Descriptive and Computational Complexity

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Leicester, 16 February 2007

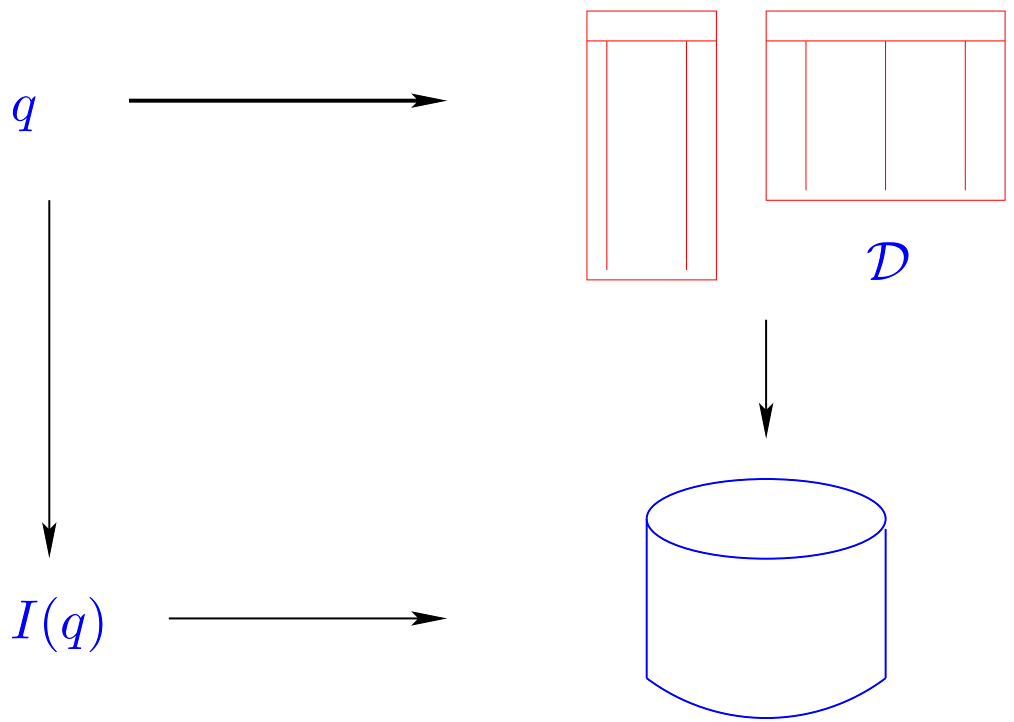
Complexity and Database Theory

Descriptive Complexity Theory arises from questions in computational complexity and in database theory.

In 1974, **Fagin** showed that the collection of problems definable in *existential second-order logic* is exactly the problems in **NP**.

In 1980, **Chandra and Harel** asked whether there a database query language in which one can express exactly the *feasible, generic* queries.

Generic Queries



A query q is *generic* if the answer to q depends only on the abstract view $\mathcal{D}$

q is *feasible* if its implementation $I(q)$ runs in time polynomial in the size of $\mathcal{D}$

Descriptive vs. Computational Complexity

Computational Complexity:

is concerned with measuring space, time or other resources on a machine model of computation.

usually defines complexity of a language - i.e. a set of strings

Descriptive Complexity:

defines the complexity of classes of structures - *e.g.* a collection of graphs, or relations.

concerned with the complexity of describing the collection in a suitable language.

Relational Databases

$$Cinema = \{Movies[3], Location[3], Guide[3]\}$$

Movies	Title	Director	Actor
	Volver	Almodovar	Cruz
	Volver	Almodovar	Maura
	Casino Royale	Campbell	Craig
	Casino Royale	Campbell	Green
	...		
	Rocky	Stallone	Stallone

Guide	Title	Cinema	Time
	Rocky	Vue	12:00
	Volver	Picturehouse	19:00
	...		
	Casino Royale	Cineworld	19:00
	Rocky	Cineworld	22:00

Location	Cinema	Address	Tel
	Picturehouse	Cambridge	504444
	Vue	Leicester	240240
	Cineworld	Cambridge	560225

Relational Algebra

In relational algebra, queries are built up from

Base relations: R

Singleton constant relations: $\{\langle a \rangle\}$

using

select: $\sigma_{j=a}(q)$ or $\sigma_{j=k}(q)$

project: $\pi_{j_1,\ldots,j_k}(q)$

join: $q_1 \bowtie q_2$

union: $q_1 \cup q_2$

difference: $q_1 - q_2$

Relational Calculus

Codd in 1972 introduced the relational calculus (based on first-order logic) and equivalent to the relational algebra.

Conjunctive Queries:

$$q(x, y) \leftarrow \textit{Movies}(z_1, \text{“Almodovar”}, z_2), \textit{Guide}(x, z_1, z_3), \textit{Location}(x, y, z_4)$$

expresses the query

$$\{x, y \mid \exists z_1, \ldots, z_4\ \textit{Movies}(z_1, \text{“Almodovar”}, z_2) \wedge \textit{Guide}(x, z_1, z_3) \wedge \textit{Location}(x, y, z_4)\}$$

Disjunction is expressed by *multiple rules*.

First-Order Logic

Adding negation and universal quantification gives us the full-power of relational algebra, or equivalently, *first-order logic*.

Note: closed-world assumption.

From now on, we speak of finite relational structures:

$$\mathbb{A} = (A, R_1, \ldots, R_m)$$

where A is a finite *domain* and each R_i is a relation on A.

And queries are given by formulas of predicate logic:

atomic formulas – $R(t_1, \ldots, t_m)$, $t_1 = t_2$

Boolean operations – $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$

first-order quantifiers – $\exists x \varphi$, $\forall x \varphi$

Complexity of First-Order Logic

A query expressed by a first-order formula φ can be evaluated in time polynomial in the size of the structure $\mathbb{A}$.

If $\psi(x_1, \ldots, x_k)$ is a sub-formula of φ,

there are at most n^k tuples satisfying this formula.

where n is the number of elements in A.

In fact, it can be shown that the query can be computed in *logarithmic space*.

Limitations of First-Order Logic

There are *polynomial-time computable* and *generic* queries that are not computable in first-order logic.

Evennness:

Is the number of elements in A even?

Transitive Closure:

In a structure (A, R) with a binary relation R, give the set of pairs (x, y) such that there is an R-path from x to y.

Second-Order Quantifiers

Existential Second-Order Quantification:

$$\exists P_1 \ldots \exists P_m \varphi$$

A structure $\mathbb{A}$ satisfies $\exists P \varphi$ if there is a relation R on the universe of $\mathbb{A}$ such that $(\mathbb{A}, R)$ satisfies φ.

ESO – existential second order logic

$$\text{ESO} \subseteq \text{NP}$$

An existential second order quantifier represents a polynomial amount of non-determinism.

Examples

Evennness

This formula is true in a structure if, and only if, the size of the domain is even.

$$
\begin{aligned}
\exists B \exists S \quad & \forall x \exists y B(x,y) \wedge \forall x \forall y \forall z B(x,y) \wedge B(x,z) \rightarrow y = z \\
& \forall x \forall y \forall z B(x,z) \wedge B(y,z) \rightarrow x = y \\
& \forall x \forall y S(x) \wedge B(x,y) \rightarrow \neg S(y) \\
& \forall x \forall y \neg S(x) \wedge B(x,y) \rightarrow S(y)
\end{aligned}
$$

Examples

Transitive Closure

This formula is true of a pair of elements a, b in a structure if, and only if, there is an R-path from a to b.

$$\begin{array}{ll} \exists P & \forall x \forall y\, P(x,y) \rightarrow R(x,y) \\ & \exists x P(a,x) \land \exists x P(x,b) \land \neg\exists x P(x,a) \land \neg\exists x P(b,x) \\ & \forall x(x \neq a \land \exists y(P(x,y) \rightarrow \forall z(P(x,z) \rightarrow y = z))) \\ & \forall x(x \neq b \land \exists y(P(y,x) \rightarrow \forall z(P(z,x) \rightarrow y = z))) \end{array}$$

Examples

3-Colourability

The following formula is true in a graph (V, E) if, and only if, it is 3-colourable.

$$
\begin{aligned}
\exists R \exists B \exists G \quad & \forall x (Rx \vee Bx \vee Gx) \wedge \\
& \forall x (\quad \neg(Rx \wedge Bx) \wedge \neg(Bx \wedge Gx) \wedge \neg(Rx \wedge Gx)) \wedge \\
& \forall x \forall y (Exy \rightarrow (\quad \neg(Rx \wedge Ry) \wedge \\
& \qquad\qquad\qquad\qquad \neg(Bx \wedge By) \wedge \\
& \qquad\qquad\qquad\qquad \neg(Gx \wedge Gy)))
\end{aligned}
$$

Note, this is an NP-complete problem and so unlikely to be computable in polynomial-time.

Fagin's Theorem

Fagin proved that *every* problem that is in the complexity class NP is definable by a formula of ESO.

NP can be defined as the class of problems decidable by guessing a polynomial number of bits, and then running a polynomial-time verification algorithm

Fagin's theorem says that the verification phase can always be replaced by a first-order formula.

Chandra and Harel's question asks whether we can similarly characterise the class P.

Recursion

We are looking for logical formalisms intermediate in expressive power between first-order and second-order logic.

One idea, considered by Chandra and Harel, is to add a recursion mechanism to first-order logic.

Example:

$$\begin{aligned} T(x,y) &\leftarrow R(x,y) \\ T(x,y) &\leftarrow R(x,z), T(z,y). \end{aligned}$$

This recursively defines a relation T that is the transitive closure of the relation R.

LFP

More generally, we allow any first-order formula on the right-hand side of the rule:

$S(\mathbf{x}) \leftarrow \varphi(S)$ where φ is positive in the symbol S.

This rule has a *least* solution for S, and this solution can be constructed in time polynomial in the size of the structure $\mathbb{A}$.

If we allow S to occur inside a negation symbol on the right, the rule may not have a solution (*viz.* $S(x) \leftarrow \neg S(x)$).

LFP is the logic that is obtained by adding a recursion operator to first-order logic.

It can still not express *Evenness*.

Counting

LFP + C is a logic formulated to add the ability to count to LFP.

A second *sort* of variables: $\nu_1, \nu_2, \ldots$ which range over *numbers* in the range

$$0, \ldots, |A|$$

If $\varphi(x)$ is a formula with free variable x, then $\nu = \#x\varphi$ denotes that ν is the number of elements of A that satisfy the formula φ.

We also have the order $\nu_1 < \nu_2$, which allows us (using recursion) to define arithmetic operations.

Evenness

There are an even number of elements satisfying $\varphi(x)$.

$$\exists\nu_1\exists\nu_2(\nu_1 = [\#x\varphi] \land (\nu_2 + \nu_2 = \nu_1))$$

Cai-Fürer-Immerman

Cai, Fürer and Immerman (1992) showed that $\mathrm{LFP} + \mathrm{C}$ is not powerful enough to express all properties in *P*.

The proof involved a contrived construction of a class of graphs on which the graph isomorphism problems is solvable in polynomial time but not definable in $\mathrm{LFP} + \mathrm{C}$.

They conjectured that adding some "group-theoretic operators" may be a solution.

Group-theoretic Operators

We (Atserias, Bulatov, D., 2007) have recently exhibited natural feasibly computable problems that are not definable in LFP $+$ C.

- Solving linear equations over a finite field; or more simply
- Solving additive equations over a finite Abelian group.

These suggest natural operators that could be added to LFP $+$ C to obtain a logic that can still only express feasibly computable properties.

Linear Equations

Consider systems of equations (with three variables per equation), over the integers $\mathrm{mod}\ 2$.

$$a_1 + a_2 + a_3 = 0$$
$$a_2 + a_3 + a_4 = 1$$

has the solution $a_1 = a_2 = a_3 = 0$, $a_4 = 1$.

This can be coded as a structure with domain $\{a_1, \ldots, a_n\}$ and ternary relations R_0 and R_1, with:

$(a_i, a_j, a_k) \in R_m$ iff $a_i + a_j + a_k = m$ is an equation in the system

There is no formula of LFP $+$ C that defines the *solvable* systems of equations.

Challenges

Prove that the extension of LFP + C with an operator for determining the *rank of a matrix* still does not express all properties in P.

Other operators have also been defined in the literature (*e.g.* symmetric choice). It remains an open problem to show that these don't capture all of P.

It's possible that P cannot be "generated from below" by a finite collection of operators. To prove this would also separate P from NP.