Games and Isomorphism in Finite Model Theory

Part 1

Anuj Dawar

University of Cambridge

Games Winter School, Champéry, 6 February 2013

Model Comparison Games

Games in *Finite Model Theory* are generally used as a tool for proving limits on the expressive power of *logics*.

In this tutorial, we focus on *Model Comparison Games*.

These are typically two-player games played on a pair of structures $\mathbb{A}$ and $\mathbb{B}$.

The games are used to establish that $\mathbb{A}$ and $\mathbb{B}$ cannot be distinguished in some logic under consideration.

Spoiler and Duplicator

The two players in our games are generally called *Spoiler* and *Duplicator*.

The game board consists of two finite structures $\mathbb{A}$ and $\mathbb{B}$.

Spoiler tries to prove that $\mathbb{A}$ and $\mathbb{B}$ are different.

Duplicator tries to pretend that they are really the same

We say the two structures are *indistinguishable* (according to the rules of the game) if *Duplicator* has a winning strategy.

If the structures *are* the same (i.e. they are *isomorphic*), then *Duplicator* necessarily has a winning strategy.

In general, the relation of *indistinguishability* gives us an *approximation* of isomorphism.

Some Games

Classes of games we will look at in this tutorial include:

> *Ehrenfeucht-Fraïssé games; pebble games; counting games; bijection games; partition games; and invertible map games*

Associated with them are various *logics* we will examine which they are used to establish inexpressiveness results.

Many of these logics arose in the long-standing *quest* for a logic for PTime.

We will also see how the indistinguishability relations defined by the games relate to isomorphism, and look at other ways to characterise these equivalences.

Expressive Power of Logics

We are interested in the *expressive power* of logics on finite structures.

We consider finite structures in a *relational vocabulary*.

A finite set A, with relations $R_1, \ldots, R_m$ and constants $c_1, \ldots, c_n$.

A *property* of finite structures is any *isomorphism-closed* class of structures.

For a logic (i.e., a *description* or *query* language) $\mathcal{L}$, we ask for which properties P, there is a sentence φ of the language such that

$$\mathbb{A} \in P \quad \text{if, and only if,} \quad \mathbb{A} \models \varphi.$$

In our examples, we will mainly confine ourselves to vocabularies with just one binary relation E.

First-Order Logic

terms – c, x

atomic formulae – $R(t_1, \ldots, t_a), t_1 = t_2$

boolean operations – $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$

first-order quantifiers – $\exists x\varphi$, $\forall x\varphi$

Graphs which contain a triangle:

$$\exists x\exists y\exists z(x \neq y \wedge y \neq z \wedge x \neq y \wedge E(x, y) \wedge E(y, z) \wedge E(x, z))$$

Unions of cycles: $\forall x(\exists! yE(x, y) \wedge \exists! zE(z, y))$

Can we define the class of *connected graphs*? No, but how to prove it?

The Power of First-Order Logic

For every finite structure $\mathbb{A}$, there is a sentence $\varphi_{\mathbb{A}}$ such that

$$\mathbb{B} \models \varphi_{\mathbb{A}} \quad \text{if, and only if,} \quad \mathbb{B} \cong \mathbb{A}$$

Given a structure $\mathbb{A}$ with n elements, we define

$$\varphi_{\mathbb{A}} = \exists x_1 \ldots \exists x_n \psi \wedge \forall y \bigvee_{1 \leq i \leq n} y = x_i$$

where, $\psi(x_1, \ldots, x_n)$ is the conjunction of all atomic and negated atomic formulas that hold in $\mathbb{A}$.

For any isomorphism-closed class of finite structures, there is a first-order theory that defines it.

First-Order Logic is too Weak

For any first-order sentence φ, its class of finite models

$$\mathrm{Mod}_{\mathcal{F}}(\varphi) = \{\mathbb{A} \mid \mathbb{A} \text{ finite, and } \mathbb{A} \models \varphi\}$$

is trivially decidable (in LOGSPACE).

There are computationally easy classes that are not defined by any first-order sentence.

- The class of sets with an even number of elements.
- The class of graphs (V, E) that are connected.

Quantifier Rank

The *quantifier rank* of a formula φ, written $\mathrm{qr}(\varphi)$ is defined inductively as follows:

1. if φ is atomic then $\mathrm{qr}(\varphi) = 0$,
2. if $\varphi = \neg\psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi)$,
3. if $\varphi = \psi_1 \vee \psi_2$ or $\varphi = \psi_1 \wedge \psi_2$ then $\mathrm{qr}(\varphi) = \max(\mathrm{qr}(\psi_1), \mathrm{qr}(\psi_2))$.
4. if $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi) + 1$

In a finite relational vocabulary, it is easily proved that in a finite vocabulary, for each q, there are (up to logical equivalence) only finitely many sentences φ with $\mathrm{qr}(\varphi) \leq q$.

Finitary Elementary Equivalence

For two structures $\mathbb{A}$ and $\mathbb{B}$, we say $\mathbb{A} \equiv_p \mathbb{B}$ if for any sentence φ with $\mathrm{qr}(\varphi) \leq p$,

$$\mathbb{A} \models \varphi \text{ if, and only if, } \mathbb{B} \models \varphi.$$

Key fact:

> a class of structures S is definable by a first order sentence if, and only if, S is closed under the relation $\equiv_p$ for some p.

In a finite relational vocabulary, for any structure $\mathbb{A}$ there is a sentence $\theta^p_{\mathbb{A}}$ such that

$$\mathbb{B} \models \theta^p_{\mathbb{A}} \quad \text{if, and only if,} \quad \mathbb{A} \equiv_p \mathbb{B}$$

Ehrenfeucht-Fraïssé Game

The p-round Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ proceeds as follows:

- There are two players called Spoiler and Duplicator.
- At the ith round, Spoiler chooses one of the structures (say $\mathbb{B}$) and one of the elements of that structure (say b_i).
- Duplicator must respond with an element of the other structure (say a_i).
- If, after p rounds, the map $a_i \mapsto b_i$ is a partial isomorphism, then Duplicator has won the game, otherwise Spoiler has won.

Theorem (Fraïssé 1954; Ehrenfeucht 1961)

Duplicator has a strategy for winning the p-round Ehrenfeucht game on $\mathbb{A}$ and $\mathbb{B}$ if, and only if, $\mathbb{A} \equiv_p \mathbb{B}$.

Proof by Example

Suppose $\mathbb{A} \not\equiv_3 \mathbb{B}$, in particular, suppose $\theta(x, y, z)$ is quantifier free, such that:

$$\mathbb{A} \models \exists x \forall y \exists z \theta \quad \text{and} \quad \mathbb{B} \models \forall x \exists y \forall z \neg\theta$$

round 1: *Spoiler* chooses $a_1 \in A$ such that $\mathbb{A} \models \forall y \exists z \theta[a_1]$.
Duplicator responds with $b_1 \in B$.

round 2: *Spoiler* chooses $b_2 \in B$ such that $\mathbb{B} \models \forall z \neg\theta[b_1, b_2]$.
Duplicator responds with $a_2 \in A$.

round 3: *Spoiler* chooses $a_3 \in A$ such that $\mathbb{A} \models \theta[a_1, a_2, a_3]$.
Duplicator responds with $b_3 \in B$.

Spoiler wins, since $\mathbb{B} \not\models \theta[b_1, b_2, b_3]$.

Using Games

To show that a class of structures S is not definable in FO, we find, for every p, a pair of structures $\mathbb{A}_p$ and $\mathbb{B}_p$ such that

- $\mathbb{A}_p \in S$, $\mathbb{B}_p \in \overline{S}$; and
- *Duplicator* wins a p round game on $\mathbb{A}_p$ and $\mathbb{B}_p$.

Example:

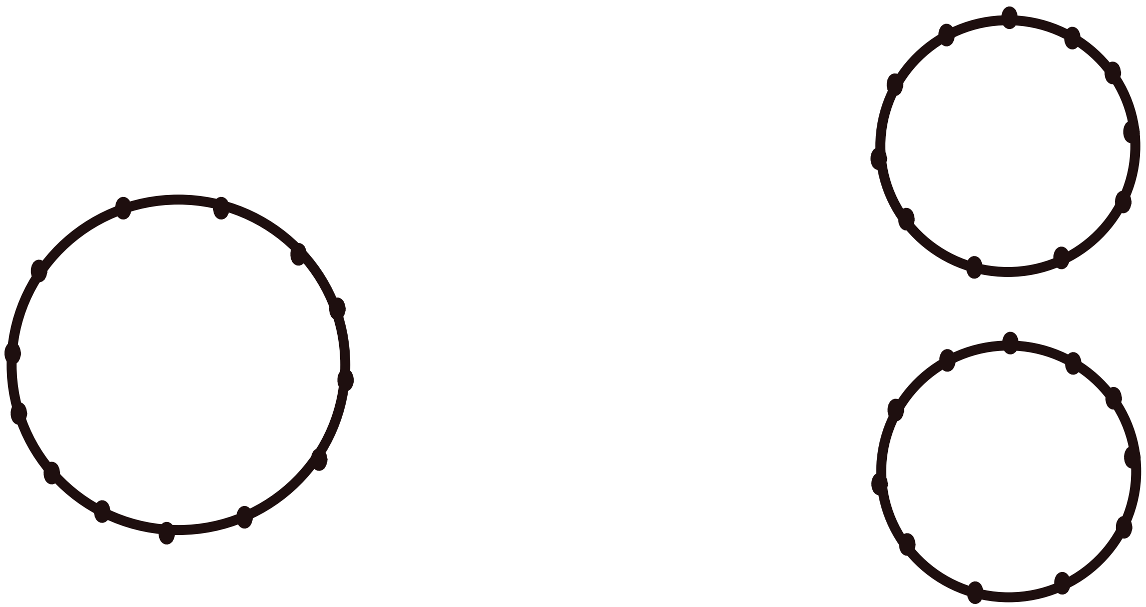
C_n—a cycle of length n.

Duplicator wins the p round game on $C_{2^p} \oplus C_{2^p}$ and C_{2^p+1}.

- 2-Colourability is not definable in FO.
- Even cardinality is not definable in FO.
- Connectivity is not definable in FO.

Using Games

An illustration of the game for undefinability of *connectivity* and *2-colourability*.



Duplicator's strategy is to ensure that after r moves, the distance between corresponding pairs of pebbles is either *equal* or $\geq 2^{p-r}$.

Stratifying Isomorphism

In order to study the expressive power of *first-order logic* on finite structures, we considered one stratification of isomorphism:

$$\mathbb{A} \equiv_q \mathbb{B}$$

if $\mathbb{A}$ and $\mathbb{B}$ cannot be distinguished by any sentence with *quantifier rank* at most q.

An alternative stratification that is useful in studying *fixed-point logics* is based on the number of variables.

$$\mathbb{A} \equiv^k \mathbb{B}$$

if $\mathbb{A}$ and $\mathbb{B}$ cannot be distinguished by any sentence with at most k distinct variables.

Inductive Definitions

Let $\varphi(R, x_1, \ldots, x_k)$ be a first-order formula in the vocabulary $\sigma \cup \{R\}$

Associate an operator Φ on a given structure $\mathbb{A}$:

$$\Phi(R^{\mathbb{A}}) = \{\mathbf{a} \mid (\mathbb{A}, R^{\mathbb{A}}, \mathbf{a}) \models \varphi(R, \mathbf{x})\}$$

We define the *increasing* sequence of relations on $\mathbb{A}$:

$$\Phi^0 = \emptyset$$

$$\Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

The *inflationary fixed point* of Φ is the limit of this sequence.

On a structure with n elements, the limit is reached after at most n^k stages.

IFP

The logic IFP is formed by closing first-order logic under the rule:

If φ is a formula of vocabulary $\sigma \cup \{R\}$ then $[\mathbf{ifp}_{R,\mathbf{x}}\varphi](\mathbf{t})$ is a formula of vocabulary σ.

The formula is read as:

the tuple $\mathbf{t}$ is in the inflationary fixed point of the operator defined by φ

LFP is the similar logic obtained using *least fixed points* of *monotone* operators defined by *positive* formulas.

LFP and IFP have the same expressive power **(Gurevich-Shelah; Kreutzer)**.

Transitive Closure

The formula

$$[\mathbf{ifp}_{T,xy}(x = y \vee \exists z(E(x,z) \wedge T(z,y)))](u,v)$$

defines the *reflexive and transitive closure* of the relation E

The expressive power of IFP properly extends that of first-order logic.

On structures which come equipped with a linear order IFP expresses exactly the properties that are in PTime.

(Immerman; Vardi)

Open Question: Is there a logic that expresses exactly the properties for *unordered* structures?

Finite Variable Logic

We write L^k for the first order formulas using only the variables $x_1, \ldots, x_k$.

$$\mathbb{A} \equiv^k \mathbb{B}$$

denotes that $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k.

For any k, $\quad \mathbb{A} \equiv^k \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_k \mathbb{B}$

However, for any q, there are $\mathbb{A}$ and $\mathbb{B}$ such that

$$\mathbb{A} \equiv_q \mathbb{B} \quad \text{and} \quad \mathbb{A} \not\equiv^2 \mathbb{B}.$$

Axiomatisability

Any class of finite structures closed under isomorphisms is *axiomatised* by a first-order theory.

A class of finite structures is closed under $\equiv_q$ (for some q) if, and only if, it is *finitely axiomatised*, i.e. defined by a single FO sentence.

A class of finite structures is closed under $\equiv^k$ if, and only if, it is axiomatisable in L^k (possibly by an infinite collection of sentences).

Every sentence of IFP is equivalent, *on finite structures*, to an L^k theory, for some k.

$$\varphi(R, x_1, \ldots, x_l) \in L^k$$

Each stage of the induction φ^m can be written as a formula in L^{k+l}.

Pebble Games

The k-pebble game is played on two structures $\mathbb{A}$ and $\mathbb{B}$, by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

Spoiler moves by picking a pebble and placing it on an element (a_i on an element of $\mathbb{A}$ or b_i on an element of $\mathbb{B}$).

Duplicator responds by picking the matching pebble and placing it on an element of the other structure

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism

If *Duplicator* has a winning strategy for q moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k of quantifier rank at most q. **(Barwise)**

Using Pebble Games

To show that a class of structures S is not definable in first-order logic:

$$\forall k\ \forall q\ \exists \mathbb{A}, \mathbb{B}\ (\mathbb{A} \in S \land \mathbb{B} \notin S \land \mathbb{A} \equiv_q^k \mathbb{B})$$

To show that S is not axiomatisable with a finite number of variables:

$$\forall k\ \exists \mathbb{A}, \mathbb{B}\ \forall q\ (\mathbb{A} \in S \land \mathbb{B} \notin S \land \mathbb{A} \equiv_q^k \mathbb{B})$$

Evenness

To show that *Evenness* is not definable in IFP, it suffices to show that:

> for every k, there are structures $\mathbb{A}_k$ and $\mathbb{B}_k$ such that $\mathbb{A}_k$ has an even number of elements, $\mathbb{B}_k$ has an odd number of elements and
>
> $$\mathbb{A} \equiv^k \mathbb{B}.$$

It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing k elements (and no other relations) and the other structure has $k + 1$ elements.

Matching

Take $K_{k,k}$—the complete bipartite graph on two sets of k vertices.

and $K_{k,k+1}$—the complete bipartite graph on two sets, one of k vertices, the other of $k+1$.

These two graphs are $\equiv^k$ equivalent, yet one has a perfect matching, and the other does not.

Stratifications of Isomorphism

In a *finite, relational* vocabulary, there are only finitely many sentences of quantifier rank at most q.

Thus, the relation $\equiv_q$ has only finitely many equivalence classes.

As approximations of isomorphism, these are very *coarse*.

The relation $\equiv^k$ has infintiely many classes for all $k \geq 2$.

Still, for any k, and *randomly chosen* graphs G_1 and G_2, we have $G_1 \equiv^k G_2$.

Indeed, there is a single $\equiv^k$-equivalence class that contains *almost all* graphs.

Fixed-point Logic with Counting

Immerman proposed IFPC—the extension of IFP with a mechanism for *counting*

Two sorts of variables:

- $x_1, x_2, \ldots$ range over $|A|$—the domain of the structure;
- $\nu_1, \nu_2, \ldots$ which range over *numbers* in the range $0, \ldots, |A|$

If $\varphi(x)$ is a formula with free variable x, then $\nu = \#x\varphi$ denotes that ν is the number of elements of A that satisfy the formula φ.

We also have the order $\nu_1 < \nu_2$, which allows us (using recursion) to define arithmetic operations.

Expressive Power of IFPC

There are an even number of elements satisfying $\varphi(x)$:

$$\exists \nu_1 \exists \nu_2 (\nu_1 = [\#x\varphi] \land (\nu_2 + \nu_2 = \nu_1))$$

Many *"obviously"* polynomial-time algorithms can be expressed in IFPC.

IFPC captures all of PTime over many interesting classes of structures, such as any *proper minor-closed class of graphs* **(Grohe 2010)**

Matching on graphs can be defined in IFPC.

- bipartite graphs **(Blass, Gurevich, Shelah 2005)**
- general graphs **(Anderson, D., Holm 2013)**

Counting Quantifiers

C^k is the logic obtained from *first-order logic* by allowing:

- allowing *counting quantifiers*: $\exists^i x\, \varphi$; and
- only the variables $x_1, \ldots . x_k$.

Every formula of C^k is equivalent to a formula of first-order logic, albeit one with more variables.

For every sentence φ of IFPC, there is a k such that if $\mathbb{A} \equiv^{C^k} \mathbb{B}$, then

$$\mathbb{A} \models \varphi \quad \text{if, and only if,} \quad \mathbb{B} \models \varphi.$$

Counting Game

Immerman and Lander (1990) defined a *pebble game* for C^k.

This is again played by *Spoiler* and *Duplicator* using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

At each move, *Spoiler* picks i and a subset of the universe (say $X \subseteq B$)

Duplicator responds with a subset of the other structure (say $Y \subseteq A$) of the same *size*.

Spoiler then places a_i on an element of Y and *Duplicator* must place b_i on an element of X.

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism

If *Duplicator* has a winning strategy for q moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of C^k of quantifier rank at most q.

Bijection Games

$\equiv^{C^k}$ is also characterised by a k-pebble *bijection game*. **(Hella 96)**.

The game is played on structures $\mathbb{A}$ and $\mathbb{B}$ with pebbles $a_1, \ldots, a_k$ on $\mathbb{A}$ and $b_1, \ldots, b_k$ on $\mathbb{B}$.

- *Spoiler* chooses a pair of pebbles a_i and b_i;
- *Duplicator* chooses a bijection $h : A \to B$ such that for pebbles a_j and $b_j (j \neq i)$, $h(a_j) = b_j$;
- *Spoiler* chooses $a \in A$ and places a_i on a and b_i on $h(a)$.

Duplicator loses if the partial map $a_i \mapsto b_i$ is not a partial isomorphism. *Duplicator* has a strategy to play forever if, and only if, $\mathbb{A} \equiv^{C^k} \mathbb{B}$.

Equivalence of Games

It is easy to see that a winning strategy for *Duplicator* in the bijection game yields a winning strategy in the counting game:

> Respond to a set $X \subseteq A$ (or $Y \subseteq B$) with $h(X)$ ($h^{-1}(Y)$, respectively).

For the other direction, consider the partition induced by the equivalence relation

$$\{(a, a') \mid (\mathbb{A}, \mathbf{a}[a/a_i]) \equiv^{C^k} (\mathbb{A}, \mathbf{a}[a'/a_i])\}$$

and for each of the parts X, take the response Y of *Duplicator* to a move where *Spoiler* would choose X.

Stitch these together to give the bijection h.

Counting Tuples of Elements

We could consider extending the counting logic with quantifiers that count *tuples* of elements.

This does not add further expressive power.

$$\exists^i \overline{xy}\ \varphi$$

is equivalent to

$$\bigvee_{f \in F} \bigwedge_{j \in \mathrm{dom}(f)} \exists^{f(j)} x\ \exists^j y\ \varphi$$

where F is the set of finite partial functions f on $\mathbb{N}$ such that $(\sum_{j \in \mathrm{dom}(f)} j f(j)) = i$.

Thus, there is no strengthening to the game if we allow *Spoiler* to move more than one pebble in a move (with *Duplicator* giving a bijection between sets of tuples.)

Cai-Fürer-Immerman Graphs

There are polynomial-time decidable properties of graphs that are not definable in IFPC. **(Cai, Fürer, Immerman, 1992)**

More precisely, we can construct a sequence of pairs of graphs $G_k, H_k (k \in \omega)$ such that:

- $G_k \equiv^{C^k} H_k$ for all k.
- There is a polynomial time decidable class of graphs that includes all G_k and excludes all H_k.

Still, IFPC is a *natural* level of expressiveness within PTime.

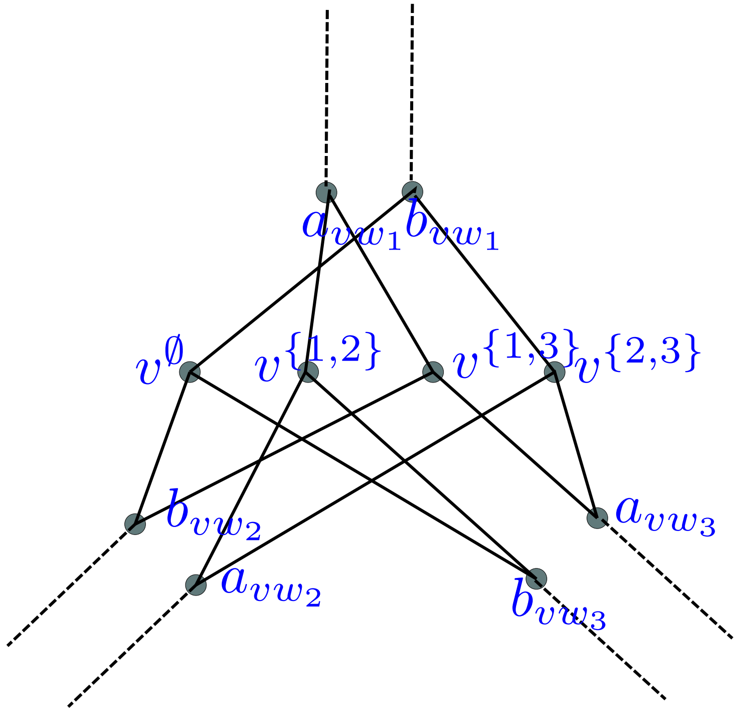
Constructing G_k and H_k

Given any graph G, we can define a graph X_G by replacing every edge with a pair of edges, and every vertex with a gadget.

The picture shows the gadget for a vertex v that is adjacent in G to vertices w_1, w_2 and w_3.

The vertex v^S is adjacent to $a_{vw_i} (i \in S)$ and $b_{vw_i} (i \notin S)$ and there is one vertex for all *even size* S.

The graph $\tilde{X}_G$ is like X_G except that at *one vertex* v, we include V^S for *odd size* S.



Properties

If G is *connected* and has *treewidth* at least k, then:

1. $X_G \not\cong \tilde{X}_G$; and
2. $X_G \equiv^{C^k} \tilde{X}_G$.

(1) allows us to construct a polynomial time property separating X_G and $\tilde{X}_G$.

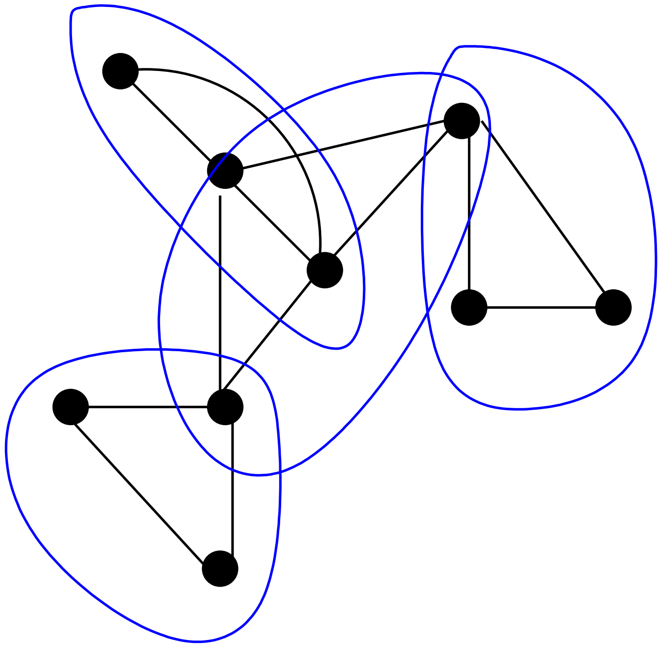
(2) is proved by a game argument.

The original proof of **(Cai, Fürer, Immerman)** relied on the existence of balanced separators in G. The characterisation in terms of treewidth is from **(D., Richerby 07)**.

TreeWidth

The *treewidth* of a graph is a measure of its interconnectedness.

A graph has treewidth k if it can be covered by subgraphs of at most $k+1$ nodes in a tree-like fashion.



TreeWidth

Formal Definition:

For a graph $G = (V, E)$, a *tree decomposition* of G is a relation $D \subset V \times T$ with a tree T such that:

- for each $v \in V$, the set $\{t \mid (v, t) \in D\}$ forms a connected subtree of T; and
- for each edge $(u, v) \in E$, there is a $t \in T$ such that $(u, t), (v, t) \in D$.

The *treewidth* of G is the least k such that there is a tree T and a tree-decomposition $D \subset V \times T$ such that for each $t \in T$,

$$|\{v \in V \mid (v, t) \in D\}| \leq k + 1.$$

Cops and Robbers

A game played on an undirected graph $G = (V, E)$ between a player controlling k *cops* and another player in charge of a *robber*.

At any point, the cops are sitting on a set $X \subseteq V$ of the nodes and the robber on a node $r \in V$.

A move consists in the cop player removing some cops from $X' \subseteq X$ nodes and announcing a new position Y for them. The robber responds by moving along a path from r to some node s such that the path does not go through $X \setminus X'$.

The new position is $(X \setminus X') \cup Y$ and s. If a cop and the robber are on the same node, the robber is caught and the game ends.

Strategies and Decompositions

Theorem (Seymour and Thomas 93):

There is a winning strategy for the *cop player* with k cops on a graph G if, and only if, the tree-width of G is at most $k - 1$.

It is not difficult to construct, from a tree decomposition of width k, a winning strategy for $k + 1$ cops.

Somewhat more involved to show that a winning strategy yields a decomposition.

Cops, Robbers and Bijections

If G has treewidth k or more, than the *robber* has a winning strategy in the k-*cops and robbers* game played on G.

We use this to construct a winning strategy for Duplicator in the k-pebble bijection game on X_G and $\tilde{X}_G$.

- A bijection $h : X_G \to \tilde{X}_G$ is *good bar* v if it is an isomorphism everywhere except at the vertices v^S.
- If h is good bar v and there is a path from v to u, then there is a bijection h' that is good bar u such that h and h' differ only at vertices corresponding to the path from v to u.
- Duplicator plays bijections that are good bar v, where v is the robber position in G when the cop position is given by the currently pebbled elements.

Undefinability Results for IFPC

Other undefinability results for IFPC have been obtained:

- Isomorphism on *multipedes*—a class of structures defined by **(Gurevich-Shelah 96)** to exhibit a *first-order definable* class of *rigid* structures with no order definable in IFPC.
- 3-colourability of graphs. **(D. 1998)**

Both proofs rely on a construction very similar to that of Cai-Fürer-Immerman.

Question: Is there a natural polynomial-time computable property that is not definable in IFPC?

Solvability of Linear Equations

It has been shown that the problem of solving linear equations over the two element field $\mathbb{Z}_2$ is not definable in IFPC. **(Atserias, Bulatov, D. 09)**

The question arose in the context of classification of *Constraint Satisfaction Problems*.

The problem is clearly solvable in polynomial time by means of Gaussian elimination.

> We see how to represent systems of linear equations as *unordered* relational structures.

Systems of Linear Equations

Consider structures over the domain $\{x_1, \ldots, x_n, e_1, \ldots, e_m\}$, (where $e_1, \ldots, e_m$ are the equations) with relations:

- unary E_0 for those equations e whose r.h.s. is 0.
- unary E_1 for those equations e whose r.h.s. is 1.
- binary M with $M(x, e)$ if x occurs on the l.h.s. of e.

Solv$(\mathbb{Z}_2)$ is the class of structures representing solvable systems.

Undefinability in IFPC

Take $\mathcal{G}$ a 3-regular, connected graph with treewidth $> k$.

Define equations $\mathbf{E}_{\mathcal{G}}$ with two variables x^e_0, x^e_1 for each edge e.

For each vertex v with edges e_1, e_2, e_3 incident on it, we have eight equations:

$$E_v : \qquad x_a^{e_1} + x_b^{e_2} + x_c^{e_3} \equiv a + b + c \pmod 2$$

$\tilde{\mathbf{E}}_{\mathcal{G}}$ is obtained from $\mathbf{E}_{\mathcal{G}}$ by replacing, for exactly one vertex v, E_v by:

$$E'_v : \qquad x_a^{e_1} + x_b^{e_2} + x_c^{e_3} \equiv a + b + c + 1 \pmod 2$$

We can show: $\mathbf{E}_{\mathcal{G}}$ is satisfiable; $\tilde{\mathbf{E}}_{\mathcal{G}}$ is unsatisfiable; $\mathbf{E}_{\mathcal{G}} \equiv^{C^k} \tilde{\mathbf{E}}_{\mathcal{G}}$

Satisfiability

Lemma $\mathbf{E}_G$ is satisfiable.

by setting the variables x_i^e to i.

Lemma $\tilde{\mathbf{E}}_G$ is unsatisfiable.

Consider the subsystem consisting of equations involving only the variables x_0^e.

The sum of all *left-hand sides* is

$$2 \sum_e x_0^e \equiv 0 \pmod 2$$

However, the sum of *right-hand sides* is 1.

Cops, Robbers and Bijections

If G has treewidth k or more, than the *robber* has a winning strategy in the k-*cops and robbers* game played on G.

We use this to construct a winning strategy for Duplicator in the k-pebble bijection game on $\mathbf{E}_{\mathcal{G}}$ and $\tilde{\mathbf{E}}_{\mathcal{G}}$.

- A bijection $h : \mathbf{E}_{\mathcal{G}} \to \tilde{\mathbf{E}}_{\mathcal{G}}$ is *good bar v* if it is an isomorphism everywhere except at the variables $x^e a$ for edges e incident on v.
- If h is good bar v and there is a path from v to u, then there is a bijection h' that is good bar u such that h and h' differ only at vertices corresponding to the path from v to u.
- Duplicator plays bijections that are good bar v, where v is the robber position in G when the cop position is given by the currently pebbled elements.