Fixed Point Logics and Infinite Trees

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Computation Trees

Deterministic computation

$$\begin{tikzcd}
(q_0, s_0) \arrow[d] \\
(q_1, s_1) \arrow[d] \\
(q_2, s_2) \arrow[d] \\
{}
\end{tikzcd}$$

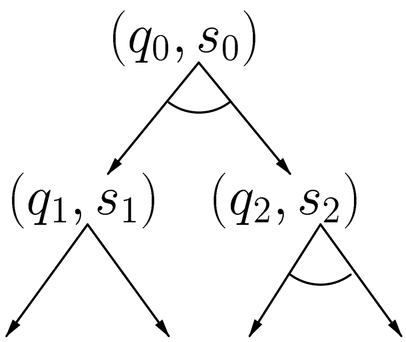
Non-deterministic computation

$$\begin{tikzcd}
& & (q_0, s_0) \arrow[dl] \arrow[dr] & & \\
& (q_1, s_1) \arrow[dl] \arrow[d] & & (q_2, s_2) \arrow[d] \arrow[dr] & \\
{} & {} & & {} & {}
\end{tikzcd}$$

If the height of the tree is bounded by a polynomial in the length of the input, we say the machine runs in polynomial time.

Alternation

Alternation allows us to combine the branches of a tree in more interesting ways than straightforward non-determinism.



The computation at a conjunctive node succeeds if all branches from that node succeed.

Fixed Point Logic

Fixed Point Logics

extensions of first order logic with fixed point operators.

- Extending study of inductive definability.
- Extensively studied in finite model theory.
- Related to questions of computational complexity.

LFP

Least fixed point logic

Close first order logic simultaneously under first order operations and

$$\mathbf{lfp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

where,

- R is a k-ary relation variable.
- $\mathbf{x}$ is a k-tuple of variables.
- $\mathbf{t}$ is a k-tuple of terms.
- R occurs only positively in φ.

The operator **lfp** binds all occurrences of R and variables in $\mathbf{x}$ that occur in φ, but not those in $\mathbf{t}$

Semantics

The semantics of the **lfp** operator are given by:

$$\mathcal{A} \models \textbf{lfp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

if

> $\mathbf{t}^{\mathcal{A}}$ is in the least fixed point of the monotone operator defined by $\varphi(R, \mathbf{x})$ on $|\mathcal{A}|^k$:

$$\Phi(R^{\mathcal{A}}) = \{\mathbf{a} \mid \mathcal{A} \models \varphi[R^{\mathcal{A}}, \mathbf{a}]\}$$

The least fixed point can be reached by iterating the operator:

$$\begin{aligned} \varphi_0 &= \emptyset \\ \varphi_{\alpha+1} &= \varphi(\varphi_\alpha / R) \\ \varphi_\lambda &= \bigcup_{\beta<\lambda} \varphi_\beta \qquad \text{for limit } \lambda \end{aligned}$$

The closure ordinal of φ is the least ordinal α such that

$$\varphi_\alpha = \varphi_{\alpha+1}.$$

Example 1

Example:

Let $\varphi(R, x, y)$ be $x = y \lor \exists z(E(x, z) \land R(z, y))$

Then, **lfp**$_{R,xy}\varphi(u, v)$ is a formula in two free variables that defines the transitive closure of E.

φ_n defines the set of pairs with paths of length at most n.

On finite graphs, the closure ordinal is the length of the longest path.

The closure ordinal is $\leq \omega$ on all structures.

Example 2

Let $\varphi(P, x)$ be

$$\forall y(y < x \to P(y)).$$

$\mathbf{lfp}_{P,x}\varphi$ defines the well-founded part of the binary relation $<$.

If $<$ is a linear order, this is the longest well-ordered initial segment.

The closure ordinal is the length of this well-order.

For any formula $\varphi(\mathbf{x})$ *of* LFP*,*

the set

$$\{(\mathcal{A}, \mathbf{c}) \mid \mathcal{A} \models \varphi[\mathbf{c}]\}$$

is decidable in polynomial time.

Conversely, in the presence of a distinguished binary relation $<$ *interpreted as a linear order on all structures,:*

Every polynomial time query is definable in LFP.

(Immerman; Vardi 1982)

Inflationary Fixed Points

IFP is similar to LFP except it has an operator

$$\textbf{ifp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

to denote that $\mathbf{t}$ is in the union of the sequence of relations:

$$\varphi_\alpha = \bigcup_{\beta<\alpha} \varphi(\varphi_\beta/R).$$

where,

$$\varphi(\varphi_\beta/R)$$

is the relation defined by φ when φ_β is fixed as the interpretation of R.

Note: R need not be positive in φ.

If φ defines a monotone operator, then

$$\mathbf{ifp}_{R,\mathbf{x}}\varphi$$

defines the same relation as

$$\mathbf{lfp}_{R,\mathbf{x}}\varphi$$

So, in general:

$$\mathsf{LFP} \leq \mathsf{IFP}$$

On finite structures:

$$\mathsf{LFP} = \mathsf{IFP}$$

(Gurevich–Shelah 1986).

Second Order Logic

A collection of finite structures is definable in Σ^1_1

if, and only if,

it is decidable in NP.

(Fagin 1974).

On finite structures:

$$\mathsf{LFP} = \mathsf{IFP} \leq \Delta^1_1.$$

either by Fagin's theorem or by a direct translation.

Translation

Translating $\textbf{ifp}_{R,\mathbf{x}}\varphi(\mathbf{t})$:

$$
\begin{array}{ll}
\forall(\exists) & \text{a linear order on } k\text{-tuples} \\
\forall(\exists) & \text{a sequence of relations } R^i \text{ indexed by } k\text{-tuples} \\
 & [R^0 = \emptyset \wedge \ R^{i+1} = R^i \cup \varphi(R^i)] \\
 & \qquad\qquad \to (\wedge)\ \exists i\, \mathbf{t} \in R^i.
\end{array}
$$

Without the restriction to finite structures:

$$\mathsf{LFP} \leq \mathsf{IFP} \leq \Delta^1_2$$

$\forall(\exists)$ a sequence of relations indexed by k-tuples
$\forall(\exists)$ a well ordering of k-tuples
$\ldots$

This cannot be improved as

$$\Pi^1_1 \cup \Sigma^1_1 \leq \mathsf{LFP}$$

on $(\omega, <)$.

Non-Deterministic Fixed Points

Consider two first order formulas:

$$\varphi_0(R, \mathbf{x}) \text{ and } \varphi_1(R, \mathbf{x})$$

We define a sequence of relations φ^b on a *finite* structure, indexed by *binary strings* b.

$$\begin{aligned} \varphi^\epsilon &= \emptyset \\ \varphi^{b\cdot 0} &= \varphi_0(\varphi^b/R) \cup \varphi^b \\ \varphi^{b\cdot 1} &= \varphi_1(\varphi^b/R) \cup \varphi^b \end{aligned}$$

$\mathbf{nfp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)$ is then the relation

$$\bigcup_b \varphi^b$$

On ordered structures:

A property is definable as:

$$\exists x\, \mathbf{nfp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)(x, \ldots, x)$$

where φ_0 and φ_1 are first order

if, and only if, it is NP.

The above is a normal form for "positive" NFP.

A property is definable in NFP (*closing under negations and* **nfp**) if, and only if, it is definable in second order logic.

Without linear order,

$$\text{NFP} < \text{SO}$$

still,

$$\text{IFP} = \text{NFP if, and only if, P} = \text{NP}.$$

(Abiteboul, Vardi and Vianu)

Infinite Structures

For any ordinal α, a binary string b of length α is a map

$$b : \alpha \to \{0, 1\}.$$

For a pair of formulas $\varphi_0(R, \mathbf{x})$ and $\varphi_1(R, \mathbf{x})$, define:

$$\begin{aligned}
\varphi^{\epsilon} &= \emptyset \\
\varphi^{b\cdot 0} &= \varphi_0(\varphi^b/R) \cup \varphi^b \\
\varphi^{b\cdot 1} &= \varphi_1(\varphi^b/R) \cup \varphi^b \\
\varphi^{b} &= \bigcup_{c<b} \varphi^c \qquad \text{when } b \text{ is of limit ordinal length}
\end{aligned}$$

here, $c < b$ means that c is a proper initial segment of b.

Well Ordered Structures

On well-ordered structures:

$$\mathsf{NFP} = \mathsf{SO}$$

NFP $\leq$ SO holds always

by a direct translation (*viz.* IFP).

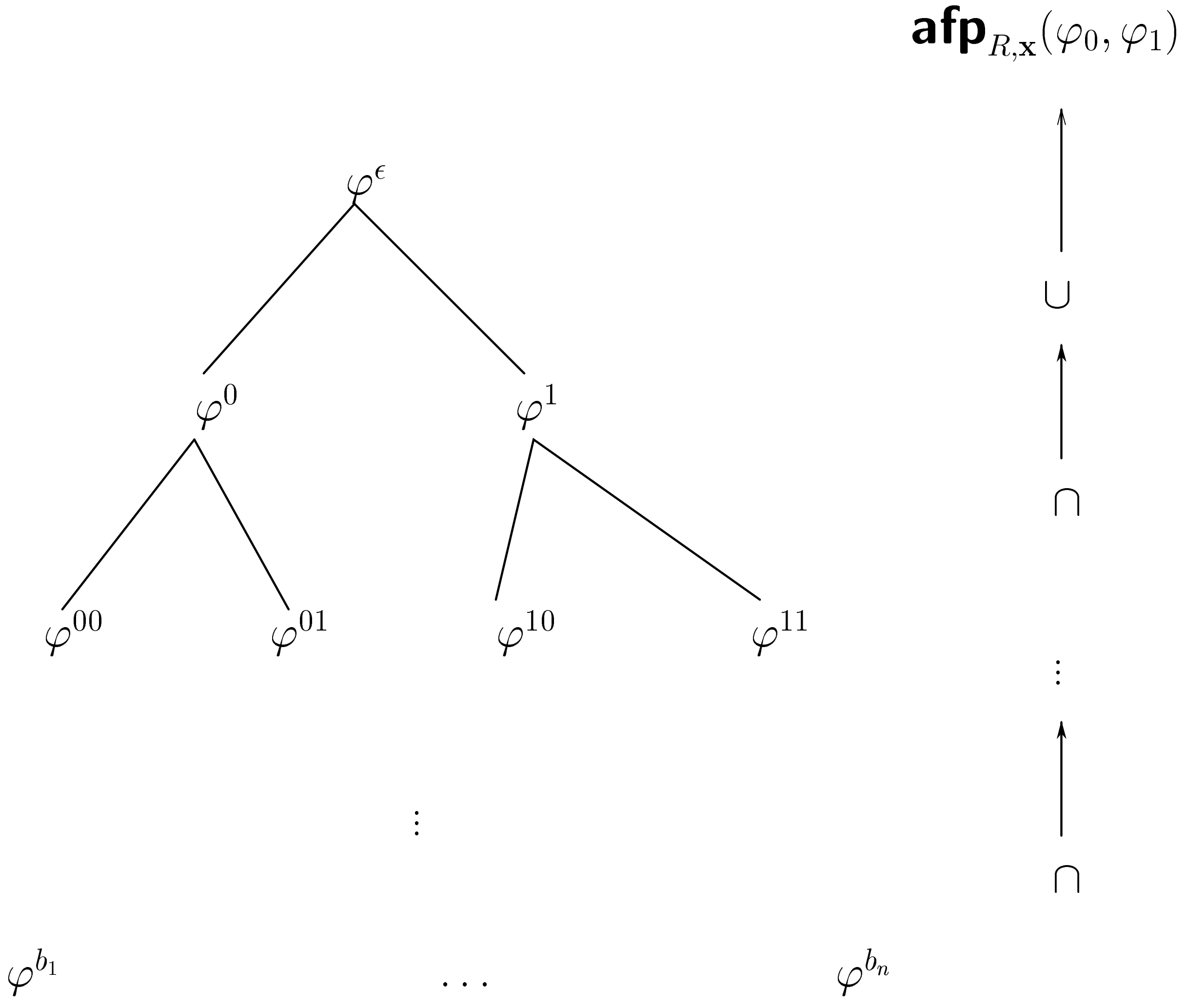
In the other direction, to translate $\exists R\,\theta$,

Intuitively, we can write φ_0 and φ_1 so that at each stage,

φ_0 adds the next element in the order to R.
φ_1 does not.

Alternations of **nfp** and $\neg$ correspond exactly to *quantifier alternations*.

Alternating Fixed Point Logic



$$\text{AFP} = \text{PFP}$$

Games

Player 1 and Player 2 take turns in deciding which of the two formulas

$$\varphi_0 \text{ or } \varphi_1$$

to apply.

Player 1 plays at limit ordinals.

A tuple $\mathbf{t}$ is in

$$\textbf{afp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)$$

if Player 1 has a winning strategy for placing it in the constructed relation.

In what situations is this game determined?

Determinacy

There are (uncountable) structures for which there are formulas with undetermined games.

If a fixed point is always reached at ω, then the game is determined.

> The set of ω-length binary strings b such that $\mathbf{x} \in \varphi^b$ is necessarily an open set

There is a countable structure and a pair of first-order formulas where the closure ordinal is not bounded by any countable ordinal

There is a countable structure and a pair of first-order formulas where the question of determinacy is independent of ZFC