Lindstöm Quantifiers and 0–1 Laws

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Asymptotic Probabilities

For a graph property P,

$\mu_n(P)$ is the proportion of graphs on n vertices that have property P

The asymptotic probability of P is $\lim_{n\to\infty} \mu_n(P)$ if this limit exists.

- μ(connectivity) $= 1$
- μ(3-colourability) $= 0$
- μ(planarity) $= 0$
- μ(Hamiltonicity) $= 1$
- μ(rigidity) $= 1$

By contrast

- μ(even number of nodes) is not defined
- μ(even number of edges) $= 1/2$

$\mathcal{G}(n,p)$ – the probability space of random graphs with edge probability $p(n)$ and universe $\{0,\ldots,n-1\}$.

$\mathcal{G}(p)$ denotes the sequence $\mathcal{G}(n,p)$, $n \in \mathbb{N}$.

For a given sequence of probabilty spaces Γ,

The asymptotic probability of a property P of graphs on Γ is $\lim_{n\to\infty} \mu_n(P)$ if this limit exists.

For instance,

$\mu(\text{Hamiltonicity}) = 1$ for $\mathcal{G}(p)$ for any positive constant p.

0–1 Laws

Theorem **(Glebskiĭ *et al.*; Fagin)**
For every property P definable in first-order logic, $\mu(P) = 0$ or 1 on $\mathcal{G}(1/2)$.

0–1 laws hold for a variety of extensions of first-order logic

In particular:

LFP **(Blass *et al.*)**

PFP, $L^{\omega}_{\infty\omega}$ **(Kolaitis and Vardi)**.

Extension Axioms

The proofs of the above 0-1 laws rely on extension axioms

An *atomic type* is a maximal consistent set of literals.

For atomic types $s \subseteq t$

$$\forall x_1 \ldots \forall x_n (s \rightarrow \exists x_{n+1} t)$$

Indeed, the following is a direct consequence of the fact that each extension axiom has asymptotic probability 1.

Every formula of first-order logic is equivalent *almost everywhere* to a quantifier free formula.

(Glebskĭi *et al.*; **Grandjean)**

There is no first-order property that implies either Hamiltonicity or rigidity and has asymptotic probability 1.

(Blass and Harary)

Is there a logic that has a 0–1 law and which can express Hamiltonicity or rigidity?

Lindström Quantifiers

Lindström Quantifiers provide a framework for studying abstract logics that can express a given property.

Given a property P –

FO$[Q_P]$ *is the minimal logic that can express* P *and has certain natural closure properties:*

- negation
- conjunction
- particularization
- substitution

A regular logic is, in addition, closed under:

- relativization

For which graph properties $\mathcal{H}$ is it the case that FO[$Q_{\mathcal{H}}$] has a 0–1 law?

FO[Rig] has a 0–1 law.

FO[Ham] does not

Moreover, if Rig' is the relativized version of Rig, then

FO[Rig'] has a 0-1 law.

There is no regular logic that can express Hamiltonicity and that has a 0–1 law.

There is such a logic in the case of rigidity.

Graph Interpretations

A formula $\psi(x, y)$ defines a map Ψ from graphs to graphs.

If $G = (V, E)$, then $\Psi G = (V, E')$,

where, $E'ab \leftrightarrow (G, a, b) \models x \neq y \wedge (\psi(x, y) \vee \psi(y, x))$.

A formula $\varphi(x, y, \bar{z})$ with additional free variables defines a graph interpretation *with parameters* Φ which maps each $(G, \bar{a})$ to a graph.

Generalized Quantifiers

With any isomorphism closed collection of graphs $\mathcal{H}$, we associate the quantifier $Q_{\mathcal{H}}$.

FO[$Q_{\mathcal{H}}$] is the logic obtained by closing first-order logic under the formation rule:

> if $\psi(x, y, \bar{z})$ is a formula with free variables among $x, y, \bar{z}$, then $Q_{\mathcal{H}} x, y\, \psi$ is a formula with free variables among $\bar{z}$

The semantics of the formula is given by:

> $(G, \bar{a}) \models Q_{\mathcal{H}} x, y\, \psi$ iff $\Psi(G, \bar{a}) \in \mathcal{H}$, where Ψ is the graph interpretation defined by ψ.

necessary conditions

When does FO[$Q_{\mathcal{H}}$] have a 0–1 law?

$\mathcal{H}$ must itself have asymptotic probability 0 or 1.

Let $\mathcal{E}$ be the class of Eulerian graphs. $\mathcal{E}$ has asymptotic probability 0.

$Q_{\mathcal{E}}x, y(x \neq y)$ does not have an asymptotic probability

FO'[$Q_{\mathcal{H}}$] — those sentences of FO[$Q_{\mathcal{H}}$] of the form $Q_{\mathcal{H}}x, y\,\psi$ (ψ is first-order).

Up to equivalence, there are only four quantifier free first-order formulas in two variables defining an irreflexive and symmetric relation.

$x \neq y$, $x \neq x$, Exy, $\neg Exy$.

Theorem
FO'[$Q_{\mathcal{H}}$] has a 0–1 law for $\mathcal{G}(p)$ if and only if $\mathcal{H}$ has asymptotic probability 0 or 1 on each of

$\mathcal{G}(0)$, $\mathcal{G}(1)$, $\mathcal{G}(p)$ and $\mathcal{G}(1-p)$,

FO'[Ham] has a 0-1 law.

$\exists z\mathsf{Ham}\, x, y\ (Exz \wedge \neg Eyz)$

is false on all graphs with an odd number of nodes

is true on a graph with $2n$ nodes just in case it contains a vertex of degree n

FO[Ham] does not have a 0–1 law on $\mathcal{G}(1/2)$

Quantifier Elimination

For each quantifier free formula ψ, there is a quantifier free formula θ such that $Q_{\mathcal{H}} x, y\, \psi$ is equivalent to θ almost everywhere, provided that $\mathcal{H}$ satisfies:

> $\mathcal{H}$ converges $quickly$ to 0 or 1 on every $interpretive\ measure$

FO[Rig] has a 0-1 law.

Let $\psi(x, y)$ be a formula defining a graph interpretation Ψ

Ψ also maps a probability space Γ_n to a new space $\Psi\Gamma_n$, which is defined by assigning to a graph H the probability

$$\nu_n(H) = \sum_{\Psi G = H} \mu_n(G)$$

The asymptotic probability of $Q_{\mathcal{H}} x, y \psi$ on Γ is given by the asymptotic probability of $\mathcal{H}$ on $\Psi\Gamma$.

For a formula $\psi(x, y, \bar{z})$ defining a graph interpretation Ψ with parameters:

$\Psi\Gamma_n$ is defined for a probability space Γ_n that assigns a probability to each $(G, \bar{a})$.

Let ψ be a quantifier free formula in the variables $x, y, z_1, \ldots, z_m$.

Also, let $t_0, \ldots, t_{r-1}$ be an enumeration of all atomic m-types.

Then,

$$\psi \equiv \bigvee_{i<r} \alpha_i,$$

where,

$$\alpha_i = \bigvee s_j,$$

and each s_j is an atomic $(m+2)$-type extending t_i.

In order to show that

$$Qx, y\psi \leftrightarrow \theta$$

holds for some quantifier free θ, almost everywhere.

It suffices to show that, for each t_i,

$$Qx, y\alpha_i \leftrightarrow t_i$$

holds almost everywhere.

For each atomic type t in m variables,

the number of tuples of type t in a graph of size n tends to n^m/c.

Define the probability space $\mathcal{G}_{n,t}(p)$ by assigning, for each $G \in \mathcal{G}_n(p)$ equal probability to $(G, \bar{a})$ for each tuple $\bar{a}$ such that $G \models t[\bar{a}]$.

For any formula $\varphi(\bar{z})$,

1. if $\mu_{n,t}(\varphi) = 1 - o(n^{-m})$ then $\mu(\forall \bar{z}\ t(\bar{z}) \to \varphi) = 1$
2. if $\mu_{n,t}(\varphi) = o(n^{-m})$ then $\mu(\exists \bar{z}\ t(\bar{z}) \wedge \varphi) = 0$

If φ is of the form $Q_{\mathcal{H}} x, y \psi$, then (1) holds if:

$\mu_n(\mathcal{H}) = 1 - o(n^{-m})$ on $\Psi \mathcal{G}_{n,t}(p)$

Quantifier Elimination

If, for every atomic type $t(\bar{z})$ and every quantifier free formula $\psi(x, y, \bar{z})$, extending t, one of the following holds

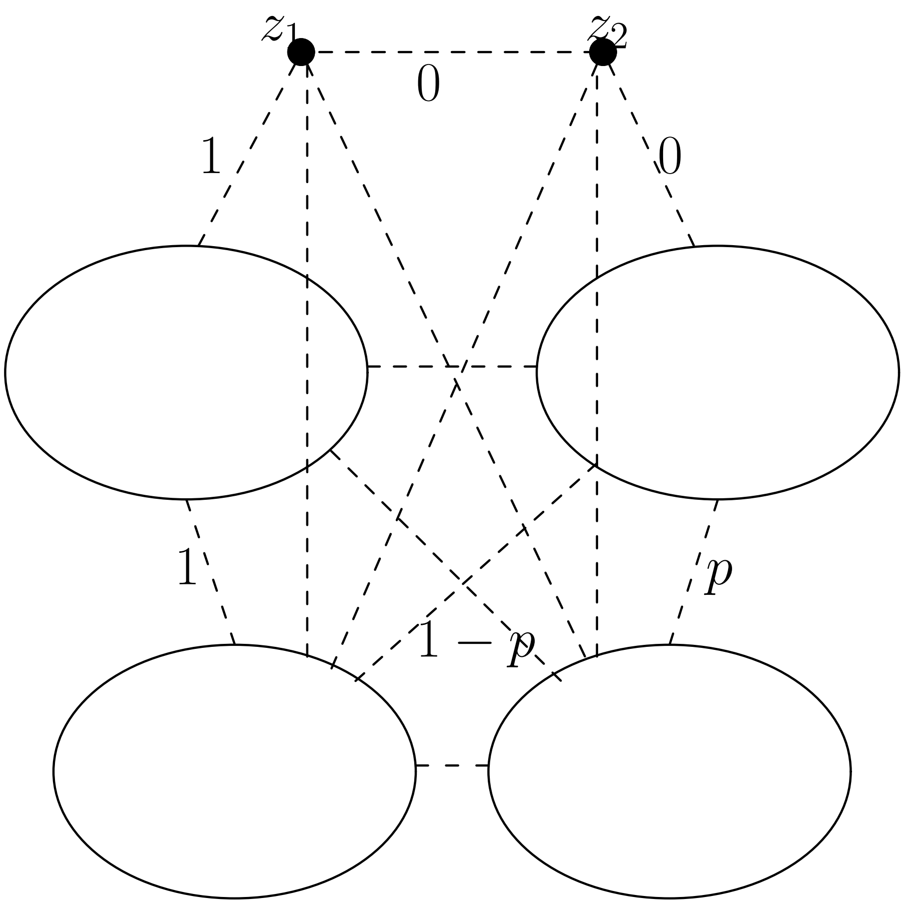
$\mu_n(\mathcal{H}) = 1 - o(n^{-m})$ on $\Psi\mathcal{G}_{n,t}(p)$

$\mu_n(\mathcal{H}) = o(n^{-m})$ on $\Psi\mathcal{G}_{n,t}(p)$

then we can prove the following:

For each quantifier free formula ψ, there is a quantifier free formula θ such that $Q_{\mathcal{H}}x, y\,\psi$ is equivalent to θ almost everywhere

$\psi(x, y, z_1, z_2)$



f: $(m \uplus \mathcal{P}(m))^2 \rightarrow \{0, 1, 1/2\}$

Interpretive measure — $\Gamma_n(m, \mathsf{f})$

Theorem
FO[$Q_{\mathcal{H}}$] has a 0–1 law if $\mathcal{H}$ converges $quickly$ to asymptotic probability 0 or 1 on each $\Gamma(m, f)$.

FO[Rig] has a 0-1 law.

Rigidity

For an interpretive measure $\Gamma_n(m, \mathsf{f})$, with
f: $(m \uplus \mathcal{P}(m))^2 \longrightarrow \{0, 1, 1/2\}$,

We have

1. if there is a permutation π on m such that
$$f(p, p') = f(\pi p, \pi p')$$
for all $p, p' \in m$, and
$$f(p, q) = f(\pi p, q)$$
for all $p \in m$, $q \in \mathcal{P}(m)$,
then π is an automorphism on all $G \in \Gamma_n(m, \mathsf{f})$.
2. if there is a $q \in Q$ such that
$$f(q, q') \in \{0, 1\}$$
for all $q' \in Q$,
then almost all $G \in \Gamma_n(m, \mathsf{f})$ are not rigid.
3. otherwise almost all $G \in \Gamma_n(m, \mathsf{f})$ are rigid.

Closure Conditions

Relativization

Rig': The relativized version of Rig binds variables in two formulas.

Rig'$w, x, y(\psi(w), \varphi(x, y))$

FO[Rig'] has a 0–1 law.

Vectorization

Plan^k: The kth vectorized version of Plan binds $2k$ variables.

Plan$^k \bar{x}, \bar{y}\, \psi$

FO[$\{$Plan$^k \mid k \in \omega\}$] has a 0–1 law

FO[$\{$3-Col$^k \mid k \in \omega\}$] has a 0–1 law

The same holds for any quantifier closed under substructures.