*(\mathrm{Id}_{\mathcal{T}'}).$$

In this way, restriction along I becomes a functor

$$I^* : \mathrm{Mod}_{\cong}(\mathcal{T}', \mathcal{K}) \to \mathrm{Mod}_{\cong}(\mathcal{T}, \mathcal{K})$$

between categories of models and isomorphisms in $\mathcal{K}$.

If $\bar{I} : \mathcal{H}(\mathcal{T}) \to \mathcal{H}(\mathcal{T}')$ denotes the Heyting pretopos morphism corresponding to I under the equivalence

$$\mathrm{Mod}_{\cong}(\mathcal{T}, \mathcal{H}(\mathcal{T}')) \cong \mathrm{LOG}_{\cong}(\mathcal{H}(\mathcal{T}), \mathcal{H}(\mathcal{T}')),$$

then on replacing the categories of models by the equivalent categories of morphisms, I^* becomes identified with *composition* with $\bar{I}$:

$$\begin{array}{ccc} \mathrm{Mod}_{\cong}(\mathcal{T}', \mathcal{K}) & \simeq & \mathrm{LOG}_{\cong}(\mathcal{H}(\mathcal{T}'), \mathcal{K}) \\ {\scriptstyle I^*}\big\downarrow & & \big\downarrow{\scriptstyle \bar{I}^* = \mathrm{LOG}_{\cong}(\bar{I}, \mathcal{K})} \\ \mathrm{Mod}_{\cong}(\mathcal{T}, \mathcal{K}) & \simeq & \mathrm{LOG}_{\cong}(\mathcal{H}(\mathcal{T}), \mathcal{K}) \end{array}$$

2.4. Composition of interpretations. If I is an interpretation of $\mathcal{T}$ in $\mathcal{T}'$ and J is an interpretation of $\mathcal{T}'$ in $\mathcal{T}''$, then we can *compose* J with I to obtain an interpretation of $\mathcal{T}$ in $\mathcal{T}''$, denoted $J \circ I$ and defined as follows:

As in 2.3, let $\bar{J} : \mathcal{H}(\mathcal{T}') \to \mathcal{H}(\mathcal{T}'')$ correspond to J under the equivalence

$$\mathrm{Mod}_{\cong}(\mathcal{T}', \mathcal{H}(\mathcal{T}'')) \simeq \mathrm{LOG}_{\cong}(\mathcal{H}(\mathcal{T}'), \mathcal{H}(\mathcal{T}'')).$$

(Thus $\bar{J}$ is determined up to unique isomorphism by the requirement $\bar{J}_*(\mathrm{Id}_{\mathcal{T}'}) \cong J$.) Then $J \circ I$ is defined to be $\bar{J}_*(I)$.

With this definition of composition, the collection of theories in IPC, interpretations between them and isomorphism between the interpretations becomes a *bicategory* in the sense of Benabou [2]. In particular, the generic model $\mathrm{Id}_{\mathcal{T}} \in \mathrm{Mod}_{\cong}(\mathcal{T}, \mathcal{H}(\mathcal{T}))$ regarded as an interpretation of $\mathcal{T}$ in itself, acts as an *identity* for composition:

$$I \circ \mathrm{Id}_{\mathcal{T}} \cong I \cong \mathrm{Id}_{\mathcal{T}'} \circ I.$$

The assignment

$$\mathcal{T} \mapsto \mathcal{H}(\mathcal{T})$$

is then the object part of a full and faithful homomorphism of bicategories to the bicategory of Heyting pretoposes, morphisms of such and natural isomorphisms. Moreover, since every Heyting pretopos is equivalent to $\mathcal{H}(\mathcal{T})$ for some theory $\mathcal{T}$ (e.g. take $\mathcal{T}$ to be as defined in Remark 1.7(iii)), this gives an *equivalence* of bicategories.

In passing from $\mathcal{T}$ to $\mathcal{H}(\mathcal{T})$, what is lost is a knowledge of the underlying language and axioms of $\mathcal{T}$. In other words, in working with Heyting pretoposes one is dealing with the *presentation-free* properties of theories and interpretations in IPC. What is gained is the fact that Heyting pretoposes are directly amenable to algebraic manipulation and also to the powerful *functorial* techniques of category theory: the proof of the conceptual completeness theorem given below illustrates both these aspects. The results so gained can then be translated back across the equivalence of bicategories mentioned above to ones about first-order logic and model theory.

2.5. Equivalent theories. Call two theories $\mathcal{T}$ and $\mathcal{T}'$ *equivalent* (and write $\mathcal{T} \simeq \mathcal{T}'$) if there are interpretations $I: \mathcal{T} \to \mathcal{T}'$ and $J: \mathcal{T}' \to \mathcal{T}$ with $J \circ I \cong \mathrm{Id}_{\mathcal{T}}$ and $I \circ J \cong \mathrm{Id}_{\mathcal{T}'}$. It follows from 2.4 that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent theories iff $\mathcal{H}(\mathcal{T})$ and $\mathcal{H}(\mathcal{T}')$ are equivalent categories. Note also that, using the operations of restricting models along interpretations defined in 2.3, equivalent theories have equivalent categories of models and isomorphisms in any Heyting pretopos.

2.6. Conservative interpretations. As usual, an interpretation $I: \mathcal{T} \to \mathcal{T}'$ will be called *conservative* if it reflects the validity of sentences, i.e. if whenever σ is a sentence of $\mathcal{T}$ which is satisfied by the model I of $\mathcal{T}$ in $\mathcal{H}(\mathcal{T}')$, then already $\mathcal{T} \vdash \sigma$. For Heyting pretoposes (or logoses), the concept corresponding to 'sentence' is 'subobject of the terminal object' (cf. Lemma 1.13(i)). Thus a morphism $\mathcal{I}: \mathcal{H} \to \mathcal{H}'$ of Heyting pretoposes will be called *conservative* if whenever $A \rightarrowtail 1$ in $\mathcal{H}$ is such that $\mathcal{I}(A) \rightarrowtail \mathcal{I}(1) \cong 1$ is the top subobject of 1 in $\mathcal{H}'$, then $A \rightarrowtail 1$ is already the top subobject of 1 in $\mathcal{H}$.

One has:

(i) $\mathcal{I}: \mathcal{H} \to \mathcal{H}'$ is conservative iff $\mathcal{I}$ reflects isomorphisms (i.e. $\mathcal{I}(f)$ an isomorphism implies f is), iff $\mathcal{I}$ is faithful (i.e. $\mathcal{I}(f) = \mathcal{I}(g)$ implies $f = g$).

(ii) An interpretation $I: \mathcal{T} \to \mathcal{T}'$ between theories is conservative iff the corresponding Heyting pretopos morphism $\bar{I}: \mathcal{H}(\mathcal{T}) \to \mathcal{H}(\mathcal{T}')$ is conservative.

2.7. Quotient theories. If $\mathcal{T}$ and $\mathcal{T}'$ are theories in IPC, $\mathcal{T}'$ is a *quotient* of $\mathcal{T}$ if it is obtained from $\mathcal{T}$ by leaving the underlying language unchanged but adding extra sentences as axioms. In this case there is a canonical interpretation $\mathcal{T} \to \mathcal{T}'$, the corresponding morphism $\mathcal{H}(\mathcal{T}) \to \mathcal{H}(\mathcal{T}')$ being essentially the identity on objects and on morphisms sending (the $\mathcal{T}$-provable equivalence classes of) $\mathcal{T}$-provably functional relations to the corresponding ($\mathcal{T}'$-provable equivalence classes of) $\mathcal{T}'$-provable functional relations. In particular such a Heyting pretopos morphism enjoys the two following properties:

(i) A Heyting pretopos morphism $\mathcal{I}: \mathcal{H} \to \mathcal{H}'$ will be called *full on subobjects* if for all objects X of $\mathcal{H}$ and subobjects $B \rightarrowtail \mathcal{I}(X)$ in $\mathcal{H}'$, there is $A \rightarrowtail X$ in $\mathcal{H}$ with $B = \mathcal{I}(A)$ in $\mathrm{Sub}_{\mathcal{H}'}(\mathcal{I}X)$.

(ii) A Heyting pretopos morphism $\mathcal{I}: \mathcal{H} \to \mathcal{H}'$ will be called *subcovering* if for all objects Y of $\mathcal{H}'$ there is an object X of $\mathcal{H}$ and a diagram of the form

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, tail, "m"'] & Y \\ \mathcal{I}(X) \end{tikzcd} \qquad (m \text{ a monomorphism, } e \text{ a cover})$$

in $\mathcal{H}'$. (In the above diagram, Y will be said to be a *subquotient* of $\mathcal{I}(X)$.)

Conversely, given $I:\mathcal{T}\to\mathcal{T}'$ such that the corresponding $\bar{I}:\mathcal{H}(\mathcal{T})\to\mathcal{H}(\mathcal{T}')$ satisfies (i) and (ii), then one can find a theory $\mathcal{T}''$ equivalent to $\mathcal{T}'$ so that $\mathcal{T}''$ is a quotient of $\mathcal{T}$ and the composition of $I:\mathcal{T}\to\mathcal{T}'$ with $\mathcal{T}'\simeq\mathcal{T}''$ is the canonical interpretation of $\mathcal{T}$ in the quotient $\mathcal{T}''$.

In fact, in the presence of condition (i), condition (ii) is equivalent to just requiring that $\mathcal{I}$ be *essentially surjective*, i.e. that for each Y in $\mathcal{H}'$ there is X in $\mathcal{H}$ with $\mathcal{I}(X)\cong Y$. For suppose that $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$ satisfies both (i) and (ii). Then given any Y in $\mathcal{H}'$, evidently one can find some X in $\mathcal{H}$ and a cover $e:\mathcal{I}(X)\twoheadrightarrow Y$. Pulling e back along itself yields an equivalence relation

$$S\rightarrowtail\mathcal{I}(X)\times\mathcal{I}(X)\cong\mathcal{I}(X\times X)$$

which by (i) is of the form $S=\mathcal{I}(R)$ for some subobject $R\rightarrowtail X\times X$. Now in $\mathcal{H}$ there is a subobject $U\rightarrowtail 1$ of the terminal object which is the interpretation of the statement "R is an equivalence relation" (written in the language $\mathcal{L}_{\mathcal{H}}$ of 1.7(iii)). It follows that $R\times U$ is an equivalence relation on $X\times U$: let

$$q:X\times U\twoheadrightarrow Q$$

be its effective coequalizer. Now $\mathcal{I}(U)=\top$ (since it is the interpretation of the statement "$\mathcal{I}R$ is an equivalence relation", which is true since $\mathcal{I}R=S$). Therefore

$$\mathcal{I}(X)\cong\mathcal{I}(X\times U)\xrightarrow{\mathcal{I}_q}\mathcal{I}(Q)$$

coequalizes $S=\mathcal{I}R\rightrightarrows\mathcal{I}X$. But so does $e:\mathcal{I}(X)\twoheadrightarrow Y$, and hence $Y\cong\mathcal{I}(Q)$, as required. (Note that this argument uses the universal quantification present in a *Heyting* pretopos. Quotient morphisms of *pretoposes* satisfy (i) and (ii), but are not necessarily essentially surjective.)

Accordingly one can define a Heyting pretopos morphism to be a *quotient morphism* if it is both full on subobjects and essentially surjective. Each Heyting pretopos morphism $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$ can be factored as $\mathcal{I}=\mathcal{K}\circ\mathcal{J}$ where $\mathcal{J}$ is a quotient morphism and $\mathcal{K}$ is conservative: cf. Section 3 of [14] for more details. Here we shall need the fact that the only morphisms that are both quotients and conservative are equivalences:

2.8. Lemma. *If a Heyting pretopos morphism $\mathcal{I}:\mathcal{H}\to\mathcal{H}$ is both conservative and a quotient, it is an equivalence of categories.*

Proof. Being a quotient, $\mathcal{I}$ is essentially surjective, being conservative, $\mathcal{I}$ is faithful (2.6(i)). So it is sufficient to show that $\mathcal{I}$ is also full. Given $g:\mathcal{I}(X)\to\mathcal{I}(X')$ in $\mathcal{H}'$, consider the graph of g

$$\langle\mathrm{id},g\rangle:\mathcal{I}(X)\to\mathcal{I}(X)\times\mathcal{I}(X')\cong\mathcal{I}(X\times X').$$

Since $\mathcal{I}$ is full on subobjects, this particular subobject of $\mathcal{I}(X\times X')$ is of the form $\mathcal{I}(F)$ for some $F\rightarrowtail X\times X'$. Now $\mathcal{I}$ is conservative and takes F to the graph of a function: hence F is already the graph of a function $f: X \to X'$ (i.e. $F \rightarrowtail X \times X' \xrightarrow{\pi} X$ is an isomorphism, because $\mathcal{I}$ sends it to an isomorphism). Then by definition of F and G, we have $\mathcal{I}(f) = g$, as required. □

2.9. Completeness. We have as yet made no *size* restrictions on theories and categories. For simplicity, let us fix a pair of (Grothendieck) universes of sets, the elements of the first being 'sets' and those of the second being 'classes'. A category is called 'small' if its collection of morphisms is a set and is called 'large' if they form a class. (A better, but less well understood framework is to take 'small' to mean 'internal to a fixed elementary topos' and 'large' to mean 'fibred over the fixed topos': cf. [3].)

We shall henceforward assume that theories have underlying languages with only a set of basic sort, function and relation symbols. Consequently for a theory $\mathcal{T}$, its classifying Heyting pretopos $\mathcal{H}(\mathcal{T})$ is a small category; and conversely any small Heyting pretopos is equivalent to one of the form $\mathcal{H}(\mathcal{T})$ for $\mathcal{T}$ a theory satisfying the above size requirement. Thus the equivalences of bicategories mentioned in 2.4 now becomes one between the bicategory of (small) theories in IPC and the bicategory of small Heyting pretoposes.

We shall say that a collection $\mathfrak{H}$ of (large) Heyting pretoposes is *complete* for theories in IPC if it satisfies that for every theory

$$\mathcal{T} \vdash \sigma \text{ just in case } M \models \sigma \text{ for all } M \in \mathrm{Mod}(\mathcal{T}, \mathcal{K}) \text{ and all } \mathcal{K} \in \mathfrak{H}.$$

The existence of generic models in classifying Heyting pretoposes implies that the collection of all small Heyting pretoposes is an example of such an $\mathfrak{H}$. Less trivially, the collection of categories of sheaves on complete Heyting algebras is also an example: cf. Fourman and Scott [5]. (If one restrict the class of theories one is interested in, then it is possible to take smaller collections; for example the *single* Heyting pretopos of sheaves on Baire space (N^N) is complete for finitely axiomatizable theories. But in contrast to the case for classical logic, one can show that no *set* of Heyting pretoposes can be complete in the above sense, i.e. for all (small) theories in IPC simultaneously.)

Replacing theories by the small Heyting pretoposes which classify them, one can characterize the above notion of completeness for a collection $\mathfrak{H}$ of large Heyting pretoposes in category-theoretic terms, as follows:

Generalising from 2.6, we shall call a collection of Heyting pretopos morphisms with common domain $\mathcal{H}$, $(\mathcal{F}_i : \mathcal{H} \to \mathcal{K}_i \mid i \in I)$, *jointly convervative* if whenever we have $A \rightarrowtail 1$ in $\mathcal{H}$ such that for each $i \in I$, $\mathcal{F}_i(A \rightarrowtail 1)$ is an isomorphism, then $A \rightarrowtail 1$ is already an isomorphism. Then a collection $\mathfrak{H}$ of Heyting pretoposes will be called *sufficient for small Heyting pretoposes* if for all small Heyting pretoposes $\mathcal{H}$, the collection of Heyting pretopos morphisms from $\mathcal{H}$ to members of $\mathfrak{H}$ is jointly conservative. Under the equivalence of the bicategory of theories and the bicategory of small Heyting pretoposes, one has: $\mathfrak{H}$ *is complete for* IPC *iff it is sufficient for small Heyting pretoposes.*

We can now state precisely the main result of this paper, first as a theorem about interpretations between theories in IPC, and then in an equivalent form as a result about morphisms of Heyting pretoposes:

2.10. Conceptual Completeness Theorem for IPC. *Let $\mathfrak{H}$ be a collection of Heyting pretoposes which is complete for theories in* IPC. *Let $I:\mathcal{T}\to\mathcal{T}'$ be an interpretation between such theories. Then*

(i) *$\mathcal{T}'$ is equivalent to a quotient theory of $\mathcal{T}$ via I, if for all $K\in\mathfrak{H}$*

$$I^*:\mathrm{Mod}_{\cong}(\mathcal{T}',\mathcal{K})\to\mathrm{Mod}_{\cong}(\mathcal{T},\mathcal{K})$$

is full and faithful.

(ii) *$\mathcal{T}$ and $\mathcal{T}'$ are equivalent theories, via I, if for all $\mathcal{K}\in\mathfrak{H}$*

$$I^*:\mathrm{Mod}_{\cong}(\mathcal{T}',\mathcal{K})\to\mathrm{Mod}_{\cong}(\mathcal{T},\mathcal{K})$$

is an equivalence of categories.

2.11. Conceptual Completeness Theorem for Heyting Pretoposes. *Let $\mathfrak{H}$ be a collection of Heyting pretoposes which is sufficient for small Heyting pretoposes. Let $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$ be a morphism of small Heyting pretoposes. Then*

(i) *$\mathcal{I}$ is a quotient morphism if for all $\mathcal{K}\in\mathfrak{H}$*

$$\mathcal{I}^*:\mathrm{LOG}_{\cong}(\mathcal{H}',\mathcal{K})\to\mathrm{LOG}_{\cong}(\mathcal{H},\mathcal{K})$$

is full and faithful.

(ii) *$\mathcal{I}$ is an equivalence if for all $\mathcal{K}\in\mathfrak{H}$*

$$\mathcal{I}^*:\mathrm{LOG}_{\cong}(\mathcal{H}',\mathcal{K})\to\mathrm{LOG}_{\cong}(\mathcal{H},\mathcal{K})$$

is an equivalence.

Proof (outline). Replacing theories by their classifying Heyting pretoposes, categories of models become identified with categories of morphisms and the functor restricting models along an interpretation becomes identified with that of pre-composing by a morphism: cf. 2.3. In this way 2.10 is a consequence of 2.11.

We shall now sketch the proof of 2.11; filling in the details will occupy the rest of the paper.

Firstly, one can show easily (see 4.5(ii)) that:

(a) If $\mathcal{I}^*$ is essentially surjective for all $\mathcal{K}\in\mathfrak{H}$, then $\mathcal{I}$ is conservative.

This is simply a consequence of the hypothesis that $\mathfrak{H}$ is sufficient for small Heyting Pretoposes. Combining (a) with Lemma 2.8, one sees that part (ii) of Theorem 2.11 follows from part (i).

Splitting the definition of quotient morphism into its two component parts (2.7(i) and (ii)), we split the proof of 2.11(i) into two implications:

(b) If $\mathcal{I}^*$ is full for all $\mathcal{K}\in\mathfrak{H}$, then $\mathcal{I}$ is full on subobjects.

(c) If $\mathcal{I}^*$ is full and faithful for all $\mathcal{K}\in\mathfrak{H}$, then $\mathcal{I}$ is subcovering.

(It is interesting to note that (c) becomes false if one drops the word 'full' from its hypothesis: a counterexample due to Makkai is presented in 4.8.)

The proof (in 4.5(i)) of statement (b) is a consequence of the *interpolation property* for pushout squares of Heyting pretoposes (4.3) proven in [14] (and which is a generalisation to arbitrary interpretations of the usual interpolation theorem for IPC). The method of proof used there was to introduce a certain functorial construction, called the *topos of filters,* which associates to each Heyting pretopos a *Grothendieck topos* (a generalisation of the notion of topological space via its category of set-valued sheaves), and associates to a Heyting pretopos morphism an *open geometric morphism* (which correspondingly generalises the notion of open continuous map). In fact, this construction allows us to deduce *both* statements (b) and (c) from properties of geometric morphisms. These properties are discussed in Section 3. In particular, the proof of (c) (see 4.7) requires the consideration of sheaves equipped with *descent data.*

The properties of the topos of filters construction which allow one to deduce (b) and (c) from the sheaf theoretic considerations of Section 3 are outlined at the beginning of Section 4. The details of the construction itself are given in an appendix. □

2.12. Remarks. (i) Just as Heyting pretoposes and morphisms can be viewed as giving a category-theoretic approach to certain properties of first order intuitionistic logic, so can Grothendieck toposes and geometric morphisms (or rather, their inverse image parts) be viewed as giving one to the *geometric* $(=, \wedge, \exists, \bigvee)$ fragment of *infinitary logic.* From this view point, our method of proof for the above conceptual completeness theorem is to use a certain nice translation of theories in IPC into theories in geometric logic. Whilst at first this might seem to complicate matters by replacing the finitary by the infinitary, the method succeeds because of the tractability of the geometric fragment compared with the full first order one. (For example, formulae of the former kind can be rearranged into a standard form ($\bigvee\exists\wedge$ atomics), but nothing of the sort is possible intuitionistically for full first-order formulae.)

(ii) It is rather easy to see that the converses of 2.11(i) and (ii) both hold: Obviously, if $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$ is an equivalence, the functorality of the $(-)^*$ operation implies that

$$\mathcal{I}^*:\mathrm{LOG}_{\cong}(\mathcal{H}',\mathcal{K})\to\mathrm{LOG}_{\cong}(\mathcal{H},\mathcal{K})$$

is also one, for any Heyting pretopos $\mathcal{K}$. Less obviously, if $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$ is a quotient, then $\mathcal{I}^*:\mathrm{LOG}_{\cong}(\mathcal{H}',\mathcal{K})\to\mathrm{LOG}_{\cong}(\mathcal{H},\mathcal{K})$ will be full and faithful for any $\mathcal{K}$. The proof of this can be broken into two stages:

(a) *If $\mathcal{I}$ is subcovering, $\mathcal{I}^*$ is faithful*: For suppose we have $\alpha,\beta:\mathcal{F}\to\mathcal{G}$ in $\mathrm{LOG}_{\cong}(\mathcal{H}',\mathcal{K})$ with $\mathcal{I}^*(\alpha)=\mathcal{I}^*(\beta)$, i.e. with $\alpha_{\mathcal{I}}=\beta_{\mathcal{I}}$. For any object Y of $\mathcal{H}'$, we can find X in $\mathcal{H}$ such that Y is a subquotient of $\mathcal{I}(X)$:

$$\begin{array}{l} B \xrightarrow{e} Y \\ m \big\downarrow \\ \mathcal{I}(X) \end{array} \qquad (m \text{ a monomorphism, } e \text{ a cover}).$$

Then by naturality of α and β one has

$$\mathcal{G}m \circ \alpha_B = \alpha_{\mathcal{I}X} \circ \mathcal{F}m = \beta_{\mathcal{I}X} \circ \mathcal{F}m = \mathcal{G}m \circ \beta_B$$

and hence $\alpha_B = \beta_B$, since $\mathcal{G}m$ is a monomorphism. But then

$$\alpha_Y \circ \mathcal{F}e = \mathcal{G}e \circ \alpha_B = \mathcal{G}e \circ \beta_B = \beta_Y \circ \mathcal{F}e$$

and hence $\alpha_Y = \beta_Y$, since $\mathcal{F}e$ is a cover and hence is an epimorphism. Thus $\alpha = \beta$.

(b) *If furthermore, $\mathcal{I}$ is full on subobjects (and hence is a quotient), then $\mathcal{I}^*$ is also full*: For suppose $\mathcal{F}, \mathcal{G} \in \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K})$ and $\gamma : \mathcal{I}^*(\mathcal{F}) \to \mathcal{I}^*(\mathcal{G})$ in $\mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})$. For each object Y of $\mathcal{H}'$ one can find an X in $\mathcal{H}$ and a cover $e : \mathcal{I}(X) \twoheadrightarrow Y$ in $\mathcal{H}'$; and hence on taking the pullback of e against itself, one obtains an equivalence relation $R \rightarrowtail X \times X$ in $\mathcal{H}$ such that

$$\mathcal{I}(R) \rightrightarrows \mathcal{I}(X) \xrightarrow{e} Y$$

is a coequalizer in $\mathcal{H}'$. Thus in the diagram below, the rows are coequalizers:

$$\begin{array}{ccccc} \mathcal{F}\mathcal{I}(R) & \rightrightarrows & \mathcal{F}\mathcal{I}(X) & \xrightarrow{\mathcal{F}e} & \mathcal{F}(Y) \\ \gamma_R \big\downarrow & & \big\downarrow \gamma_X & & \big\downarrow \alpha_Y \\ \mathcal{G}\mathcal{I}(R) & \rightrightarrows & \mathcal{G}\mathcal{I}(x) & \xrightarrow[\mathcal{G}e]{} & \mathcal{G}(Y) \end{array}$$

Let α_Y be the unique morphism $\mathcal{F}(Y) \to \mathcal{G}(Y)$ making the right-hand square above commute. It follows easily that α_Y is natural in Y, is an isomorphism (because γ is) and satisfies $\mathcal{I}^*(\alpha) = \gamma$.

3. Some geometric morphisms

In this section we shall develop those properties of geometric morphisms between Grothendieck toposes which will be needed to prove the conceptual completeness theorem for IPC. We refer the reader to [1] and [6] for general information about toposes. All but one (Theorem 3.5) of the results about geometric morphisms can be found in Johnstone [7, 8] and Joyal and Tierney [9].

3.1. Notation. (i) GTOP will denote the bicategory of Grothendieck toposes, geometric morphisms and natural transformations (between inverse image functors).

(ii) For $f: \mathscr{F} \to \mathscr{E}$ in GTOP, the inverse image functor $f^*: \mathscr{E} \to \mathscr{F}$ induces order preserving maps on subobject lattices

$$\begin{aligned} f^*: \mathrm{Sub}_{\mathscr{E}}(X) &\to \mathrm{Sub}_{\mathscr{F}}(f^*X), \\ (A \rightarrowtail X) &\mapsto (f^*A \rightarrowtail f^*X). \end{aligned}$$

The left adjoint of the above map, if it exists, will be denoted by

$$f_X: \mathrm{Sub}_{\mathscr{F}}(f^*X) \to \mathrm{Sub}_{\mathscr{E}}(X)$$

(Since the subobject lattices are complete, f_X will exist just in case $f^*: \mathrm{Sub}_{\mathscr{E}}(X) \to \mathrm{Sub}_{\mathscr{F}}(f^*X)$ preserves arbitrary meets.)

3.2. Open geometric morphisms. Each Grothendieck topos is in particular a Heyting pretopos. But if $f: \mathscr{F} \to \mathscr{E}$ is a geometric morphism, $f^*: \mathscr{E} \to \mathscr{F}$ is not in general a morphism of Heyting pretoposes, since it does not necessarily preserve universal quantification of subobjects along maps. An important class of geometric morphisms for which this is the case is that comprising the *open* geometric morphisms: a geometric morphism $f: \mathscr{F} \to \mathscr{E}$ is open just in case the left adjoints f_X mentioned in 3.1(ii) all exist, are natural in X and satisfy 'Frobenius reciprocity', i.e.

$$f_X(B \wedge f^*A) = f_X(B) \wedge A.$$

This is equivalent to requiring f^* to preserve *infinitary first-order logic*, i.e. the maps

$$f^*: \mathrm{Sub}_{\mathscr{E}}(X) \to \mathrm{Sub}_{\mathscr{F}}(f^*X)$$

should preserve all meets and universal quantification along maps (and so in particular, preserve implication: cf. 1.2(v)); see [7] and [9].

We will be particularly concerned with the properties of open geometric morphisms under *pullback*. given two geometric morphisms

$$\mathscr{F} \xrightarrow{f} \mathscr{E} \xleftarrow{g} \mathscr{G},$$

their pullback will be denoted $\mathscr{F} \times_{\mathscr{E}} \mathscr{G}$; thus there is a diagram in GTOP of the form

$$\begin{tikzcd} \mathscr{F} \times_{\mathscr{E}} \mathscr{G} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\theta}{\cong}"] & \mathscr{G} \arrow[d, "g"] \\ \mathscr{F} \arrow[r, "f"'] & \mathscr{E} \end{tikzcd} \tag{10}$$

such that for each Grothendieck topos $\mathscr{H}$, the functor

$$\begin{gathered} \mathrm{GTOP}(\mathscr{H}, \mathscr{F} \times_{\mathscr{E}} \mathscr{G}) \to \mathrm{GTOP}(\mathscr{H}, \mathscr{F}) \times_{\mathrm{GTOP}(\mathscr{H}, \mathscr{E})} \mathrm{GTOP}(\mathscr{H}, \mathscr{G}), \\ h \mapsto (fh, \theta_h, gh) \end{gathered}$$

is an equivalence of categories. (The codomain of this functor is a pullback category: its objects are triples (b, α, c) where $b: \mathscr{H} \to \mathscr{F}$, $c: \mathscr{H} \to \mathscr{G}$ and $\alpha: fb \cong gc$, and whose morphisms $(b, \alpha, c) \to (b', \alpha', c')$ are pairs (β, γ) where $\beta: b \to b'$, $\gamma: c \to c'$ and $\alpha' \circ f\beta = g\gamma \circ \alpha$. (See 4.2 below.)

3.3. Proposition. *In the pullback square* (10) *if f is an open geometric morphism, then:*

(i) *q is also open.*

(ii) *For any object X of $\mathscr{E}$*

$$\begin{tikzcd}
\mathrm{Sub}_{\mathscr{F}}(f^*X) \arrow[r, "f_X"] \arrow[d, "p^*"'] & \mathrm{Sub}_{\mathscr{E}}(X) \arrow[dd, "g^*"] \\
\mathrm{Sub}_{\mathscr{F}\times_{\mathscr{E}}\mathscr{G}}(p^*f^*X) \arrow[d, "\exists\theta_X"', "\cong"] & \\
\mathrm{Sub}_{\mathscr{F}\times_{\mathscr{E}}\mathscr{G}}(q^*g^*X) \arrow[r, "q_{g^*X}"'] & \mathrm{Sub}_{\mathscr{G}}(g^*X)
\end{tikzcd}$$

commutes.

(iii) *If f is also a surjection, then so is q.*

(iv) *Interpolation property: for any object X of $\mathscr{E}$ and subobjects $B \rightarrowtail f^*X$ in $\mathscr{F}$ and $C \rightarrowtail g^*X$ in $\mathscr{G}$, if*

$$(p^*B \rightarrowtail p^*f^*X \cong q^*g^*X) \leq (q^*C \rightarrowtail q^*g^*X)$$

in $\mathrm{Sub}_{\mathscr{F}\times_{\mathscr{E}}\mathscr{G}}(q^*g^*X)$, *then there is a subobject $A \rightarrowtail X$ in $\mathscr{E}$ with $B \leq f^*A$ in* $\mathrm{Sub}_{\mathscr{F}}(f^*X)$ *and $g^*A \leq C$ in* $\mathrm{Sub}_{\mathscr{G}}(g^*X)$.

Proof. For (i), (ii) and (iii) see Chapter VII of [9] or Section 4 of [7]. For (iv), take $A = f_X(B)$: then $B \leq f^*A$ is automatic and $g^*A \leq C$ follows from (ii). □

Open geometric morphisms will be important for us because of the logical operations which their inverse image functors preserve; but as their name suggests, they were originally introduced as a generalisation of the usual notion of open continuous map between topological spaces. Similarly, other classes of geometric morphisms, named for their 'geometrical' properties, are of significance here because of the corresponding 'logical' properties of their inverse image functors. We introduced in 2.6 and 2.7 the notions of being conservative, being full on subobjects, being subcovering and being a quotient morphism of Heyting pretoposes. These notions make sense at the level of pretoposes and in particular for inverse image parts of geometric morphisms. Then for a geometric morphism $f: \mathscr{F} \to \mathscr{E}$ one has:

3.4. Lemma. (i) *f is a surjection iff f^* is conservative.*

(ii) *f is an inclusion iff f^* is a quotient (i.e. both full on subobjects and subcovering).*

(iii) *f is localic iff f^* is subcovering.*

(iv) *f is hyperconnected iff f^* is conservative and full on subobjects.*

Proof. For the definitions of surjection and inclusion, see Chapter 4 or [6]. For the definitions of localic and hyperconnected morphisms see [8] or VI.5 of [9] (where localic morphisms are called 'spatial'). Proofs of (i) to (iv) can be found in, or easily deduced from these references. □

We shall need the following characterisation of localic geometric morphisms:

3.5. Proposition. *Given a geometric morphism $f: \mathcal{F} \to \mathcal{E}$ between Grothendieck toposes, form the pullback square*

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\pi}{\cong}"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd}$$

and let $d: \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ be the diagonal geometric morphism, i.e. the morphism (defined uniquely up to unique isomorphism) for which there are isomorphisms

$$\delta_0 : p_0 d \cong id_{\mathcal{F}} \quad \text{and} \quad \delta_1 : p_1 d \cong id_{\mathcal{F}}$$

making

$$\begin{tikzcd} fp_0 d \arrow[r, "\pi d"] \arrow[d, "f\delta_0"'] & fp_1 d \arrow[d, "f\delta_1"] \\ f \arrow[r, "id_f"'] & f \end{tikzcd}$$

commute. Then

(i) *Given an object Y of $\mathcal{F}$, it is a subquotient of an object in the image of f^*, i.e. there is a diagram in $\mathcal{F}$ of the form*

$$\begin{tikzcd} A \arrow[r, "e", two heads] \arrow[d, "m"', tail] & Y \\ f^* X & \end{tikzcd} \tag{11}$$

with X an object in $\mathcal{E}$, m a monomorphism and e a cover, iff there is a subobject

$D \rightarrowtail p_0^* Y \times p_1^* Y$ in $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ with

$$\begin{array}{ccc} d^*D & \rightarrowtail & d^*(p_0^*Y \times p_1^*Y) \\ \cong\downarrow & & \cong\downarrow (\delta_0)_Y \times (\delta_1)_Y \\ Y & \underset{\langle \mathrm{id}, \mathrm{id} \rangle}{\rightarrowtail} & Y \times Y \end{array} \tag{12}$$

in $\mathcal{F}$.

(ii) *f is localic iff d is an inclusion.*

Proof. (i) A somewhat indirect proof in the spirit of [9] can be given by first relativizing to $\mathcal{E}$, regarding the pullback as a product of toposes over $\mathcal{E}$, and then analysing subobjects in such a product using the fact [9, VI.5.2] that finite products are preserved by localic reflection. Instead, we give a rather more direct argument suggested by Makkai.

First observe that every subobject $D(y_1, y_2)$ of $p_0^* Y \times p_1^* Y$ in $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is of the form

$$D(y_1, y_2) = \bigvee_{i \in I} \exists x_i \, [p_0^* B_i(y_1, x_i) \wedge p_1^* C_i(\pi_{X_i}(x_i), y_2)] \tag{13}$$

where $\{X_i \mid i \in I\}$ is a set of objects in $\mathcal{E}$ and $B_i \rightarrowtail Y \times f^* X_i$, $C_i \rightarrowtail f^* X_i \times Y$ are subobjects in $\mathcal{F}$. (To see that this is so, consider the characterisation of $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ as the classifying topos of the following geometric theory: two models of the theory classified by $\mathcal{F}$ plus an isomoprhism between the restrictions of these models along f^* to models of the theory classified by $\mathcal{E}$.) Taking the disjoint coproduct $X = \coprod_{i \in I} X_i$, we can further assume that the subobject in (13) is of the form

$$D(y_1, y_2) = \exists x \, [p_0^* B(y_1, x) \wedge p_1^* C(\pi_X(x), y_2)]$$

for subobjects $B \rightarrowtail Y \times f^* X$, $C \rightarrowtail f^* X \times Y$ (obtained from the B_i and C_i by taking coproducts).

Now for such a subobject, $d^* D$ is

$$\exists x \in X \, [B(y_1, x) \wedge C(x, y_2)]$$

and so (12) holds iff $\mathcal{F}$ satisfies

$$y_1 = y_2 \leftrightarrow \exists x \in X \, [B(y_i, x) \wedge C(x, y_2)]. \tag{14}$$

But if the subobject $B(y, x) \wedge C(x, y)$ of $f^* X \times Y$ is represented by the monomorphism

$$\langle m, e \rangle : A \rightarrowtail f^* X \times Y,$$

then (14) immediately implies that m is a monomorphism and e a cover. Thus the existence of a D satisfying (12) implies the existence of a diagram of the form (11).

Conversely, given m and e as in (11), $A(x, y)$ satisfies

$$y_1 = y_2 \leftrightarrow \exists x \in X\,[A(x, y_1) \wedge A(x, y_2)].$$

So as above, we can take $D(y_1, y_2)$ to be

$$\exists x\,[p_0^*A(x, y_1) \wedge p_1^*(\pi_X(x), y_2)]$$

to satisfy (12). This proves (i).

(ii) If d is an inclusion, by Lemma 3.4(ii), d^* is full on subobjects and hence by part (i), f^* is subcovering; thus by 3.4(iii), f is localic.

Conversely, suppose that f is localic. Note that since d is split by p_i, d^* is essentially surjective, hence is subcovering. So by Lemma 3.4(ii), to see that d is an inclusion it suffices to show that d^* is full on subobjects. Since f is localic, by 3.4(iii), f^* is subcovering and hence by part (i), the diagonal subobject

$$Y \xrightarrow{\langle \mathrm{id},\mathrm{id}\rangle} Y \times Y \cong d^*(p_0^*Y \times p_1^*Y)$$

is in the image of d^*. It follows that *any* subobject of $d^*(p_0^*Y \times p_1^*Y)$ is in the image of d^*. Hence d^* is full on subobjects, since every object of $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is a subquotient of one of the form $p_0^*Y \times p_1^*Y$ (Y in $\mathcal{F}$). $\square$

If $f: \mathcal{F} \to \mathcal{E}$ is an open geometric morphism, it does not follow that the diagonal $d: \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is also open. In proving Theorem 2.11 (step (c)) we shall essentially have to show that a certain open geometric morphism is localic: the fact that the corresponding diagonal morphism is not necessarily open will block a direct application of Proposition 3.5. To get round this problem, we shall now develop a sufficient condition (3.7) for a morphism to be localic which eliminates consideration of the diagonal morphism in favour of further pullbacks and projections:

Given a geometric morphism $f: \mathcal{F} \to \mathcal{E}$, form the pullback squares

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\substack{\pi \\ \cong}"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd} \tag{15}$$

and

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\substack{\pi_1 \\ \cong}"] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[d, "p_0"] \\ \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, "p_1"'] & \mathcal{F} \end{tikzcd} \tag{16}$$

in GTOP. Let $p_{02}: \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ be the third projection, defined by the requirement that there be isomorphisms

$$\pi_0 : p_0 p_{02} \cong p_0 p_{01} \quad \text{and} \quad \pi_2 : p_1 p_{02} \cong p_1 p_{12}$$

making the following diagram commute:

$$
\begin{tikzcd}
f p_0 p_{02} \arrow[r, "\pi p_{02}"] \arrow[d, "f\pi_0"'] & f p_1 p_{02} \arrow[d, "f\pi_2"] \\
f p_0 p_{01} \arrow[r, "\pi p_{12}\cdot f\pi_1\cdot \pi p_{01}"'] & f p_1 p_{12}
\end{tikzcd}
\tag{17}
$$

Now suppose Y is an object of $\mathscr{F}$ and $y: p_0^*Y \to p_1^*Y$ a morphism in $\mathscr{F}\times_{\mathscr{E}}\mathscr{F}$. Then (Y, y) is said to satisfy the *cocycle condition* if

$$
\begin{tikzcd}
p_{02}^* p_0^*(Y) \arrow[r, "p_{02}^*(y)"] \arrow[d, "(\pi_0)_Y"'] & p_{02}^* p_2^*(Y) \arrow[d, "(\pi_2)_Y"] \\
p_{01}^* p_0^*(Y) \arrow[r, "p_{12}^*(y)\cdot(\pi_1)_Y\cdot p_{01}^*(y)"'] & p_{12}^* p_1^*(Y)
\end{tikzcd}
\tag{18}
$$

commutes in $\mathscr{F}\times_{\mathscr{E}}\mathscr{F}\times_{\mathscr{E}}\mathscr{F}$. For example, comparing (17) with (18), it is evident that $(f^*(X), \pi_X)$ satisfies the cocycle condition for any object X of $\mathscr{E}$.

3.6. Lemma. *Suppose that* (Y, y) *satisfies the cocycle condition and that* d, δ_0, δ_1 *are defined as in Proposition* 3.5. *Then*

$$
\begin{tikzcd}
d^* p_0^*(Y) \arrow[r, "d^*(y)"] \arrow[d, "(\delta_0)_Y"'] & d^* p_1^*(Y) \arrow[d, "(\delta_1)_Y"] \\
Y \arrow[r, tail, "\mathrm{id}_Y"'] & Y
\end{tikzcd}
\tag{19}
$$

commutes iff y *is an isomorphism.*

Proof. The lemma is an easy consequence of the fact that

$$
\begin{tikzcd}
\mathscr{F}\times_{\mathscr{E}}\mathscr{F}\times_{\mathscr{E}}\mathscr{F} \arrow[r, shift left=2, "p_{ij}"] \arrow[r] \arrow[r, shift right=2] & \mathscr{F}\times_{\mathscr{E}}\mathscr{F} \arrow[r, shift left, "p_i"] \arrow[r, shift right] & \mathscr{F} \arrow[l, bend left, "d"]
\end{tikzcd}
$$

is essentially a *groupoid* object in GTOP, the complication being that the relevant diagrams commute only up to (coherent) isomorphisms, of which one has to keep track. Leaving the book-keeping to the reader, we indicate the essential steps in the argument:

Let $t: \mathscr{F}\times_{\mathscr{E}}\mathscr{F} \to \mathscr{F}\times_{\mathscr{E}}\mathscr{F}$ be the twist morphism: i.e. there are isomorphisms

$$\tau_0: p_0 t \cong p_1 \quad \text{and} \quad \tau_1: p_1 t \cong p_0$$

such that

$$
\begin{tikzcd}
f p_0 t \arrow[r, "\pi t"] \arrow[d, "f\tau_0"'] & f p_1 t \arrow[d, "f\tau_1"] \\
f p_1 \arrow[r, "\pi^{-1}"'] & f p_0
\end{tikzcd}
$$

commutes. Then on applying $(d \times_{\mathscr{E}} id_{\mathscr{F}})^* \circ (id_{\mathscr{F}} \times_{\mathscr{E}} t)^*$ and $(id_{\mathscr{F}} \times_{\mathscr{E}} d)^* \circ (t \times_{\mathscr{E}} id_{\mathscr{F}})^*$ to (18), one obtains respectively that

$$\begin{tikzcd} & p_1^*Y & \\ p_0^*Y \arrow[ur, "y"] \arrow[rr, "p_0^*((\delta_1)_Y \cdot d^*(y) \cdot (\delta_0)_Y^{-1})"'] & & p_0^*Y \arrow[ul, "(\tau_1)_Y \cdot t^*(y) \cdot (\tau_0)_Y^{-1}"'] \end{tikzcd} \tag{20}$$

and

$$\begin{tikzcd} & p_0^*Y \arrow[dr, "y"] & \\ p_1^*Y \arrow[ur, "(\tau_1)_Y \cdot t^*y \cdot (\tau_0)_Y^{-1}"] \arrow[rr, "p_1^*((\delta_1)_Y \cdot d^*(y) \cdot (\delta_0)_Y^{-1})"'] & & p_1^*Y \end{tikzcd}$$

commute. Thus if (19) commutes, i.e. if $(\delta_1)_Y \circ d^*(y) \circ (\delta_0)_y^{-1} = \mathrm{id}_Y$, then the above diagrams show that y is an isomorphism with inverse $(\tau_1)_Y \circ t^*(y) \circ (\tau_0)_Y^{-1}$. Conversely, if y is an isomorphism, then so is $(\delta_1)_Y \circ d^*(y) \circ (\delta_0)_Y^{-1}$; but on applying d^* to (20) one finds that the latter map is also idempotent and hence must be the identity map for Y. $\square$

3.7. Proposition. *Suppose that $f: \mathscr{F} \to \mathscr{E}$ is a geometric morphism between Grothendieck toposes and that Y is an object of $\mathscr{F}$. Suppose further that there is an isomorphism $y : p_0^*Y \cong p_0^*Y$ in $\mathscr{F} \times_{\mathscr{E}} \mathscr{F}$ satisfying the cocycle condition* (18). *Then Y is subcovered via f^*, i.e. Y is a subquotient of an object in the image of f^*.*

(Thus in particular, a sufficient condition for f to be localic is that one can find such an isomorphism y for each object Y of $\mathscr{F}$.)

Proof. Given such a $y : p_0^*Y \cong p_1^*Y$, by Lemma 3.6 we have that $(\delta_1)_Y \circ d^*(y) \circ (\delta_0)_Y^{-1} = \mathrm{id}_Y$, and this is equivalent to asserting that the graph of y is sent by d^* to the diagonal subobject of $Y \times Y \cong d^*(p_0^*Y \times p_1^*Y)$. Hence by Proposition 3.5(i), Y is subcovered by an object in the image of f^*, as required. $\square$

3.8. Remark. A morphism $y : p_0^*Y \to p_1^*Y$ satisfying the cocycle condition (18) and the equivalent conditions of Lemma 3.6, is called *descent data* for Y. The main part of Joyal and Tierney's Descent theorem [9, VIII.2.1] says that when f is an open surjection, every object of $\mathscr{F}$ equipped with descent data 'descends' in the sense that there is an object X in $\mathscr{E}$ and an isomorphism $x : f^*(X) \cong Y$ with $y = p_1^*(x) \circ \pi_X \circ p_0^*(x^{-1})$. (We noted above that π_X is always descent data for f^*X.) So in this case Y is actually in the essential image of f^*, rather than just being a subquotient of an object in the image. When we come (in 4.7) to use Proposition 3.7, it will indeed by the case that the f involved is an open surjection; however, since we shall only need to conclude that f is localic, the rather easy result 3.7 is sufficient. So we do not need to appeal here to Joyal and Tierney's theorem.

4. Conceptual completeness

In order to use the results of the previous section to prove Theorem 2.11, we shall use the 'topos of filters' of a Heyting pretopos. This construction was introduced by the author in [14] and used there to prove a general interpolation property of Heyting pretoposes. In fact this property plays a key role in our proof of conceptual completeness and we will recall its statement below (4.3).

In [14] the definition and properties of the topos of filters of a (Heyting) pretopos were developed in terms of 'indexed lattice theory' and the theory of internal locales in toposes. This level of sophistication undoubtedly provides a quick and elegant road to these results. However, in an appendix to this paper we give the construction rather more concretely, in terms of sites and sheaves. (This has the advantage of making more explicit the connection that exists between the topos of filters and Makkai's 'topos of types' [11].) We shall now state the properties of the construction that we need, referring the reader to the appendix for proofs:

4.1. The topos of filters. If $\mathcal{H}$ is a small Heyting pretopos, its *topos of filters* $\Phi(\mathcal{H})$ is a certain Grothendieck topos which contains $\mathcal{H}$ as a full sub-Heyting pretopos, i.e. there is a full and faithful morphism of Heyting pretoposes

$$\mathcal{I}_H : \mathcal{H} \hookrightarrow \Phi(\mathcal{H})$$

The assignment $\mathcal{H} \mapsto \Phi(\mathcal{H})$ is functorial in the sense that for morphisms $\mathcal{I} : \mathcal{H} \to \mathcal{H}'$ of small Heyting pretoposes, there are geometric morphisms

$$\Phi(\mathcal{I}) : \Phi(\mathcal{H}') \to \Phi(\mathcal{H})$$

satisfying: $\Phi(\mathcal{I}d) \cong id$ and $\Phi(\mathcal{J} \circ \mathcal{I}) \cong \Phi(\mathcal{I}) \circ \Phi(\mathcal{J})$ (these isomorphisms satisfying the usual coherence conditions). It is shown in the appendix that Φ has the following properties:

(i) *$\Phi(\mathcal{I})$ is an open geometric morphism* (*see* 3.2); *so in particular its inverse image functor $(\Phi\mathcal{I})^*$ is a morphism of Heyting pretoposes $\Phi(\mathcal{H}) \to \Phi(\mathcal{H}')$.*

(ii) *$\mathcal{I}_{\mathcal{H}} : \mathcal{H} \hookrightarrow \Phi(\mathcal{H})$ is pseudonatural in $\mathcal{H}$, i.e.*

$$\begin{tikzcd} \mathcal{H} \arrow[r, hook, "\mathcal{I}_{\mathcal{H}}"] \arrow[d, "\mathcal{I}"'] \arrow[dr, phantom, "\cong"] & \Phi\mathcal{H} \arrow[d, "(\Phi\mathcal{I})^*"] \\ \mathcal{H}' \arrow[r, hook, "\mathcal{I}_{\mathcal{H}'}"'] & \Phi(\mathcal{H}') \end{tikzcd}$$

commutes up to (*coherent*) *isomorphism.*

(iii) *If $\mathcal{I} : \mathcal{H} \to \mathcal{H}'$ is conservative* (*see* 2.6), *then $\Phi(\mathcal{I})$ is a surjection, i.e. $(\Phi\mathcal{I})^*$ is also conservative.* (*The converse of this is implied by* (ii).)

(iv) *Suppose Y is an object of $\mathcal{H}'$ and that $\mathcal{I}_{\mathcal{H}'}(Y)$ is a subquotient of an object of $\Phi(\mathcal{H}')$ which is in the image of $(\Phi\mathcal{I})^*$. Then Y is a subquotient in $\mathcal{H}'$ of an* *object in the image of $\mathcal{I}$. (In particular, if $\Phi(\mathcal{I})$ is localic, then $\mathcal{I}$ is subcovering (see* 2.7(ii)); *the converse of this is also true.)*

4.2. Pullbacks and pushouts. Given three categories $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$ and two functors $\mathcal{F}_i : \mathcal{B}_i \to \mathcal{A}$ $(i = 1, 2)$, by the *pullback category* $\mathcal{B}_1 \times_{\mathcal{A}} \mathcal{B}_2$, we shall mean the category whose objects are triples (b_1, a, b_2) where b_i is an object of $\mathcal{B}_i$ and $a : \mathcal{F}_1(b_1) \cong \mathcal{F}_2(b_2)$ is an isomorphism in $\mathcal{A}$, and whose morphisms from (b_1, a, b_2) to (b_1', a', b_2') are pairs (g_1, g_2) where $g_i : b_i \to b_i'$ in $\mathcal{B}_i$ and

$$\begin{tikzcd} \mathcal{F}_1(b_1) \arrow[r, "a"] \arrow[d, "\mathcal{F}_1(g_1)"'] & \mathcal{F}_2(b_2) \arrow[d, "\mathcal{F}_2(g_2)"] \\ \mathcal{F}_1(b_1') \arrow[r, "a'"'] & \mathcal{F}_2(b_2') \end{tikzcd}$$

commutes in $\mathcal{A}$. Composition and identity morphism are defined from those in $\mathcal{B}_1$ and $\mathcal{B}_2$ in the obvious way.

Now given two morphisms $\mathcal{I}_i : \mathcal{H} \to \mathcal{H}_i$ $(i = 1, 2)$ between (small) Heyting pretoposes, their *pushout* is a (small) Heyting pretopos $\mathcal{H}_1 +_{\mathcal{H}} \mathcal{H}_2$ together with $\mathcal{M}_1$, $\mathcal{M}_2$

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "\mathcal{M}_1"] & \mathcal{H}_1 +_{\mathcal{H}} \mathcal{H}_2 \\ \mathcal{H} \arrow[u, "\mathcal{I}_1"] \arrow[r, "\mathcal{I}_2"'] \arrow[ur, phantom, "\overset{h}{\cong}"] & \mathcal{H}_2 \arrow[u, "\mathcal{M}_2"'] \end{tikzcd} \tag{21}$$

and a natural isomorphism $h : \mathcal{M}_1\mathcal{I}_1 \cong \mathcal{M}_2\mathcal{I}_2$, satisfying the following universal property:

For any Heyting pretopos $\mathcal{K}$, the functor

$$\mathrm{LOG}_{\cong}(\mathcal{H}_1 +_{\mathcal{H}} \mathcal{H}_2, \mathcal{K}) \to \mathrm{LOG}_{\cong}(\mathcal{H}_1, \mathcal{K}) \times_{\mathrm{LOG}_{\cong}(\mathcal{H},\mathcal{K})} \mathrm{LOG}_{\cong}(\mathcal{H}_2, \mathcal{K}) \tag{22}$$
$$\mathcal{N} \mapsto (\mathcal{N}\mathcal{M}_1, \mathcal{N}h, \mathcal{N}\mathcal{M}_2)$$

is an equivalence of categories. (Recall that $\mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})$ denotes the category of logos morphisms $\mathcal{H} \to \mathcal{K}$ and natural isomorphisms and that this is the same as the category of Heyting pretopos morphisms and isomorphisms when $\mathcal{H}$ and $\mathcal{K}$ are Heyting pretoposes.)

In [14], the topos of filters construction (and in particular, properties 4.1(i), (ii) and (iii)) is used to deduce the following property of pushouts of Heyting pretoposes from the properties of open geometric morphisms given in Proposition 3.3:

4.3. Interpolation for Heyting Pretoposes. *In the pushout square* (21), *suppose that X is an object of $\mathcal{H}$ and that $B_i \rightarrowtail \mathcal{I}_i(X)$ $(i = 1, 2)$ are subobjects in $\mathcal{H}_i$ with the property that the subobject*

$$\mathcal{M}_1 B_1 \rightarrowtail \mathcal{M}_1\mathcal{I}_1(X) \overset{h_X}{\cong} \mathcal{M}_2\mathcal{I}_2(X)$$

is contained in the subobject

$$\mathcal{M}_2 B_2 \rightarrowtail \mathcal{M}_2 \mathcal{I}_2(X).$$

Then there is a subobject $A \rightarrowtail X$ *in* $\mathcal{H}$ *with*

$$B_1 \leqslant \mathcal{I}_1 A \quad \textit{in} \quad \mathrm{Sub}_{\mathcal{H}_1}(\mathcal{I}_1 X) \quad \textit{and}$$
$$\mathcal{I}_2 A \leqslant B_2 \quad \textit{in} \quad \mathrm{Sub}_{\mathcal{H}_2}(\mathcal{I}_2 X).$$

4.4. Remarks. (i) More generally, the topos of filters construction can be applied to logoses and can be used to prove that pushout squares of logoses have the above interpolation property. Alternatively, this can be deduced from 4.3 using the Heyting pretopos completion of a logos and its particular properties given in Lemma 1.11.

(ii) The usual interpolation theorem of IPC (for many-sorted languages) can be recovered as a special case of 4.3 using the classifying Heyting pretopos of a theory and Lemma 3.13. (See also Section 3 of [14].) This depends upon the following description of pushouts of Heyting pretoposes in terms of amalgamating theories:

Suppose that $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are the classifying Heyting pretoposes of theories $\mathcal{T}$, $\mathcal{T}_1$, $\mathcal{T}_2$. (By 1.7(iii), we can always find such theories.) Then as in 2.3, the morphisms $\mathcal{H} \to \mathcal{H}_i$ correspond to interpretations $I_i : \mathcal{T} \to \mathcal{T}_i$. One can take for the pushout $\mathcal{H}_1 +_{\mathcal{H}} \mathcal{H}_2$ the classifying Heyting pretopos of the theory consisting of the disjoint union of the theories $\mathcal{T}_1$, $\mathcal{T}_2$ together with new function symbols for the graphs of the components of h plus axioms saying that these components are natural and are isomorphisms.

4.5. Proposition. *Let* $\mathfrak{H}$ *be a collection of Heyting pretoposes which is sufficient for small Heyting pretoposes* (*in the sense of* 2.9) *and suppose that* $\mathcal{I} : \mathcal{H} \to \mathcal{H}'$ *is a morphism of small Heyting pretoposes.*

(i) *If for all* $\mathcal{K} \in \mathfrak{H}$

$$\mathcal{I}^* : \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \to \mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})$$

is full, then $\mathcal{I}$ *is full on subobjects.*

(ii) *If for all* $\mathcal{K} \in \mathfrak{H}$, $\mathcal{I}^*$ *is essentially surjective, then* $\mathcal{I}$ *is conservative.*

Proof. (i) We use the interpolation property 4.3. Form the pushout square

$$\begin{tikzcd} \mathcal{H}' \arrow[r, "\mathcal{M}_1"] & \mathcal{H}' +_{\mathcal{H}} \mathcal{H}' \\ \mathcal{H} \arrow[u, "\mathcal{I}"] \arrow[r, "\mathcal{I}"'] \arrow[ur, phantom, "h \atop \cong"] & \mathcal{H}' \arrow[u, "\mathcal{M}_2"'] \end{tikzcd}$$

To show that $\mathcal{I}$ is full on subobjects, given any object X in $\mathcal{H}$ and subobject $B \rightarrowtail \mathcal{I}(X)$ in $\mathcal{H}'$ we have to find $A \rightarrowtail X$ in $\mathcal{H}$ with $B = \mathcal{I}(A)$ in $\mathrm{Sub}_{\mathcal{H}'}(\mathcal{I}X)$, i.e.

with $B \leqslant \mathscr{I}(A)$ and $\mathscr{I}(A) \leqslant B$. By 4.3, it is enough to show that

$$\mathscr{M}_1 B \rightarrowtail \mathscr{M}_1\mathscr{I}(X) \cong \mathscr{M}_2\mathscr{I}(X)$$

is contained in $\mathscr{M}_2 B \rightarrowtail \mathscr{M}_2\mathscr{I}(X)$, i.e. that

$$\exists h_X(\mathscr{M}_1 B) \leqslant \mathscr{M}_2 B \quad \text{in} \quad \mathrm{Sub}_{\mathscr{H}' +_{\mathscr{H}} \mathscr{H}'}(\mathscr{M}_2\mathscr{I}X).$$

Since $\mathfrak{H}$ is sufficient for small Heyting pretoposes, this holds if for all $\mathscr{K} \in \mathfrak{H}$ and all $\mathscr{N} \in \mathrm{LOG}_{\cong}(\mathscr{H}' +_{\mathscr{H}} \mathscr{H}', \mathscr{K})$ we have

$$\mathscr{N}(\exists h_X(\mathscr{M}_1 B)) \leqslant \mathscr{N}\mathscr{M}_2 B, \quad \text{i.e.} \quad \exists \mathscr{N}h_X(\mathscr{N}\mathscr{M}_1 B) \leqslant \mathscr{N}\mathscr{M}_2 B.$$

Now $\mathscr{N}h$ is an isomorphism between $\mathscr{I}^*(\mathscr{N}\mathscr{M}_1)$ and $\mathscr{I}^*(\mathscr{N}\mathscr{M}_2)$: so by hypothesis on $\mathscr{I}^*$, $\mathscr{N}h = k_{\mathscr{I}}$ for some $k : \mathscr{N}\mathscr{M}_1 \cong \mathscr{N}\mathscr{M}_2$. But then by naturality of k

$$\begin{tikzcd}
\mathscr{N}\mathscr{M}_1 B \arrow[r, tail] \arrow[d, "k_B"'] & \mathscr{N}\mathscr{M}_1\mathscr{I}X \arrow[d, "k_{\mathscr{I}X} = \mathscr{N}h_X"] \\
\mathscr{N}\mathscr{M}_2 B \arrow[r, tail] & \mathscr{N}\mathscr{M}_2\mathscr{I}X
\end{tikzcd}$$

commutes, and thus $\exists \mathscr{N}h_X(\mathscr{N}\mathscr{M}_1 B) \leqslant \mathscr{N}\mathscr{M}_2 B$ as required.

(ii) This is simply a consequence of $\mathfrak{H}$ being sufficient for small Heyting pretoposes. For suppose we have $A \rightarrowtail X$ in $\mathscr{H}$ with $\mathscr{I}(A \rightarrowtail X)$ an isomorphism. to show that $A \rightarrowtail X$ is already an isomorphism, it is enough to show that $\mathscr{N}(A \rightarrowtail X)$ is an isomorphism for each $\mathscr{N} : \mathscr{H} \to \mathscr{K}$ with $\mathscr{K} \in \mathfrak{H}$. But by hypothesis $\mathscr{N} = \mathscr{M}\mathscr{I}$ for some $\mathscr{M} : \mathscr{H}' \to \mathscr{K}$, and then $\mathscr{N}(A \rightarrowtail X)$ is an isomorphism because $\mathscr{M}\mathscr{I}(A \rightarrowtail X)$ is. $\square$

4.6. Lemma. *Suppose that $\mathfrak{H}$ is a collection of Heyting pretoposes which is sufficient for small Heyting pretoposes, and suppose that $\mathscr{I} : \mathscr{H} \to \mathscr{H}'$ is a morphism of small Heyting pretoposes. For*

$$\mathscr{I}^* : \mathrm{LOG}_{\cong}(\mathscr{H}', \mathscr{K}) \to \mathrm{LOG}_{\cong}(\mathscr{H}, \mathscr{K})$$

to be full and faithful for all Heyting pretoposes $\mathscr{K}$, it is sufficient that it be so for all $\mathscr{K} \in \mathfrak{H}$.

Proof. We use the following simple observation about pullbacks of functors between groupoids:

Given a functor $\mathscr{F} : \mathscr{B} \to \mathscr{A}$ between groupoids, forming the pullback category $\mathscr{B} \times_{\mathscr{A}} \mathscr{B}$ as in 4.2, consider the *diagonal functor*

$$\Delta : \mathscr{B} \to \mathscr{B} \times_{\mathscr{A}} \mathscr{B}$$

defined by

$$\Delta\left(b \xrightarrow{g} b'\right) = (b, \mathrm{id}_{\mathscr{F}b}, b) \xrightarrow{(g,g)} (b' \,\mathrm{id}_{\mathscr{F}b'}, b').$$

Then one has:

$\mathcal{F}$ *is full and faithful iff* Δ *is an equivalence.* (23)

We apply this in the following way. Form the pushout of Heyting pretoposes

$$\begin{array}{ccc} \mathcal{H}' & \xrightarrow{\mathcal{M}_1} & \mathcal{H}' +_{\mathcal{H}} \mathcal{H}' \\ {\scriptstyle \mathcal{I}}\uparrow & \overset{h}{\cong} & \uparrow{\scriptstyle \mathcal{M}_2} \\ \mathcal{H} & \xrightarrow[\mathcal{I}]{} & \mathcal{H}' \end{array}$$

By its universal property (4.2), there is a Heyting pretopos morphism

$$\nabla : \mathcal{H}' +_{\mathcal{H}} \mathcal{H}' \to \mathcal{H}'$$

called the *codiagonal* morphism, together with isomorphisms

$$m_1 : \nabla \mathcal{M}_1 \cong \mathcal{I}d \quad \text{and} \quad m_2 : \nabla \mathcal{M}_2 = \mathcal{I}d$$

such that

$$\begin{array}{ccc} \nabla \mathcal{M}_1 \mathcal{I} & \xrightarrow{\nabla h} & \nabla \mathcal{M}_2 \mathcal{I} \\ {\scriptstyle (m_1)_{\mathcal{I}}}\downarrow & & \downarrow{\scriptstyle (m_2)_{\mathcal{I}}} \\ \mathcal{I} & \xrightarrow[id_{\mathcal{I}}]{} & \mathcal{I} \end{array}$$

commutes.

Now for any $\mathcal{K} \in \mathfrak{H}$, $\mathcal{I}^* : \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \to \mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})$ is full and faithful. So by (23), the diagonal functor

$$\mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \to \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \times_{\mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})} \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K})$$

is an equivalence. Composing this with the equivalence (22), by construction gives

$$\nabla^* : \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \to \mathrm{LOG}_{\cong}(\mathcal{H}' +_{\mathcal{H}} \mathcal{H}', \mathcal{K})$$

which is therefore also an equivalence. Therefore, by Proposition 4.5, we have that ∇ is full on subobjects and conservative. Since ∇ is split by $\mathcal{M}_i$, it is also essentially surjective: hence it is both a quotient and conservative, and so by Lemma 2.8 is an equivalence. Consequently the diagonal functor

$$\mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \xrightarrow{\nabla^*} \mathrm{LOG}_{\cong}(\mathcal{H}' +_{\mathcal{H}}{}', \mathcal{K})$$

$$\simeq \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \times_{\mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})} \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K})$$

is an equivalence for any $\mathcal{K}$. But then by (23) again,

$$\mathcal{I}^* : \mathrm{LOG}_{\cong}(\mathcal{H}', \mathcal{K}) \to \mathrm{LOG}_{\cong}(\mathcal{H}, \mathcal{K})$$

is full and faithful, for any $\mathcal{K}$. □

4.7. Proposition. *Suppose that $\mathfrak{H}$ is a collection of Heyting pretoposes which is sufficient for small Heyting pretoposes, and suppose that $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$ is a morphism of small Heyting pretoposes. If for all $\mathcal{K}\in\mathfrak{H}$*

$$\mathcal{I}^*:\mathrm{LOG}_{\cong}(\mathcal{H}',\mathcal{K})\to\mathrm{LOG}_{\cong}(\mathcal{H},\mathcal{K})$$

is full and faithful, then $\mathcal{I}$ is subcovering.

Proof. Applying the topos of filters construction to $\mathcal{I}:\mathcal{H}\to\mathcal{H}'$, as in 4.1 one obtains an open geometric morphism

$$\Phi\mathcal{I}:\Phi(\mathcal{H}')\to\Phi(\mathcal{H})$$

and a square of Heyting pretopos morphisms

$$\begin{tikzcd} \mathcal{H} \arrow[r, hook, "\mathcal{I}_{\mathcal{H}}"] \arrow[d, "\mathcal{I}"'] \arrow[dr, phantom, "\overset{\phi}{\cong}"] & \Phi(\mathcal{H}) \arrow[d, "(\Phi\mathcal{I})^*"] \\ \mathcal{H}' \arrow[r, hook, "\mathcal{I}_{\mathcal{H}'}"'] & \Phi(\mathcal{H}') \end{tikzcd} \tag{24}$$

commuting up to isomorphism. With notation as in Section 3, take successive pullbacks in GTOP:

$$\Phi\mathcal{H}'\times_{\Phi\mathcal{H}}\Phi\mathcal{H}'\times_{\Phi\mathcal{H}}\Phi\mathcal{H}' \begin{array}{c}\xrightarrow{p_{01}}\\ \xrightarrow{p_{02}}\\ \xrightarrow{p_{12}}\end{array} \Phi\mathcal{H}'\times_{\Phi\mathcal{H}}\Phi\mathcal{H}' \begin{array}{c}\xrightarrow{p_{0}}\\ \xrightarrow{p_{1}}\end{array} \Phi\mathcal{H}' \xrightarrow{\Phi\mathcal{I}} \Phi\mathcal{H}$$

(with isomorphisms $\pi:\Phi(\mathcal{I})p_0\cong\Phi(\mathcal{I})p_1$, etc.). By Proposition 3.3(i), p_0, p_1, p_{01}, p_{12} and p_{02} are all open geometric morphisms, since they are obtained by pullback from open ones. (This is by definition for the first four: cf. (15) and (16). For the last, p_{02} is the first projection for the pullback of p_1 against itself.) Therefore $p_0^*\mathcal{I}_{\mathcal{H}'}$ and $p_1^*\mathcal{I}_{\mathcal{H}'}$ are both elements of

$$\mathrm{LOG}_{\cong}(\mathcal{H}',\Phi\mathcal{H}'\times_{\Phi\mathcal{H}}\Phi\mathcal{H}').$$

Now by Lemma 4.6,

$$\mathcal{I}^*:\mathrm{LOG}_{\cong}(\mathcal{H}',\Phi\mathcal{H}'\times_{\Phi\mathcal{H}}\Phi\mathcal{H}')\to\mathrm{LOG}_{\cong}(\mathcal{H},\Phi\mathcal{H}'\times_{\Phi\mathcal{H}}\Phi\mathcal{H}')$$

is full. Thus the isomorphism

$$p_0^*\mathcal{I}_{\mathcal{H}'}\mathcal{I}\underset{p_0^*(\phi)}{\cong}p_0^*(\Phi\mathcal{I})^*\mathcal{I}_{\mathcal{H}}\underset{\pi_{\mathcal{I}_{\mathcal{H}}}}{\cong}p_1^*(\Phi\mathcal{I})^*\mathcal{I}_{\mathcal{H}}\underset{p_1^*(\phi)}{\cong}p_1^*\mathcal{I}_{\mathcal{H}'}\mathcal{I}$$

(where ϕ is as in (24)) is of the form $y_{\mathcal{I}}$, for some isomorphism

$$y:p_0^*\mathcal{I}_{\mathcal{H}'}\cong p_1^*\mathcal{I}_{\mathcal{H}'}.$$

Now again by Lemma 4.6

$$\mathcal{I}^*: \mathrm{LOG}_{\cong}(\mathcal{H}', \Phi\mathcal{H}' \times_{\Phi\mathcal{H}} \Phi\mathcal{H}' \times_{\Phi\mathcal{H}} \Phi\mathcal{H}') \to \mathrm{LOG}_{\cong}(\mathcal{H}, \Phi\mathcal{H}' \times_{\Phi\mathcal{H}} \Phi\mathcal{H}' \times_{\Phi\mathcal{H}} \Phi\mathcal{H}')$$

is faithful. But on applying this functor to the diagram

$$\begin{array}{ccc} p_{02}^* p_1^* \mathcal{I}_{\mathcal{H}'} & \xrightarrow{\quad p_{02}^*(y) \quad} & p_{02}^* p_0^* \mathcal{I}_{\mathcal{H}'} \\ {\scriptstyle (\pi_0)_{\mathcal{I}_{\mathcal{H}'}}} \downarrow & & \downarrow {\scriptstyle (\pi_2)_{\mathcal{I}_{\mathcal{H}'}}} \\ p_{01}^* p_0^* \mathcal{I}_{\mathcal{H}'} & \xrightarrow[p_{12}^*(y) \cdot (\pi_1)_{\mathcal{I}_{\mathcal{H}'}} \cdot p_{01}^*(y)]{} & p_{12}^* p_1^* \mathcal{I}_{\mathcal{H}'} \end{array} \tag{25}$$

by definition of y we obtain the diagram (17) (with $p = \Phi(\mathcal{I})$) evaluated at $\mathcal{I}_{\mathcal{H}}$, which commutes by construction. Hence (25) also commutes. Thus for each object Y of $\mathcal{H}'$,

$$y_Y : p_0^* \mathcal{I}_{\mathcal{H}'}(Y) \cong p_1^* \mathcal{I}_{\mathcal{H}'}(Y)$$

is an isomorphism satisfying the cocycle condition (18). Therefore by Proposition 3.7, $\mathcal{I}_{\mathcal{H}'}(Y)$ is a subquotient of an object of $\Phi(\mathcal{H}')$ which is in the image of $(\Phi\mathcal{I})^*$. But then by 4.1(iv), Y is a subquotient in $\mathcal{H}'$ of an object in the image of $\mathcal{I}$. Since Y was an arbitrary object of $\mathcal{H}'$, we have that $\mathcal{I}$ is subcovering, as required. □

This completes the proof of the conceptual completeness Theorem 2.11: 4.5(i) and 4.7 together give 2.11(i), and then 2.11(ii) follows by 4.5(ii) (and 2.8).

We noted in Remark 2.12(ii) that when $\mathcal{I}$ is subcovering, the functors $\mathcal{I}^*$ (restricting models along $\mathcal{I}$) are always faithful. It is natural to wonder whether the converse of this holds, i.e. whether the word 'full' can be dropped from the hypothesis of Proposition 4.7. For pretoposes (i.e. for the $=$, $\wedge$, $\vee$, $\exists$ fragment of first order logic) the equivalent proposition does hold: a morphism of pretoposes is subcovering if the induced functor on models is faithful (see [15, Proposition 2.9] and [10, Theorem 7.1.6]). Nevertheless, the corresponding statement for Heyting pretoposes *fails*. We present an example, due to Makkai, which shows this:

4.8. Example. We will define an interpretation $I: \mathcal{T} \to \mathcal{T}'$ between two theories in IPC with the following properties:

(a) For every Heyting pretopos $\mathcal{K}$

$$I^*: \mathrm{Mod}_{\cong}(\mathcal{T}', \mathcal{K}) \to \mathrm{Mod}_{\cong}(\mathcal{T}, \mathcal{K})$$

is a faithful functor.

(b) The morphism $\bar{I}: \mathcal{H}(\mathcal{T}) \to \mathcal{H}(\mathcal{T}')$ between classifying Heyting pretoposes corresponding to I, is not subcovering.

$\mathcal{T}$ is the theory with a single sort symbol X, no function and relation symbols and no axioms.

$\mathcal{T}'$ is the theory with two sort symbols X, S together with a binary relation symbol $E \rightarrowtail X \times S$ satisfying the axiom

$$\forall s, s' \in S\, [s = s' \leftrightarrow \forall x \in X\, (E(x, s) \leftrightarrow E(x, s'))]. \tag{26}$$

Evidently models of $\mathcal{T}$ in a Heyting pretopos correspond to the selection of an arbitrary object. Then the interpretation $I : \mathcal{T} \to \mathcal{T}'$ is to be given by the model of $\mathcal{T}$ in $\mathcal{H}(\mathcal{T}')$ which selects the object corresponding to the sort symbol X of $\mathcal{T}'$.

Proof of (a). Suppose that $f, g : M \cong N$ are isomorphisms of $\mathcal{T}'$-models in a Heyting pretopos $\mathcal{H}$ satisfying $I^*(f) = I^*(g)$, i.e. with $f_X = g_X$. Thus we have to show also that $f_S = g_S$ to deduce that $f = g$. But (phrasing the argument in $\mathcal{H}$ in informal language), if $y \in NX$, $s \in MS$ and $NE(y, f_S(s))$, then since f_X is an isomorphism there is $x \in MX$ with $f_X(x) = y$: hence $NE(f_X(x), f_S(s))$ and therefore $ME(x, s)$ (since f is an isomorphism). Thus we also have $NE(g_X(x), g_S(s))$ (since g is a homomorphism) and hence $NE(y, g_S(s))$, since $y = f_X(x) = g_X(x)$. We have therefore shown that

$$\forall y \in NX\, [NE(y, f_S(s)) \to NE(y, g_S(s))]$$

holds in $\mathcal{H}$, and the converse implication holds by a similar argument. Then because N satisfies (26), we have

$$\forall s \in NS\, [f_S(s) = g_S(s)],$$

so that $f_S = g_S$ and thus $f = g$.

Proof of (b). If $\bar{I} : \mathcal{H}(\mathcal{T}) \to \mathcal{H}(\mathcal{T}')$ were subcovering, then for any model M of $\mathcal{T}'$ in the category of sets, MS would have to be a subquotient of a finite coproduct of finite powers of MX (cf. 1.13(ii)). Hence MS would be countable if MX were. But there is a model of $\mathcal{T}'$ with MX the set of natural numbers, MS the powerset of MX and E the membership relation.

Appendix: the topos of filters construction

The construction will be given in terms of sites and sheaves. The reader is referred to Chapter 1 of [10] and to Makkai's 'categories for the working logician' (Part 2 of [12]) for background information. See also [11].

Although we are here concerned with applying the topos of filters construction to theories in full first-order intuitionistic logic (via their classifying Heyting pretoposes), initially it is best described at the level of the $=$, $\vee$, $\wedge$, $\exists$ fragment of first order logic. Because of its initimate connections with Grothendieck's theory of coherent toposes (cf. [6, Section 7.3]), this fragment has been called *coherent* logic. In Section 1 we outlined the correspondence between theories in full first-order IPC and logoses. In exactly the same way, theories in coherent logic correspond to a variety of category termed 'logical' in [10] but now commonly called 'coherent'.

A.1. Definition. A category $\mathscr{C}$ is called *coherent* if it has finite limits, finite joins of subobjects which are preserved under pullbacks, and has existential quantification of subobjects along morphisms, satisfying the Beck–Chavalley condition (see 1.1(iii) and (iv)). Given two such categories $\mathscr{C}$ and $\mathscr{D}$, a functor $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a *morphism of coherent categories* if it preserves finite limits, finite joins of subobjects and existential quantification of subobjects along morphisms (cf. 1.5).

Thus a logos is a coherent category which also has universal quantification of subobjects along morphisms.

A.2. Notation. We shall be using some properties of filters and ideals in distributive lattices, and begin by fixing the notation. (The particular properties we want are developed in detail in [13].)

(i) We shall only consider distributive lattices D with greatest and least elements: thus in addition to binary meets ($\wedge$) and binary joints ($\vee$), D has the empty meet ($\top$) and the empty join ($\perp$). Similarly, a morphism of distributive lattices will mean a map preserving $\top$ and $\perp$ as well as $\wedge$ and $\vee$.

(ii) A *filter* δ on a distributive lattice D is a subset of D which is upward closed ($d'\geqslant d\in\delta\Rightarrow d'\in\delta$) and closed under finite meets ($\top\in\delta; d, d'\in\delta\Rightarrow d\wedge d'\in\delta$). $\mathfrak{F}(D)$ will denote the collection of filters on D partially ordered by *reverse inclusion*: $\delta\leqslant\delta'\Leftrightarrow\delta'\subseteq\delta$. Thus $\mathfrak{F}(D)$ is again a distributive lattice with

$$\top=\{\top\},\qquad \delta\wedge\delta'=\{d\wedge d'\mid d\in\delta \text{ and } d'\in\delta'\},$$
$$\perp=D\quad\text{and}\quad \delta\vee\delta'=\{d\mid d\in\delta \text{ and } d\in\delta'\}.$$

A morphism $\phi:D\to D'$ of distributive lattices induces a distributive lattice morphism $\mathfrak{F}(\phi):\mathfrak{F}(D)\to\mathfrak{F}(D')$ where

$$\mathfrak{F}(\phi)(\delta)=\{d'\in D'\mid \exists d\in\delta\ \phi(d)\leqslant d'\}.$$

Note that $\mathfrak{F}(\phi)$ has a *left* adjoint $\phi^{-1}:\mathfrak{F}(D')\to\mathfrak{F}(D)$, given by

$$\phi^{-1}(\delta')=\{d\in D\mid f(d)\in\delta'\}.$$

(iii) Replacing D by its opposite D^{op} in (ii), dually one obtains the notion of an *ideal* on D. The distributive lattice of ideals of D partially ordered by inclusion, will be denoted $\mathfrak{I}(D)$. (Thus $\mathfrak{I}(D)=(\mathfrak{F}(D^{\mathrm{op}}))^{\mathrm{op}}$). As above, $\phi:D\to D'$ induces $\mathfrak{I}(\phi):\mathfrak{I}(D)\to\mathfrak{I}(D')$ which this time has a *right* adjoint, ϕ^{-1}.

A.3. Definition. Let $\mathscr{C}$ be a coherent category.

(i) The *category of filters* of $\mathscr{C}$, denoted $\Lambda\mathscr{C}$, is defined as follows:

The objects of $\Lambda\mathscr{C}$ are pairs (X, ξ) where X is an object of $\mathscr{C}$ and ξ is a filter on the distributive lattice of subobjects of X in $\mathscr{C}$, $\mathrm{Sub}_{\mathscr{C}}(X)$.

Given objects (X, ξ) and (Y, η) of $A\mathscr{C}$, consider the partial maps in $\mathscr{C}$ from X to Y:

$$\begin{tikzcd} D(f) \arrow[r, "f"] \arrow[d, tail] & Y \\ X & \end{tikzcd}$$

Call such an f 'admissible' if for all $B \in \eta$, $f^{-1}B \rightarrowtail D(f) \rightarrowtail X$ is in ξ. There is an equivalence relation on the collection of admissible partial maps from X to Y given by

$$f \sim g \quad \text{iff} \quad \exists A \in \xi \text{ with } A \leqslant D(f) \wedge D(g) \text{ and } f|_A = g|_A.$$

(Here $f|_A$ denotes the partial map

$$\begin{tikzcd} A \wedge D(f) \arrow[r, tail] \arrow[d, tail] & D(f) \arrow[r, "f"] & Y.) \\ X & & \end{tikzcd}$$

Then the $\sim$-equivalence class $[f]$ of an admissible partial map from X to Y determines a typical morphism $(X, \xi) \to (Y, \eta)$ in $A\mathscr{C}$. The identity morphism on (X, ξ) is $[\mathrm{id}_X]$; and the composition of $[f]:(X, \xi) \to (Y, \eta)$ and $[g]:(Y, \eta) \to (Z, \xi)$ is $[g \circ f]$ where $g \circ f$ is the partial map

$$\begin{tikzcd} f^{-1}D(g) \arrow[r] \arrow[d, tail] & D(g) \arrow[r, "g"] & Z \\ D(f) \arrow[d, tail] & & \\ X & & \end{tikzcd}$$

(ii) There is a functor $[-]:\mathscr{C} \to A\mathscr{C}$ defined on objects X of $\mathscr{C}$ by

$$[X] = (X, \top)$$

(where as usual, $\top \in \mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(X))$ denotes the greatest element) and on morphisms $f: X \to Y$ by

$$[f] = (X, \top) \xrightarrow{[f]} (Y, \top)$$

(where f is first regarded as a partial map from X to Y in the usual way).

The following lemma collects together some easy consequences of Definition A.3:

A.4. Lemma. (i) *If (X, ξ) is an object in $A\mathscr{C}$ and $m: A \rightarrowtail X$ is a monomorphism in $\mathscr{C}$ whose corresponding subobject of X is in ξ, then $[m]:(A, \xi|_A) \to (X, \xi)$ is an* *isomorphism in* $\Lambda\mathscr{C}$, *where*

$$\xi|_A = \{B \rightarrowtail A \mid (B \rightarrowtail A \overset{m}{\rightarrowtail} X) \in \xi\}.$$

(ii) *Given finitely many maps in* $\Lambda\mathscr{C}$ *with common domain*

$$(X, \xi) \to (Y_i, \eta_i) \quad (i < n),$$

by changing (X, ξ) *up to isomorphism as in* (i), *we can assume that the maps are represented by **total** maps* $X \to Y_i$ *in* $\mathscr{C}$.

(iii) *For any object* (X, ξ) *in* $\Lambda\mathscr{C}$, *the unique map from* X *to the terminal object* 1 *of* $\mathscr{C}$ *determines a map* $(X, \xi) \to [1]$ *in* $\Lambda\mathscr{C}$ *and it is the unique such.*

(iv) *Given*

$$(Y, \eta) \xrightarrow{[f]} (X, \xi) \xrightarrow{[g]} (Z, \xi)$$

in $\Lambda\mathscr{C}$ *with* $Y \xrightarrow{f} X \xrightarrow{g} Z$ ***total*** *maps in* $\mathscr{C}$, *letting*

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "k"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

be a pullback square in $\mathscr{C}$ *and* $\pi = \mathfrak{F}(k^{-1})(\eta) \wedge \mathfrak{F}(h^{-1})(\xi) \in \mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(P))$, *then*

$$\begin{tikzcd} (P, \pi) \arrow[r, "{[h]}"] \arrow[d, "{[k]}"'] & (Z, \xi) \arrow[d, "{[g]}"] \\ (Y, \eta) \arrow[r, "{[f]}"'] & (X, \xi) \end{tikzcd}$$

is a pullback square in $\Lambda\mathscr{C}$.

(v) *For* (X, ξ) *an object of* $\Lambda\mathscr{C}$, $\mathrm{id}_X : X \to X$ *represents a monomorphism* $[\mathrm{id}_X] : (X, \xi) \rightarrowtail [X]$ *in* $\Lambda\mathscr{C}$; *let* $\lambda_X(\xi)$ *denote the corresponding subobject of* $[X]$ *in* $\Lambda\mathscr{C}$. *Then the map*

$$\mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(X)) \to \mathfrak{F}(\mathrm{Sub}_{\Lambda\mathscr{C}}([X])),$$
$$\xi \mapsto \lambda_X(\xi)$$

is an order-preserving bijection which is natural in X, *in the sense that for* $f : X \to Y$ *and* $\eta \in \mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(Y))$ *one has*

$$\lambda_X(\mathfrak{F}(f^{-1})(\eta)) = [f]^{-1}(\lambda_Y(\eta)) \qquad \textit{in } \mathrm{Sub}_{\Lambda\mathscr{C}}([X]).$$

(*vi*) *If* (X, ξ) *is an object of* $\Lambda\mathscr{C}$, *then composition with the monomorphism* $[\mathrm{id}_X] : (X, \xi) \rightarrowtail [X]$ *induces a (necessarily) injective map*

$$\mathrm{Sub}_{\Lambda\mathscr{C}}(X, \xi) \rightarrowtail \mathrm{Sub}_{\Lambda\mathscr{C}}[X].$$

The image of this map corresponds under the bijection mentioned in (v) *with the subset*

$$\downarrow(\xi) = \{\xi' \in \mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(X)) \mid \xi' \leqslant \xi\}$$

of $\mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(X))$. *Thus there is an isomorphism of partially ordered sets*

$$\mathrm{Sub}_{\Lambda\mathscr{C}}(X, \xi) \cong \downarrow(\xi).$$

Moreover, if $[f]:(X, \xi)\to(Y, \eta)$ *is a map in* $\Lambda\mathscr{C}$ *with* $f:X\to Y$ *a total map in* $\mathscr{C}$, *then*

$$[f]^{-1}:\mathrm{Sub}_{\Lambda\mathscr{C}}(Y, \eta)\to\mathrm{Sub}_{\Lambda\mathscr{C}}(X, \xi)$$

is identified under the above isomorphisms with the map

$$\begin{aligned}&\downarrow(\eta)\to\downarrow(\xi),\\ &\quad \eta' \mapsto \xi \wedge \mathfrak{F}(f^{-1})(\eta').\end{aligned}$$

Using the above lemma, one can deduce the following properties of $[-]:\mathscr{C}\to\Lambda\mathscr{C}$:

A.5. Proposition. *Let* $\mathscr{C}$ *be a (small) coherent category.*

(i) *Its category of filters* $\Lambda\mathscr{C}$ *is a (small) coherent category in which any collection of subobjects of an object has a meet.*

(ii) *The functor* $[-]:\mathscr{C}\to\Lambda\mathscr{C}$ *is full and faithful and a morphism of coherent categories.*

(iii) *Every object of* $\Lambda\mathscr{C}$ *is a subobject of one in the image of* $[-]$.

(iv) *For each object* X *of* $\mathscr{C}$, *every subobject of* $[X]$ *in* $\Lambda\mathscr{C}$ *is a meet of subobjects in the image of* $[-]$; *moreover, the map*

$$\begin{aligned}&\mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(X))\to\mathrm{Sub}_{\Lambda\mathscr{C}}([X]),\\ &\qquad \xi\mapsto\bigwedge\{[A]\mid A\in\xi\}\end{aligned}$$

is a bijection.

We now extend the definition of Λ to morphisms of coherent categories:

A.6. Definition. Let $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ be a morphism of coherent categories. For (X, ξ) an object of $\Lambda\mathscr{C}$, let $\mathscr{F}(\xi)\in\mathfrak{F}(\mathrm{Sub}_D(\mathscr{F}X))$ be the filter

$$\mathscr{F}(\xi) = \{B \rightarrowtail \mathscr{F}X \mid \exists A\in\xi\ \mathscr{F}A\leqslant B\}.$$

Now define a functor $\Lambda\mathscr{F}:\Lambda\mathscr{C}\to\Lambda\mathscr{D}$ on objects (X, ξ) by

$$\Lambda\mathscr{F}(X, \xi) = (\mathscr{F}X, \mathscr{F}(\xi))$$

and on morphisms $[f]:(X, \xi)\to(Y, \eta)$ by

$$\Lambda\mathscr{F}[f] = [\mathscr{F}f].$$

(One checks easily that $\mathscr{F}f$ is admissible when f is, and that if $f \sim f'$ then $\mathscr{F}f \sim \mathscr{F}f'$.)

A.7. Proposition. *Let $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ be a morphism of coherent categories.*

(i) *$\Lambda\mathscr{F}: \Lambda\mathscr{C} \to \Lambda\mathscr{D}$ is also a morphism of coherent categories and preserves arbitrary meets of subobjects.*

(ii)
$$\begin{tikzcd} \mathscr{C} \arrow[r, "\lfloor - \rfloor"] \arrow[d, "\mathscr{F}"'] & \Lambda\mathscr{C} \arrow[d, "\Lambda\mathscr{F}"] \\ \mathscr{D} \arrow[r, "\lfloor - \rfloor"'] & \Lambda\mathscr{D} \end{tikzcd}$$
commutes.

(iii) *$\Lambda(\mathscr{I}d) = \mathscr{I}d$ and $\Lambda(\mathscr{G} \circ \mathscr{F}) = \Lambda\mathscr{G} \circ \Lambda\mathscr{F}$.*

Proof. (i) That Λ preserves the terminal object and pullbacks follows from the fact that $\mathscr{F}$ does together with Lemma A.4. Under the isomorphisms of A.4(vi)

$$\Lambda\mathscr{F}: \mathrm{Sub}_{\Lambda\mathscr{C}}(X, \xi) \to \mathrm{Sub}_{\Lambda\mathscr{D}}(\mathscr{F}X, \mathscr{F}(\xi))$$

is identified with the map

$$\downarrow(\xi) \to \downarrow(\mathscr{F}\xi),$$
$$\xi' \mapsto \mathscr{F}(\xi').$$

From this it follows that $\Lambda\mathscr{F}$ preserves finite joins and existential quantification of subobjects.

(ii) and (iii) are easy consequences of Definition A.6. □

We next recall the definition and some properties of the classifying topos of a small coherent category:

A.8. Coherent toposes. Let $\mathscr{C}$ be a small coherent category. A family of maps $(f_i: X_i \to X \mid i \in I)$ in $\mathscr{C}$ is called *finite epimorphic* if I is a finite set and

$$\top = \bigvee_{i \in I} \exists f_i(\top)$$

in $\mathrm{Sub}_{\mathscr{C}}(X)$. This property of families is preserved under pullback and the Grothendieck topology generated by such families is called the *precanonical topology* on $\mathscr{C}$. The *classifying topos* of $\mathscr{C}$, $\mathscr{E}(\mathscr{C})$, is the category of sheaves on $\mathscr{C}$ for this topology.

Representable presheaves are sheaves for the precanonical topology: so the Yoneda embedding $\mathscr{C} \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ restricts to a full and faithful functor $\mathscr{Y}_{\mathscr{C}}: \mathscr{C} \to \mathscr{E}(\mathscr{C})$ which is a morphism of coherent categories. ($\mathscr{E}(\mathscr{C})$ being a (Grothendieck) topos, it is in particular a coherent category.)

$\mathscr{E}(\mathscr{C})$ is in fact the reflection of $\mathscr{C}$ into the bicategory of Grothendieck toposes, in the sense that $\mathcal{Y}_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{E}(\mathscr{C})$ has the following universal property:

For each Grothendieck topos $\mathscr{F}$, the functor

$$\mathcal{Y}^*_{\mathscr{C}} : \mathrm{GTOP}(\mathscr{F}, \mathscr{E}(\mathscr{C})) \to \mathrm{COH}(\mathscr{C}, \mathscr{F}),$$
$$f \mapsto f^* \circ \mathcal{Y}_{\mathscr{C}}$$

gives an equivalence between the categories $\mathrm{GTOP}(\mathscr{F}, \mathscr{E}(\mathscr{C}))$ of geometric morphisms $\mathscr{F} \to \mathscr{E}(\mathscr{C})$ and the category $\mathrm{COH}(\mathscr{C}, \mathscr{F})$ of morphisms of coherent categories $\mathscr{C} \to \mathscr{F}$. In particular, it follows that the assignment

$$\mathscr{C} \mapsto \mathscr{E}(\mathscr{C})$$

extends to a homomorphism of bicategories $\mathrm{COH}^{\mathrm{op}} \to \mathrm{GTOP}$ and that $\mathcal{Y}_{\mathscr{C}} ; \mathscr{C} \hookrightarrow \mathscr{E}(\mathscr{C})$ is pseudonatural in $\mathscr{C}$.

We shall need the following particular properties of the classifying topos of a small coherent category:

(i) *The topos $\mathscr{E}(\mathscr{C})$ is generated by the objects in the image of $\mathcal{Y}_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{E}(\mathscr{C})$. Thus every object of $\mathscr{E}(\mathscr{C})$ is the codomain of an epimorphism from a small coproduct of such objects.*

(ii) *For X an object of $\mathscr{C}$, the map*

$$\mathfrak{I}(\mathrm{Sub}_{\mathscr{C}}(X)) \to \mathrm{Sub}_{\mathscr{E}(\mathscr{C})}(\mathcal{Y}_{\mathscr{C}}X),$$
$$U \mapsto \bigvee \{\mathcal{Y}_{\mathscr{C}}(A) \mid A \in U\}$$

is an order-preserving bijection.

(iii) *If $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a morphism of small coherent categories and $\mathscr{E}(\mathscr{F}) : \mathscr{E}(\mathscr{D}) \to \mathscr{E}(\mathscr{C})$ the induced geometric morphism between classifying toposes, then the inverse image functor $\mathscr{E}(\mathscr{F})^* : \mathscr{E}(\mathscr{C}) \to \mathscr{E}(\mathscr{D})$ gives a map*

$$\mathscr{E}(\mathscr{F})^* : \mathrm{Sub}_{\mathscr{E}(\mathscr{C})}(\mathcal{Y}_{\mathscr{C}}X) \to \mathrm{Sub}_{\mathscr{E}(\mathscr{D})}(\mathscr{E}(\mathscr{F})^* \mathcal{Y}_{\mathscr{C}}X) \cong \mathrm{Sub}_{\mathscr{E}(\mathscr{D})}(\mathcal{Y}_{\mathscr{D}}(\mathscr{F}X))$$

for each object X in $\mathscr{C}$. Under the bijections mentioned in (ii), *this map is identified with the map*

$$\mathfrak{I}(\mathrm{Sub}_{\mathscr{C}}(X)) \to \mathfrak{I}(\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X))$$

sending an ideal U to the ideal

$$\mathscr{F}(U) = \{B \rightarrowtail \mathscr{F}X \mid \exists A \in U\, B \leqslant \mathscr{F}A\}.$$

We are now in a position to give the definition of the topos of filters construction:

A.9. Definition. Let $\mathscr{C}$ be a small coherent category. The *topos of filters* of $\mathscr{C}$, denoted $\Phi(\mathscr{C})$, is defined to be the classifying topos of the coherent category of filters of $\mathscr{C}$:

$$\Phi(\mathscr{C}) = \mathscr{E}(\Lambda\mathscr{C}).$$

If $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a morphism of small coherent categories, then $\Phi(\mathscr{F}):\Phi(\mathscr{D})\to\Phi(\mathscr{C})$ is the geometric morphism

$$\Phi(\mathscr{F})=\mathscr{E}(\Lambda\mathscr{F}).$$

Composing $[-]:\mathscr{C}\hookrightarrow\Lambda\mathscr{C}$ with $\mathscr{Y}_{\Lambda\mathscr{C}}:\Lambda\mathscr{C}\hookrightarrow\mathscr{E}(\Lambda\mathscr{C})$, one gets

$$\mathscr{I}_{\mathscr{C}}:\mathscr{C}\to\Phi(\mathscr{C})$$

a full and faithful morphism of coherent categories.

The functorial properties of Λ and $\mathscr{E}$ combine to give that Φ is a homomorphism of bicategories $\mathrm{COH}^{\mathrm{op}}\to\mathrm{GTOP}$ and that $\mathscr{I}_{\mathscr{C}}$ is pseudonatural in $\mathscr{C}$. Moreover, combining A.5, A.7 and A.8, we have:

A.10. Proposition. *Let $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ be a morphism of small coherent categories.*

(i) *Every object X of $\Phi(\mathscr{C})$ if the subquotient of a small coproduct of objects in the image of $\mathscr{I}_{\mathscr{C}}:\mathscr{C}\hookrightarrow\Phi(\mathscr{C})$.*

(ii) *For each object X of $\mathscr{C}$, the map*

$$\mathfrak{IF}(\mathrm{Sub}_{\mathscr{C}}(X))\to\mathrm{Sub}_{\Phi(\mathscr{C})}(\mathscr{I}_{\mathscr{C}}X)$$

sending an ideal U of filters of $\mathrm{Sub}_{\mathscr{C}}(X)$ *to the subobject*

$$\bigvee_{\alpha\in U}\bigwedge_{A\in\alpha}\mathscr{I}_{\mathscr{C}}(A),$$

is a lattice isomorphism. In particular, every subobject of $\mathscr{I}_{\mathscr{C}}(X)$ is expressible as a join of meets of subobjects in the image of $\mathscr{I}_{\mathscr{C}}$. Note also that given $A\rightarrowtail X$ in $\mathscr{C}$, $\mathscr{I}_{\mathscr{C}}(A)\rightarrowtail\mathscr{I}_{\mathscr{C}}(X)$ corresponds to the principal ideal

$$\downarrow(\uparrow(A))=\{\alpha\in\mathfrak{F}\,\mathrm{Sub}_{\mathscr{C}}(X)\mid A\in\alpha\}.$$

(iii) *The inverse image part of the geometric morphism $\Phi(\mathscr{F}):\Phi(\mathscr{D})\to\Phi(\mathscr{C})$ gives, for each object X of $\mathscr{C}$, a map*

$$(\Phi\mathscr{F})^*:\mathrm{Sub}_{\Phi\mathscr{C}}(\mathscr{I}_{\mathscr{C}}X)\to\mathrm{Sub}_{\Phi\mathscr{D}}(\Phi(\mathscr{F})^*\mathscr{I}_{\mathscr{C}}X)\cong\mathrm{Sub}_{\Phi\mathscr{D}}(\mathscr{I}_{\mathscr{D}}(\mathscr{F}X)),$$

which under the isomorphisms of (ii) *is identified with the map*

$$\mathfrak{IF}(\mathrm{Sub}_{\mathscr{C}}(X))\to\mathfrak{IF}(\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X))$$

sending an ideal U of filters to the ideal

$$\mathfrak{IF}(\mathscr{F})(U)=\{\beta\in\mathfrak{F}\,\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X)\mid\exists\alpha\in U\,\forall A\in\alpha\;\mathscr{F}(A)\in\beta\}.$$

A.11. Corollary. (i) *If $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a conservative morphism of small coherent categories, then $\Phi(\mathscr{F}):\Phi(\mathscr{D})\to\Phi(\mathscr{C})$ is a surjective geometric morphism.*

(ii) *Suppose that $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a morphism of small coherent categories and that $\mathscr{C}$ has finite disjoint coproducts* (*cf.* 1.8(i)). *If Y is an object of $\mathscr{D}$ and $\mathscr{I}_{\mathscr{D}}(Y)$ is a subquotient in $\Phi(\mathscr{D})$ of an object in the image of $\Phi(\mathscr{F})^*$, then Y is a subquotient in $\mathscr{D}$ of an object in the image of $\mathscr{F}$.*

Proof. (i) By A.10(i), it is sufficient to prove that for each object X of $\mathscr{C}$, the map

$$\Phi(\mathscr{F})^* : \mathrm{Sub}_{\Phi\mathscr{C}}(\mathscr{I}_{\mathscr{C}}X) \to \mathrm{Sub}_{\Phi\mathscr{D}}(\Phi(\mathscr{F})^* \mathscr{I}_{\mathscr{C}}X) \cong \mathrm{Sub}_{\Phi\mathscr{D}}(\mathscr{I}_{\mathscr{D}}(\mathscr{F}X))$$

is a monomorphism; and then by A.10(iii), for this it is sufficient to prove that the map

$$\mathfrak{I}\mathfrak{F}(\mathscr{F}) : \mathfrak{I}\mathfrak{F}(\mathrm{Sub}_{\mathscr{C}}(X)) \to \mathfrak{I}\mathfrak{F}(\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X))$$

is a monomorphism. But the functors $\mathfrak{F}$ and $\mathfrak{I}$ (as defined in A.2) preserve distributive lattice monomorphisms and by hypothesis

$$\mathscr{F} : \mathrm{Sub}_{\mathscr{C}}(X) \to \mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X)$$

is one.

(ii) By A.10(i) (and using the fact that $\Phi(\mathscr{F})^* \mathscr{I}_{\mathscr{C}} \cong \mathscr{I}_{\mathscr{D}}\mathscr{F}$), we may assume that $\mathscr{I}_{\mathscr{D}}(Y)$ is a subquotient of the (disjoint) coproduct of objects $(\mathscr{I}_{\mathscr{D}}(\mathscr{F}X_i) \mid i \in I)$ where I is a set and the X_i are objects of $\mathscr{C}$:

$$\begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d, tail] & \mathscr{I}_{\mathscr{D}}(Y) \\ \coprod_{i \in I} \mathscr{I}_{\mathscr{D}}(\mathscr{F}X_i). \end{tikzcd} \tag{27}$$

Now by A.10(ii), the lattice of subobjects of $\mathscr{I}_{\mathscr{D}}(Y)$ is isomorphic to the lattice of ideals of a distributive lattice (namely $\mathfrak{F}\,\mathrm{sub}_{\mathscr{D}}(Y)$), and hence in particular is compact: i.e. if the greatest element of the lattice is equal to the join of a collection of elements, then it is equal to the join of a *finite* subcollection of those elements. It follows that in (27) we can take I to be a finite set. Then since $\mathscr{F}$ and $\mathscr{I}_{\mathscr{D}}$ (being morphisms of coherent categories) preserve finite disjoint coproducts and since $\mathscr{C}$ is assumed to have them, $\mathscr{I}_{\mathscr{D}}(Y)$ is a subquotient of a *single* object of the form $\mathscr{I}_{\mathscr{D}}(\mathscr{F}X)$:

$$\begin{tikzcd} B \arrow[r, two heads] \arrow[d, tail] & \mathscr{I}_{\mathscr{D}}(Y) \\ \mathscr{I}_{\mathscr{D}}(\mathscr{F}X) \end{tikzcd}$$

By A.10(ii)

$$B = \bigvee_{\alpha \in U} \bigwedge_{A \in \alpha} \mathscr{I}_{\mathscr{D}}(A)$$

for some $U \in \mathfrak{I}\mathfrak{F}(\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X \times Y))$. Then since B satisfies

$$\vdash \exists x \in \mathscr{I}_{\mathscr{D}}\mathscr{F}X\, B(x, y) \quad (y \in \mathscr{I}_{\mathscr{D}}Y)$$

in $\Phi(\mathscr{D})$, using the (natural) isomorphisms of A.10(ii), we have that U satisfies

$$\mathfrak{I}\mathfrak{F}(\exists \pi_2)(U) = \top \quad \text{in } \mathfrak{I}\mathfrak{F}(\mathrm{Sub}_{\mathscr{D}}(Y));$$

and hence there is $\alpha \in U$ with

$$\exists\pi_2(A) = \top, \quad \text{for all } A \in \alpha.$$

But B also satisfies

$$B(x, y) \wedge B(x, y') \vdash y = y' \quad (x \in \mathscr{I}_{\mathscr{D}}\mathscr{F}X; y, y' \in \mathscr{I}_{\mathscr{D}}Y)$$

and hence α satisfies

$$\mathfrak{F}(\langle\pi_1, \pi_2\rangle^{-1})\alpha \wedge \mathfrak{F}(\langle\pi_1, \pi_3\rangle^{-1})\alpha \leqslant \mathfrak{F}(\exists(\mathrm{id} \times \Delta))(\top)$$

in $\mathfrak{F}\,\mathrm{Sub}_D(\mathscr{F}X \times Y \times Y)$ (where the π_i are the projections from $\mathscr{F}X \times Y \times Y$ and $\Delta : Y \to Y \times Y$ is the diagonal). Since

$$\mathfrak{F}(\exists(\mathrm{id} \times \Delta))(\top) = \uparrow(\exists \mathrm{id} \times \Delta(\top))$$

is a principal filter, it follows that there is $A \in \alpha$ with

$$\langle\pi_1, \pi_2\rangle^{-1}A \wedge \langle\pi_1, \pi_3\rangle^{-1}A \leqslant \exists(\mathrm{id} \times \Delta)\top$$

in $\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X \times Y \times Y)$. By choice of α, A also satisfies

$$\exists\pi_2(A) = \top$$

in $\mathrm{Sub}_{\mathscr{D}}(Y)$. Thus A presents Y as a subquotient of $\mathscr{F}X$ in $\mathscr{D}$:

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, tail] & Y \\ \mathscr{F}X \end{tikzcd}$$

as required. $\square$

We have now demonstrated all the properties of the topos of filters construction mentioned in 4.1 except for 4.1(i) and the fact that $\mathscr{I}_{\mathscr{C}}$ is a logos morphism when $\mathscr{C}$ is a logos. Again, these are actually corollaries of Proposition A.10:

A.12. Corollary. (i) *If $\mathscr{C}$ is a logos, then $\mathscr{I}_{\mathscr{C}} : \mathscr{C} \hookrightarrow \Phi(\mathscr{C})$ is a logos morphism.*
(ii) *If $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a morphism of logoses, then $\Phi(\mathscr{F}) : \Phi(\mathscr{D}) \to \Phi(\mathscr{C})$ is an open geometric morphism.*

Proof. (i) Given $f : X \to Y$ in $\mathscr{C}$, since $\Phi(\mathscr{C})$ is a topos, the map

$$(\mathscr{I}_{\mathscr{C}}f)^{-1} : \mathrm{Sub}_{\Phi\mathscr{C}}(\mathscr{I}_{\mathscr{C}}Y) \to \mathrm{Sub}_{\Phi\mathscr{C}}(\mathscr{I}_{\mathscr{C}}X)$$

has both left and right adjoints and so preserves all meets and joins. It follows that under the isomorphisms of A.10(ii), this map is identified with the map

$$\mathfrak{I}\mathfrak{F}\,\mathrm{Sub}_{\mathscr{C}}(Y) \to \mathfrak{I}\mathfrak{F}\,\mathrm{Sub}_{\mathscr{C}}(X)$$

sending an ideal V of filters to the ideal

$$\mathfrak{I}\mathfrak{F}(f^{-1})(V) = \{\alpha \in \mathfrak{F}\,\mathrm{Sub}_{\mathscr{C}}(X) \mid \exists\beta \in V \;\forall B \in \beta\; f^{-1}B \in \alpha\}$$

Since $f^{-1}: \mathrm{Sub}_{\mathscr{C}}(Y) \to \mathrm{Sub}_{\mathscr{C}}(X)$ itself has a right adjoint $\forall f$, then the right adjoint to $(\mathscr{I}_{\mathscr{C}} f)^{-1}$ can be described as sending an ideal U of filters to

$$\mathscr{I}\mathfrak{F}(\forall f)(U) = \{\beta \in \mathscr{F}\,\mathrm{Sub}_{\mathscr{C}}(Y) \mid \exists \alpha \in U\ \forall \alpha \in \alpha\ \forall f A \in \beta\}.$$

Note that this map sends the principal ideal $\downarrow(\uparrow(A))$ to the principal ideal $\downarrow(\uparrow(\forall f A))$: by A.10(ii), these principal ideals correspond to $\mathscr{I}_{\mathscr{C}}(A)$ and $\mathscr{I}_{\mathscr{C}}(\forall f A)$ respectively. Therefore

$$\mathscr{I}_{\mathscr{C}}(\forall f A) = \forall(\mathscr{I}_{\mathscr{T}} f)(\mathscr{I}_{\mathscr{C}} A),$$

i.e. $\mathscr{I}_{\mathscr{C}}$ preserves universal quantification. It is thus a logos morphism. In fact there is a more fundamental reason why this is so. $\mathscr{I}_{\mathscr{C}}$ is the composition of $[-]: \mathscr{C} \to \Lambda(\mathscr{C})$ with $Y_{\Lambda\mathscr{C}}: \Lambda(\mathscr{C}) \to \mathscr{E}(\Lambda\mathscr{C}) = \Phi(\mathscr{C})$; and in the general (coherent) case, these functors actually preserve any universal quantifications which happen to exist. (But in general $\Lambda(\mathscr{C})$ is not itself a logos when $\mathscr{C}$ is.)

(ii) For each object X in $\mathscr{C}$

$$\mathscr{F}: \mathrm{Sub}_{\mathscr{C}}(X) \to \mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X)$$

is a morphism of Heyting algebras. It follows (cf. [13] Proposition 2.2) that

$$\mathfrak{I}\mathfrak{F}(\mathscr{F}): \mathfrak{I}\mathfrak{F}\,\mathrm{Sub}_{\mathscr{C}}(X) \to \mathfrak{I}\mathfrak{F}\,\mathrm{Sub}_{\mathscr{D}}(\mathscr{F}X)$$

has a left adjoint satisfying Frobenius reciprocity (see 3.2); moreover, since $\mathscr{F}$ preserves universal quantification, these left adjoints are natural in X. Therefore by A.10(iii),

$$(\Phi\mathscr{F})^*: \mathrm{Sub}_{\Phi\mathscr{C}}(Z) \to \mathrm{Sub}_{\Phi\mathscr{D}}((\Phi\mathscr{F})^* Z)$$

has a left adjoint satisfying Frobenius reciprocity when Z is in the image if $\mathscr{I}_{\mathscr{C}}$; and moreover, these left adjoints are natural for maps in the image of $\mathscr{I}_{\mathscr{C}}$. But then by A.10(i), these properties extend to all objects and maps of $\Phi(\mathscr{C})$: hence $\Phi\mathscr{F}$ is open. $\square$

References

[1] M. Artin, A. Grothendieck and J.L. Verdier, Théorie des Topos et Cohomologie Etale des Schémas (SGA4), Lecture Notes in Math. 269, 270 (Springer, Berlin, 1972).

[2] J. Bénabou, Introduction to bicategories, in: Reports of the Midwest Category Seminar, Lecture Notes in Math. 47 (Springer, Berlin, 1967).

[3] J. Bénabou, Fibred categories and the foundations of naive category theory. J. Symbolic Logic 50 (1985) 10–37.

[4] A. Boileau and A. Joyal, La logique des topos, J. Symbolic logic 46 (1981) 6–16.

[5] M. Fourman and D.S. Scott, Sheaves and logic, in: Applications of Sheaves, Proceedings Durham 1977, Lecture Notes in Math. 753 (Springer, Berlin, 1979) 302–401.

[6] P.T. Johnstone, Topos Theory (Academic Press, London, 1977).

[7] P.T. Johnstone, Open maps of toposes, Manuscripta Math. 31 (1980) 217–247.

[8] P.T. Johnstone, Factorization theorems for geometric morphisms, I, Cahiers Topologie Géom. Différentielle 22 (1981) 3–17.

[9] A. Joyal and M. Tierney, An extension of the Galois theory of Grothendieck, Mem. Amer. Math. Soc. 51 (1984) 309.
[10] M. Makkai and G.E. Reyes, First Order Categorical Logic, Lecture Notes in Math. 611 (Springer, Berlin, 1977).
[11] M. Makkai, The topos of types, in: Logic Year 1979–80, Lecture Notes in Math. 859 (Springer, Berlin, 1981) 157–201.
[12] M. Makkai, Ultraproducts and categorical logic, in; C. DiPrisco, ed., Methods in Mathematical Logic (Proceedings, Caracas 1985), Lecture Notes in Math. 1130 (Springer, Berlin, 1987) 222–309.
[13] A.M. Pitts, Amalgamation and interpolation in the category of Heyting algebras, J. Pure Appl. Algebra 29 (1983) 155–165.
[14] A.M. Pitts, An application of open maps to categorical logic, J. Pure Appl. Algebra 29 (1983) 313–326.
[15] A.M. Pitts, Interpolation and conceptual completeness for pretoposes via category theory, in: D.W. Keuker et al., eds., Mathematical Logic and Theoretical Computer Science, Lecture Notes Pure Appl. Math. 106 (Marcel Dekker, New York, 1987) 301–327.