Descriptive Complexity: Part 1

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Outline

Lecture 1: Introduction to Descriptive Complexity.

- Proving inexpressibility in Logics.
- Characterizing complexity classes.
- FPC and the Cai-Fürer-Immerman construction.

Lecture 2: FPC and its connections with:

- circuit complexity
- extension polytopes
- hardness of approximation

Descriptive Complexity

Descriptive Complexity provides an alternative perspective on Computational Complexity.

Computational Complexity

- Measure use of resources (space, time, *etc.*) on a machine model of computation;
- Complexity of a language—i.e. a set of strings.

Descriptive Complexity

- Complexity of a class of structures—e.g. a collection of graphs.
- Measure the complexity of describing the collection in a formal logic, using resources such as variables, quantifiers, higher-order operators, *etc.*

There is a fascinating interplay between the views.

First-Order Logic

Consider *first-order predicate logic*.

> *Fix a vocabulary* σ *of relation symbols* $(R_1, \ldots, R_m)$ *and a collection* X *of variables.*

The formulas are given by

$$R_i(\mathbf{x}) \mid x = y \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \exists x\varphi \mid \forall x\varphi$$

First-Order Logic

For a first-order sentence φ, we ask what is the *computational complexity* of the problem:

Given: a structure $\mathbb{A}$
Decide: if $\mathbb{A} \models \varphi$

In other words, how complex can the collection of finite models of φ be?

In order to talk of the complexity of a class of finite structures, we need to fix some way of representing finite structures as strings.

Encoding Structures

We use an alphabet $\Sigma = \{0, 1, \#\}$.
For a structure $\mathbb{A} = (A, R_1, \ldots, R_m)$, fix a linear order $<$ on $A = \{a_1, \ldots, a_n\}$.

R_i (of arity k) is encoded by a string $[R_i]_<$ of 0s and 1s of length n^k.

$$[\mathbb{A}]_< = \underbrace{1 \cdots 1}_{n} \# [R_1]_< \# \cdots \# [R_m]_<$$

The exact string obtained depends on the choice of order.

Invariance

Note that the decision problem:

Given a string $[\mathbb{A}]_<$ *decide whether* $\mathbb{A} \models \varphi$

has a natural invariance property.

It is invariant under the following equivalence relation

Write $w_1 \sim w_2$ *to denote that there is some structure* $\mathbb{A}$ *and orders* $<_1$ *and* $<_2$ *on its universe such that*

$$w_1 = [\mathbb{A}]_{<_1} \text{ and } w_2 = [\mathbb{A}]_{<_2}$$

Note: deciding the equivalence relation $\sim$ is just the same as deciding structure isomorphism.

Naïve Algorithm

Back to evaluating $\mathbb{A} \models \varphi$.
The straightforward algorithm proceeds recursively on the structure of φ:

- Atomic formulas by direct lookup.
- Boolean connectives are easy.
- If $\varphi \equiv \exists x\, \psi$ then for each $a \in \mathbb{A}$ check whether

$$(\mathbb{A}, x \mapsto a) \models \psi.$$

This runs in time $O(ln^p)$ and $O(p \log n)$ space, where l is the length of φ and p is the nesting depth of quantifiers in φ.

$$\mathrm{Mod}(\varphi) = \{\mathbb{A} \mid \mathbb{A} \models \varphi\}$$

is in *logarithmic space* and *polynomial time*.

Limitations of First-Order Logic

There are computationally easy properties that are not definable in first-order logic.

- There is no sentence φ of first-order logic such that $\mathbb{A} \models \varphi$ if, and only if, $|A|$ is even.
- There is no formula $\varphi(E, x, y)$ that defines the transitive closure of a binary relation E.
- There is no sentence φ of first-order logic such that for any graph G, $G \models \varphi$ if, and only if, G is 2-colourable.

Quantifier Rank

The *quantifier rank* of a formula φ, written $\mathrm{qr}(\varphi)$ is defined inductively as follows:

1. if φ is atomic then $\mathrm{qr}(\varphi) = 0$,
2. if $\varphi = \neg\psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi)$,
3. if $\varphi = \psi_1 \vee \psi_2$ or $\varphi = \psi_1 \wedge \psi_2$ then $\mathrm{qr}(\varphi) = \max(\mathrm{qr}(\psi_1), \mathrm{qr}(\psi_2))$.
4. if $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi) + 1$

It is easily proved that in a finite vocabulary, for each p, there are (up to logical equivalence) only finitely many sentences φ with $\mathrm{qr}(\varphi) \leq p$.

Finitary Elementary Equivalence

For two structures $\mathbb{A}$ and $\mathbb{B}$, we say $\mathbb{A} \equiv_p \mathbb{B}$ if for any sentence φ with $\mathrm{qr}(\varphi) \leq p$,

$$\mathbb{A} \models \varphi \text{ if, and only if, } \mathbb{B} \models \varphi.$$

Key fact:

> *a class of structures S is definable by a first order sentence if, and only if, S is closed under the relation $\equiv_p$ for some p.*

In a *finite* relational vocabulary, for any structure $\mathbb{A}$ there is a sentence $\theta^p_{\mathbb{A}}$ such that

$$\mathbb{B} \models \theta^p_{\mathbb{A}} \quad \text{if, and only if,} \quad \mathbb{A} \equiv_p \mathbb{B}$$

Partial Isomorphisms

The equivalence relations $\equiv_p$ can be characterised in terms of sequences of partial isomorphisms

(Fraïssé 1954)

or two player games.

(Ehrenfeucht 1961)

A *partial isomorphism* is an injective partial function f from $\mathbb{A}$ to $\mathbb{B}$ such that:

- for any constant c: $f(c^{\mathbb{A}}) = c^{\mathbb{B}}$; and
- for any tuple $\mathbf{a}$ of elements of $\mathbb{A}$ such that all elements of $\mathbf{a}$ are in $\mathrm{dom}(f)$ and any relation R we have

$$R^{\mathbb{A}}(\mathbf{a}) \quad \Leftrightarrow \quad R^{\mathbb{B}}(f(\mathbf{a}))$$

Ehrenfeucht-Fraïssé Game

The p-round Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ proceeds as follows:

- There are two players called *Spoiler* and *Duplicator*
- At the ith round, *Spoiler* chooses one of the structures (say $\mathbb{B}$) and one of the elements of that structure (say b_i).
- *Duplicator* must respond with an element of the other structure (say a_i).
- If, after p rounds, the map $a_i \mapsto b_i$ is a partial isomorphism, then *Duplicator* has won the game, otherwise *Spoiler* has won.

Theorem (Fraïssé 1954; Ehrenfeucht 1961)
Duplicator has a strategy for winning the p-round Ehrenfeucht game on $\mathbb{A}$ and $\mathbb{B}$ if, and only if, $\mathbb{A} \equiv_p \mathbb{B}$.

Proof by Example

Suppose $\mathbb{A} \not\equiv_3 \mathbb{B}$, in particular, suppose $\theta(x, y, z)$ is quantifier free, such that:

$$\mathbb{A} \models \exists x \forall y \exists z \theta \quad \text{and} \quad \mathbb{B} \models \forall x \exists y \forall z \neg\theta$$

round 1: Spoiler chooses $a_1 \in A$ *such that* $\mathbb{A} \models \forall y \exists z \theta[a_1]$.
Duplicator responds with $b_1 \in B$.

round 2: Spoiler chooses $b_2 \in B$ *such that* $\mathbb{B} \models \forall z \neg\theta[b_1, b_2]$.
Duplicator responds with $a_2 \in A$.

round 3: Spoiler chooses $a_3 \in A$ *such that* $\mathbb{A} \models \theta[a_1, a_2, a_3]$.
Duplicator responds with $b_3 \in B$.

Spoiler wins, since $\mathbb{B} \not\models \theta[b_1, b_2, b_3]$.

Using Games

To show that a class of structures S is not definable in FO, we find, for every p, a pair of structures $\mathbb{A}_p$ and $\mathbb{B}_p$ such that

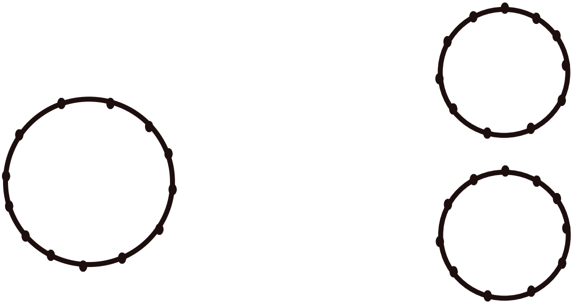
- $\mathbb{A}_p \in S$, $\mathbb{B}_p \in \overline{S}$; and
- *Duplicator* wins a p-round game on $\mathbb{A}_p$ and $\mathbb{B}_p$.

Example:
C_n—a cycle of length n.
Duplicator wins the p-round game on $C_{2^p} \oplus C_{2^p}$ and C_{2^p+1}.

- 2-Colourability is not definable in FO.
- Even cardinality is not definable in FO.
- Connectivity is not definable in FO.

Using Games

An illustration of the game for undefinability of *connectivity* and *2-colourability*.



Duplicator's strategy is to ensure that after r moves, the distance between corresponding pairs of pebbles is either *equal* or $\geq 2^{p-r}$.

Second-Order Logic

Second-Order Logic extends first-order logic with quantification over *relations*.

$$\exists X \, \varphi$$

where X has arity m is true in a structure $\mathbb{A}$ if, and only if, $\mathbb{A}$ can be expanded by an m-ary relation interpreting X to satisfy φ.

ESO or Σ^1_1—*existential second-order logic* consists of those formulas of second-order logic of the form:

$$\exists X_1 \cdots \exists X_k \, \varphi$$

where φ is a first-order formula.

Examples

Evennness

This formula is true in a structure if, and only if, the size of the domain is even.

$$\exists B \exists S \quad \forall x \exists y B(x, y) \wedge \forall x \forall y \forall z B(x, y) \wedge B(x, z) \rightarrow y = z$$
$$\forall x \forall y \forall z B(x, z) \wedge B(y, z) \rightarrow x = y$$
$$\forall x \forall y S(x) \wedge B(x, y) \rightarrow \neg S(y)$$
$$\forall x \forall y \neg S(x) \wedge B(x, y) \rightarrow S(y)$$

Examples

Transitive Closure

Each of the following formulas is true of a pair of elements a, b in a structure if, and only if, there is an E-path from a to b.

$$\forall S\big(S(a) \land \forall x \forall y[S(x) \land E(x,y) \to S(y)] \to S(b)\big)$$

$$\begin{array}{ll} \exists P & \forall x \forall y\, P(x,y) \to E(x,y) \\ & \exists x P(a,x) \land \exists x P(x,b) \land \neg\exists x P(x,a) \land \neg\exists x P(b,x) \\ & \forall x \forall y (P(x,y) \to \forall z (P(x,z) \to y = z)) \\ & \forall x \forall y (P(x,y) \to \forall z (P(z,y) \to x = z)) \\ & \forall x ((x \neq a \land \exists y P(x,y)) \to \exists z P(z,x)) \\ & \forall x ((x \neq b \land \exists y P(y,x)) \to \exists z P(x,z)) \end{array}$$

Examples

3-Colourability

The following formula is true in a graph (V, E) if, and only if, it is 3-colourable.

$$\begin{aligned}\exists R \exists B \exists G \quad & \forall x (Rx \vee Bx \vee Gx) \wedge \\ & \forall x (\quad \neg(Rx \wedge Bx) \wedge \neg(Bx \wedge Gx) \wedge \neg(Rx \wedge Gx)) \wedge \\ & \forall x \forall y (Exy \rightarrow (\quad \neg(Rx \wedge Ry) \wedge \\ & \qquad\qquad\qquad\qquad \neg(Bx \wedge By) \wedge \\ & \qquad\qquad\qquad\qquad \neg(Gx \wedge Gy)))\end{aligned}$$

Fagin's Theorem

Theorem (Fagin)
A class $\mathcal{C}$ of finite structures is definable by a sentence of *existential second-order logic* if, and only if, it is decidable by a *nondeterminisitic machine* running in polynomial time.

$$\mathsf{ESO} = \mathsf{NP}$$

$S = \mathrm{Mod}(\varphi)$ for some φ in ESO *if, and only if,* $\{[\mathbb{A}]_{\leq} \mid \mathbb{A} \in S\}$ is in NP

Fagin's Theorem

If φ is $\exists R_1 \cdots \exists R_m \theta$ for a *first-order* θ.

To decide $\mathbb{A} \models \varphi$, *guess* an interpretation for the relations $R_1, \ldots, R_m$ and then evaluate θ in the expanded structure.

Given a *nondeterministic* machine M and a polynomial p:

> $\exists \leq$ *a linear order*
> $\exists H, T, S$ *that code an accepting computation of* M *of length* p *starting with* $[\mathbb{A}]_{\leq}$.

Is there a logic for P?

The major open question in *Descriptive Complexity* (first asked by Chandra and Harel in 1982) is whether there is a logic $\mathcal{L}$ such that

> *for any class of finite structures $\mathcal{C}$, $\mathcal{C}$ is definable by a sentence of $\mathcal{L}$ if, and only if, $\mathcal{C}$ is decidable by a deterministic machine running in polynomial time.*

Formally, we require $\mathcal{L}$ to be a *recursively enumerable* set of sentences, with a computable map taking each sentence to a Turing machine M and a polynomial time bound p such that (M, p) accepts a *class of structures*.

(Gurevich 1988)

Inductive Definitions

Let $\varphi(R, x_1, \ldots, x_k)$ be a first-order formula in the vocabulary $\sigma \cup \{R\}$
Associate an operator Φ on a given σ-structure $\mathbb{A}$:

$$\Phi(R^{\mathbb{A}}) = \{\mathbf{a} \mid (\mathbb{A}, R^{\mathbb{A}}, \mathbf{a}) \models \varphi(R, \mathbf{x})\}$$

We define the *non-decreasing* sequence of relations on $\mathbb{A}$:

$$\Phi^0 = \emptyset$$

$$\Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

The *inflationary fixed point* of Φ is the limit of this sequence.
On a structure with n elements, the limit is reached after at most n^k stages.

FP

The logic FP is formed by closing first-order logic under the rule:

If φ *is a formula of vocabulary* $\sigma \cup \{R\}$ *then* $[\mathbf{ifp}_{R,\mathbf{x}}\varphi](\mathbf{t})$ *is a formula of vocabulary* σ.

The formula is read as:

the tuple $\mathbf{t}$ *is in the inflationary fixed point of the operator defined by* φ

LFP is the similar logic obtained using *least fixed points* of *monotone* operators defined by *positive* formulas.
LFP and FP have the same expressive power **(Gurevich-Shelah 1986; Kreutzer 2004)**.

Transitive Closure

The formula

$$[\textbf{ifp}_{T,xy}(x = y \vee \exists z(E(x,z) \wedge T(z,y)))](u,v)$$

defines the *transitive closure* of the relation E

The expressive power of FP properly extends that of first-order logic.

Still, every property definable in FP is decidable in *polynomial time*.

> *On a structure with n elements, the fixed-point of an induction of arity k is reached in at most n^k steps.*

Immerman-Vardi Theorem

Theorem

On structures which come equipped with a linear order FP expresses exactly the properties that are in P.

(Immerman; Vardi 1982)

Recall from *Fagin's theorem*:

$\exists \leq$ *a linear order*

$\exists H, T, S$ *that code an accepting computation of* M *of length* p *starting with* $[\mathbb{A}]_{\leq}$.

FP vs. Ptime

The order cannot be built up inductively.
It is an open question whether a *canonical* string representation of a structure can be constructed in polynomial-time.

> *If it can, there is a logic for P (and also graph isomorphism is in P).*
> *If not, then $P \neq NP$.*

All P classes of structures can be expressed by a sentence of FP with $<$, which is invariant under the choice of order. The set of all such sentences is not *r.e.*

FP by itself is too weak to express all properties in P.
Evenness is not definable in FP.

Finite Variable Logic

We write L^k for the first order formulas using only the variables $x_1, \ldots, x_k$.

A first order formula φ is equivalent to one of L^k if no sub-formula of φ contains more than k *free* variables.

$$\mathbb{A} \equiv^k \mathbb{B}$$

denotes that $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k.

For any k, $\quad \mathbb{A} \equiv^k \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_k \mathbb{B}$

However, for any p, there are $\mathbb{A}$ and $\mathbb{B}$ such that

$$\mathbb{A} \equiv_p \mathbb{B} \quad \text{and} \quad \mathbb{A} \not\equiv^2 \mathbb{B}.$$

Definability and Invariance

A class of structures is closed under $\equiv_p$ (for some p) if, and only if, it is *defined* by a FO sentence.

A class of finite structures is closed under $\equiv^k$ if, and only if, it is axiomatizable in L^k (possibly by an infinite collection of sentences).

In a *finite, relational* vocabulary, there are only finitely many sentences of quantifier rank at most p.
Thus, the relation $\equiv_p$ has only finitely many equivalence classes.

The relation $\equiv^k$ has infintiely many classes for all $k \geq 2$.

Finite Variable Logic

If $\varphi(R, \mathbf{x})$ has k variables all together, then each of the relations in the sequence:

$$\Phi^0 = \emptyset; \; \Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

is definable in L^{2k}.

Proof by induction, using *substitution* and *renaming* of bound variables.

On structures of a fixed size n, $[\mathbf{ifp}_{R,\mathbf{x}}\varphi](\mathbf{t})$ is equivalent to a formula of L^{2k}.

For any sentence φ of FP there is a k such that the property defined by φ is invariant under $\equiv^k$.

Pebble Games

The k-pebble game is played on two structures $\mathbb{A}$ and $\mathbb{B}$, by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

> *Spoiler moves by picking a pebble and placing it on an element (a_i on an element of $\mathbb{A}$ or b_i on an element of $\mathbb{B}$).*
>
> *Duplicator responds by picking the matching pebble and placing it on an element of the other structure*
>
> *Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism*
>
> *If Duplicator has a winning strategy for p moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k of quantifier rank at most p.* **(Barwise)**

Using Pebble Games

To show that a class of structures S is not definable in first-order logic:
$\forall k \, \forall p \, \exists \mathbb{A}, \mathbb{B} \, (\mathbb{A} \in S \wedge \mathbb{B} \notin S \wedge \mathbb{A} \equiv^k_p \mathbb{B})$

To show that S is not axiomatisable with a finite number of variables:
$\forall k \, \exists \mathbb{A}, \mathbb{B} \, \forall p \, (\mathbb{A} \in S \wedge \mathbb{B} \notin S \wedge \mathbb{A} \equiv^k_p \mathbb{B})$

Evenness

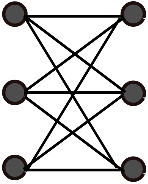
Evenness is not axiomatizable with a finite number of variables.

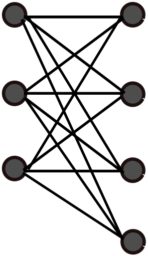
> *for every k, there are structures $\mathbb{A}_k$ and $\mathbb{B}_k$ such that $\mathbb{A}_k$ has an even number of elements, $\mathbb{B}_k$ has an odd number of elements and*
>
> $$\mathbb{A} \equiv^k \mathbb{B}.$$

It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing k elements (and no other relations) and the other structure has $k+1$ elements.

Matching

Take $K_{k,k}$—the complete bipartite graph on two sets of k vertices.
and $K_{k,k+1}$—the complete bipartite graph on two sets, one of k vertices, the other of $k+1$.





These two graphs are $\equiv^k$ equivalent, yet one has a perfect matching, and the other does not. One contains a Hamiltonian cycle, the other does not.

Inexpressibility in FP

The following are not definable in FP:

- *Evenness*;
- *Perfect Matching*;
- *Hamiltonicity*.

The examples showing these inexpressibility results all involve some form of *counting*.

Fixed-point Logic with Counting

Immerman proposed FPC—the extension of FP with a mechanism for *counting*

Two sorts of variables:

- $x_1, x_2, \ldots$ range over $|A|$—the domain of the structure;
- $\nu_1, \nu_2, \ldots$ which range over *non-negative integers*.

If $\varphi(x)$ is a formula with free variable x, then $\#x\varphi$ is a *term* denoting the *number* of elements of $\mathbb{A}$ that satisfy φ.

We have arithmetic operations $(+, \times)$ on *number terms*.

Quantification over number variables is *bounded*: $(\exists x < t)\, \varphi$

Counting Quantifiers

C^k is the logic obtained from *first-order logic* by allowing:

- *counting quantifiers*: $\exists^i x \, \varphi$; and
- only the variables $x_1, \ldots . x_k$.

Every formula of C^k is equivalent to a formula of first-order logic, albeit one with more variables.

For every sentence φ of FPC, there is a k such that if $\mathbb{A} \equiv^{C^k} \mathbb{B}$, then

$$\mathbb{A} \models \varphi \quad \text{if, and only if,} \quad \mathbb{B} \models \varphi.$$

Limits of FPC

It was proved **(Cai, Fürer, Immerman 1992)** that there are polynomial-time graph properties that are *not* expressible in FPC.

A number of other results about the limitations of FPC followed.

In particular, it has been shown that the problem of solving linear equations over the two element field $\mathbb{Z}_2$ is not definable in FPC.

(Atserias, Bulatov, D. 09)

The problem is clearly solvable in polynomial time by means of Gaussian elimination.

Systems of Linear Equations

We see how to represent systems of linear equations as unordered relational structures.

Consider structures over the domain $\{x_1, \ldots, x_n, e_1, \ldots, e_m\}$, (where $e_1, \ldots, e_m$ are the equations) with relations:

- unary E_0 for those equations e whose r.h.s. is 0.
- unary E_1 for those equations e whose r.h.s. is 1.
- binary M with $M(x, e)$ if x occurs on the l.h.s. of e.

Solv($\mathbb{Z}_2$) is the class of structures representing solvable systems.

Undefinability in FPC

To show that the *satisfiability* of systems of equations is not definable in FPC it suffices to show that for each k, we can construct a two systems of equations

$$E_k \text{ and } F_k$$

such that:

- E_k is satisfiable;
- F_k is unsatisfiable; and
- $E_k \equiv^{C^k} F_k$

Constructing systems of equations

Take G a 4-regular, connected graph.
Define equations $\mathbf{E}_G$ with two variables x_0^e, x_1^e for each edge e.
For each vertex v with edges e_1, e_2, e_3, e_4 incident on it, we have 16 equations:

$$E_v: \qquad x_a^{e_1} + x_b^{e_2} + x_c^{e_3} + x_d^{e_4} \equiv a + b + c + d \pmod{2}$$

$\tilde{\mathbf{E}}_G$ is obtained from $\mathbf{E}_G$ by replacing, for exactly one vertex v, E_v by:

$$E'_v: \qquad x_a^{e_1} + x_b^{e_2} + x_c^{e_3} + x_d^{e_4} \equiv a + b + c + d + 1 \pmod{2}$$

We can show: $\mathbf{E}_G$ is satisfiable; $\tilde{\mathbf{E}}_G$ is unsatisfiable.

Satisfiability

Lemma $\mathbf{E}_G$ is satisfiable.

by setting the variables x_i^e to i.

Lemma $\tilde{\mathbf{E}}_G$ is unsatisfiable.

Consider the subsystem consisting of equations involving only the variables x_0^e.
The sum of all left-hand sides is

$$2\sum_e x_0^e \equiv 0 \pmod 2$$

However, the sum of right-hand sides is 1.

Now we show that, for each k, we can find a graph G such that $\mathbf{E}_G \equiv^{C^k} \tilde{\mathbf{E}}_G$.

Counting Game

Immerman and Lander (1990) defined a *pebble game* for C^k.
This is again played by *Spoiler* and *Duplicator* using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$ on a pair of structures $\mathbb{A}$ and $\mathbb{B}$

At each move, Spoiler picks i and a set of elements of one structure (say $X \subseteq B$)

Duplicator responds with a set of vertices of the other structure (say $Y \subseteq A$) of the same size.

Spoiler then places a_i on an element of Y and Duplicator must place b_i on an element of X.

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism

If Duplicator has a winning strategy for p moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of C^k of quantifier rank at most p.