Parameterized Complexity Classes Under Logical Reductions

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Abstract. The parameterized complexity classes of the W-hierarchy are usually defined as the problems reducible to certain natural complete problems by means of fixed-parameter tractable (*fpt*) reductions. We investigate whether the classes can be characterised by means of weaker, logical reductions. We show that each class $W[t]$ has complete problems under slicewise bounded-variable first-order reductions. These are a natural weakening of slicewise bounded-variable LFP reductions which, by a result of Flum and Grohe, are known to be equivalent to *fpt*-reductions. If we relax the restriction on having a bounded number of variables, we obtain reductions that are too strong and, on the other hand, if we consider slicewise quantifier-free first-order reductions, they are considerably weaker. These last two results are established by considering the characterisation of $W[t]$ as the closure of a class of Fagin-definability problems under *fpt*-reductions. We show that replacing these by slicewise first-order reductions yields a hierarchy that collapses, while allowing only quantifier-free first-order reductions yields a hierarchy that is provably strict.

1 Introduction

In the theory of parameterized complexity, the W-hierarchy plays a role similar to NP in classical complexity theory in that many natural parameterized problems are shown intractable by being complete for some level $W[t]$ of the hierarchy. However, one difference between the two, perhaps no more than a historical accident, is that NP was originally defined in terms of resource bounds on a machine model, and the discovery that it has complete problems under polynomial-time reductions (and indeed that many natural combinatorial problems are NP-complete) came as a major advance, which also shows the robustness of the class. On the other hand, the classes $W[t]$ were originally defined as the sets of problems reducible to certain natural complete problems by means of fixed-parameter tractable (*fpt*) reductions [5]. These classes therefore have complete problems by construction. It was only later that a characterisation of these classes in terms of resource-bounded machines was obtained [1]. The robustness of the definition of NP is also demonstrated by the fact that many NP-complete problems are still complete under reductions much weaker than polynomial-time reductions. For instance,

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SAT is NP-complete, even under quantifier-free first-order projections, which are reductions even weaker than AC_0 reductions. Thus, the class NP can be characterized as the class of problems reducible to SAT under polynomial-time reductions, or equivalently as the class of problems reducible to SAT under quantifier-free first-order projections. The work we report in this paper is motivated by the question of whether similar robustness results can be shown for the classes $W[t]$. We investigate whether the classes can be characterised by means of weaker reductions, just like NP can.

We concentrate on reductions defined in terms of logical formulas. By a result of Flum and Grohe [7], it is known that *fpt*-reductions can be equivalently characterised, on ordered structures, as slicewise bounded-variable LFP reductions. We consider reductions defined in terms of first-order interpretations and introduce a number of parameterized versions of these. Our main result is that each class $W[t]$ has complete problems under *slicewise bounded-variable first-order* reductions, which are a natural first-order counterpart to slicewise bounded-variable LFP reductions. If we relax the restriction on having a bounded number of variables, we obtain *slicewise first-order* reductions, which are not necessarily *fpt*. Indeed we are able to show that all Fagin definability problems in $W[t]$ are reducible to problems in FPT under such reductions. On the other hand, we show that *slicewise quantifier-free first-order* reductions are considerably weaker in that there are Fagin-definability problems in $W[t+1]$ that cannot reduce to such problems in $W[t]$ under these reductions. This last class of reductions can be seen as the natural parametrization of quantifier-free first-order reductions, for which NP does have complete problems. Thus, our result shows that the definition of $W[t]$ is not quite as robust as that of NP.

We present the necessary background and preliminaries in Section 2. The various kinds of logical reductions are defined in Section 3. Section 4 shows that $W[t]$ contains complete problems under *slicewise bounded-variable first-order* reductions. Section 5 considers the case of the two other kinds of reductions we use. For space reasons, we only give sketches of proofs, omitting details which are often long and tedious coding of reductions as first-order formulas.

2 Preliminaries

We rely on standard definitions and notation from finite model theory (see [6,12]) and the theory of parameterized complexity [9]. We briefly recall some of the definitions we need, but we assume the reader is familiar with this literature.

A relational signature σ consists of a finite collection of relation and constant symbols. A decision problem over σ-structures is an isomorphism-closed class of finite σ-structures. In general, we assume that our structures are ordered. That is to say, that σ contains a distinguished binary relation symbol $\leq$ which is interpreted in every structure as a linear order of the universe. We are often interested in decision problems where the input is naturally described as a structure with additional integer parameters. For instance, the Clique problem requires, given a graph G and an integer k, to decide whether G contains a clique on k vertices. In all such cases that we will be interested in, the value of the integer parameter is bounded by the size of the structure, so it is safe to assume that it is given as an additional constant c in the signature σ, and the position in the linear order $\leq$ of $c^{\mathbb{A}}$ codes the value. However, where it is notationally convenient, we may still write the inputs as pairs $(\mathbb{A}, k)$, where $\mathbb{A}$ is a structure and k is an integer, with the understanding that they are to be understood as such coded structures.

A *parameterized problem* is a pair (Q, κ) where Q is a decision problem over σ-structures and κ a function that maps σ-structures to natural numbers. We say that (Q, κ) is *fixed-parameter tractable* (FPT) if Q is decidable by an algorithm which, given a σ-structure $\mathbb{A}$ of size n runs in time $f(\kappa(\mathbb{A}))n^c$ for some constant c and some computable function f.

Given a pair of parameterized problems, (Q, κ) and (Q', κ') where Q is a decision problem over σ-structures and Q' is a decision problem over σ'-structures, a reduction from (Q, κ) to (Q', κ') is a computable function r from σ-structures to σ'-structures such that:

- for any σ-structure $\mathbb{A}$, $r(\mathbb{A}) \in Q'$ if, and only if, $\mathbb{A} \in Q$; and
- there is a computable function g such that $\kappa'(r(\mathbb{A})) \leq g(\kappa(\mathbb{A}))$.

The reduction r is an *fpt*-reduction if, in addition, r is computable in time $f(\kappa(\mathbb{A}))|\mathbb{A}|^c$ for some constant c and some computable function f. If there is an *fpt*-reduction from (Q, κ) to (Q', κ'), we write $(Q, \kappa) \leq^{\mathtt{fpt}} (Q', \kappa')$ and say that (Q, κ) is *fpt*-reducible to (Q', κ') .

FPT is the complexity class of parameterized problems that are regarded as tractable. Above it, there is a hierarchy of complexity classes into which problems that are believed to be intractable are classified. In particular, the W-hierarchy is an increasing (or, at least, non-decreasing) sequence of complexity classes $W[t]$ ($t \geq 1$) which contain many natural hard problems. These classes were originally defined as the classes of problems *fpt*-reducible to certain weighted satisfiability problems. We use, instead, the equivalent definition from [9] in terms of weighted Fagin-definability, which we give next. For a first-order formula $\varphi(X)$ with a free relational variable X of arity s, we define the weighted Fagin-definability problem for φ as the following parameterized problem.

p-WD$_\varphi$	
Input:	A structure $\mathbb{A}$ and $k \in \mathbb{N}$
Parameter:	k
Problem:	Decide whether there is a relation $S \subseteq A^s$ with $\lvert S \rvert = k$ such that $(\mathbb{A}, S) \models \varphi$.

The complexity class $W[t]$ is then defined as the class of parameterized problems that are *fpt*-reducible to p-WD$_\varphi$ for some Π_t formula φ. Recall that φ is Π_t just in case it is in prenex normal form and its quantifier prefix consists of t alternating blocks, starting with a universal block. These classes are closed under *fpt*-reductions by definition. Indeed, to quote Flum and Grohe [9, p.95]: "for a robust theory, one has to close [...] p-WD-Π_t under *fpt*-reductions". Our aim in this paper is to test this robustness by varying the reductions used in the definition to see whether we still obtain the same classes. We specifically aim to investigate logical reductions and for these, it is convenient to work with descriptive characterisations of the complexity classes. We summarise below such characterisations that have been obtained by Flum and Grohe [7,8].

Recall that LFP is the extension of first-order logic (FO) with an operator **lfp** for least fixed-points of positive operators. We write FO^s for the collection of first-order formulas *with at most s distinct variables* and LFP^s for the collection of formulas of LFP of the form $[\mathbf{lfp}_{X,\mathbf{x}}\varphi](\mathbf{t})$ where $\varphi \in \mathsf{FO}^s$ and $\mathbf{t}$ is a tuple of at most s terms. For any collection Θ of formulas, we say that a parameterized problem (Q, κ) is *slicewise-Θ* definable if, and only if, there is a computable function $\delta : \mathbb{N} \to \Theta$ such that for all $\mathbb{A}$, we have $\mathbb{A} \models \delta(\kappa(\mathbb{A}))$ if, and only if, $\mathbb{A} \in Q$.

Theorem 1 ([7]). *A parameterized problem over* ordered *structures is in FPT if, and only if, for some s it is slicewise-LFP^s definable.*

For a similar characterisation of the classes of the W-hierarchy, we need to introduce some further notation (from [8]). We write $\Sigma_{t,u}$-Bool(LFP^s) for the collection of formulas of LFP of the form

$$\exists x_{11} \cdots \exists x_{1l_1} \forall x_{21} \cdots \forall x_{2l_2} \;\cdots\; Qx_{t1} \cdots Qx_{tl_t}\ \chi \tag{1}$$

where χ is a Boolean combination of formulas of LFP^s and for $i \geq 2$, $l_i \leq u$. In other words, the formula consists of a sequence of t alternating blocks of quantifiers, starting with an existential, with the length of all blocks except the first bounded by u, followed by a Boolean combination of LFP^s formulas. Note that all of the variables in the quantifier prefix may appear inside χ though any given formula in the Boolean combination may use at most s of them.

Theorem 2 ([8]). *A parameterized problem over* ordered *structures is in $W[t]$ if, and only if, for some s and u it is slicewise-$\Sigma_{t,u}$-Bool(LFP^s) definable.*

The key to the definition of $\Sigma_{t,u}$-Bool(LFP^s) is the interaction between the unbounded number of variables introduced by the first quantifier block, and the bounded number of variables available inside each LFP^s formula in χ. This is best illustrated with a simple example. The parameterized dominating set problem takes as input a graph G and a parameter k and asks whether G contains a set S of at most k vertices such that every vertex of G is either in S or a neighbour of a vertex in S. For fixed k, this is defined by the following first-order formula.

$$\exists x_1 \cdots \exists x_k \forall y \big(\bigvee_{1 \leq i \leq k} (y = x_i \vee E(y, x_i)) \big)$$

Here, since each of the formulas $(y = x_i \vee E(y, x_i))$ has only two variables, the whole formula is in $\Sigma_{2,1}$-Bool(LFP^2).

We can somewhat simplify the form of formulas used in Theorem 2. To be precise, we write $\Sigma_{t,u}$-Conj(LFP^s) for those formulas of the form (1) where χ is a conjunction of LFP^s formulas and $\Sigma_{t,u}$-Disj(LFP^s) for those where it is a disjunction. Then, we have the following characterisation.

Theorem 3. *For any* even $t \geq 1$*, a parameterized problem over* ordered *structures is in $W[t]$ if, and only if, for some s and u it is slicewise-$\Sigma_{t,u}$-Disj(LFP^s) definable.*

For any odd $t \geq 1$*, a parameterized problem over* ordered *structures is in $W[t]$ if, and only if, for some s and u it is slicewise-$\Sigma_{t,u}$-Conj(LFP^s) definable.*

Proof. (sketch): Consider the case of odd t, as the other case is dual. The Boolean combination χ in the formula (1) can be written in disjunctive normal form. Now, any formula $\neg\varphi$ where φ is a formula of LFP^s is equivalent to a formula of LFP^{8s}. This follows from Immerman's proof of a normal form for LFP [11]. In particular, one just needs to observe that the increase in the number of variables is bounded by a multiplicative constant. Thus, χ is equivalent to a disjunction of conjunctions of formulas of LFP^{8s}. The idea is now to replace the outermost disjunction with an existential quantifier. For each i, we can write a first-order formula $\varphi_i(x)$ (with just three variables) that asserts that x is the ith element of the linear order $\leq$ (see [4]). We use these to index the m disjuncts in χ. This requires increasing the arity in each fixed-point formula by 1, and the number of variables by at most 3. We thus obtain a formula with one existential quantifier followed by a conjunction of formulas of LFP^{8s+3} that is equivalent to χ on all ordered structures with at least m elements. We can then add a further conjunct to take care of the finitely many small structures. The existential quantifier at the front of the formula is then absorbed into the final block in the prefix, resulting in an increase of the value of u by 1.

For the case of odd t, we begin with a formula of conjunctive normal form and convert the outer conjunction to a universal quantifier. □

3 Logical Reductions

In this section, we introduce reductions that are defined by logical formulas.

Suppose we are given two relational signatures σ and τ and a set of formulas Θ. An m-ary *Θ-interpretation* of τ in σ (with parameters $\mathbf{z}$) is a sequence of formulas of Θ in the signature σ consisting of:

- a formula $\upsilon(\mathbf{x}, \mathbf{z})$;
- a formula $\eta(\mathbf{x}, \mathbf{y}, \mathbf{z})$;
- for each relation symbol R in τ of arity a, a formula $\rho^R(\mathbf{x}_1, \ldots, \mathbf{x}_a, \mathbf{z})$; and
- for each constant symbol c in τ, a formula $\gamma^c(\mathbf{x}, \mathbf{z})$,

where each $\mathbf{x}, \mathbf{y}$ or $\mathbf{x}_i$ is an m-tuple of free variables. We call m the *width* of the interpretation. We say that an interpretation Φ associates a τ-structure $\mathbb{B}$ to a pair $(\mathbb{A}, \mathbf{c})$ where $\mathbb{A}$ is a σ-structure and $\mathbf{c}$ a tuple of elements interpreting the parameters $\mathbf{z}$, if there is a surjective map h from the m-tuples $\{\mathbf{a} \in A^m \mid \mathbb{A} \models \upsilon[\mathbf{a}, \mathbf{c}]\}$ to $\mathbb{B}$ such that:

- $h(\mathbf{a}_1) = h(\mathbf{a}_2)$ if, and only if, $\mathbb{A} \models \eta[\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}]$;
- $R^{\mathbb{B}}(h(\mathbf{a}_1), \ldots, h(\mathbf{a}_a))$ if, and only if, $\mathbb{A} \models \rho^R[\mathbf{a}_1, \ldots, \mathbf{a}_a, \mathbf{c}]$;
- $h(\mathbf{a}) = c^{\mathbb{B}}$ if, and only if, $\mathbb{A} \models \gamma^c[\mathbf{a}, \mathbf{c}]$.

Note that an interpretation Φ associates a τ-structure with $(\mathbb{A}, \mathbf{c})$ only if η defines an equivalence relation on A^m that is a congruence with respect to the relations defined by the formulas ρ^R and γ^c. In such cases however, $\mathbb{B}$ is uniquely determined up to isomorphism and we write $\mathbb{B} = \Phi(\mathbb{A}, \mathbf{c})$. We will only be interested in interpretations that associate a τ-structure to every $(\mathbb{A}, \mathbf{c})$.

We say that a map r from σ-structures to τ-structures is Θ-definable if there is a Θ-interpretation Φ *without parameters* such that for all σ-structures $\mathbb{A}$, $r(\mathbb{A}) = \Phi(\mathbb{A})$.

Thus, we can ask whether a given reduction is LFP-definable or FO^s-definable, for example. It is an easy consequence of the fact that LFP captures P on ordered structures that a reduction is LFP definable *with order* if, and only if, it is a polynomial-time reduction. In the case of the complexity class NP, we know there are complete problems under much weaker reductions such as those defined by quantifier-free formulas in the presence of order, or first-order formulas even without order (see [2,13]).

For reductions between parameterized problems, it is more natural to consider the slicewise definition of interpretations. We say that a reduction r between parameterized problems (Q, κ) and (Q', κ') is *slicewise Θ-definable* if there is an m and a function δ that takes each natural number k to an m-ary Θ-interpretation $\delta(k)$ such that for any σ-structure $\mathbb{A}$ with $r(\mathbb{A}) = \delta(\kappa(\mathbb{A}))(\mathbb{A})$. Note, in particular, that the width m of the interpretation is the same for all k. It is an easy consequence of the proof of Theorem 1 in [7] that a reduction r is an *fpt*-reduction if, and only if, for some s, it is slicewise LFP^s-definable on ordered structures.

The following definition introduces some useful notation for the different classes of reductions we consider.

Definition 1. *For parameterized problems (Q, κ) and (Q', κ'), we write*

1. $(Q, \kappa) \leq^{s\text{-}fo} (Q', \kappa')$ *if there is a reduction from (Q, κ) to (Q', κ') that is* slicewise FO-definable*;*
2. $(Q, \kappa) \leq^{s\text{-}bfo} (Q', \kappa')$ *if there is a reduction from (Q, κ) to (Q', κ') that is* slicewise FO^s-definable *for some s;*
3. $(Q, \kappa) \leq^{s\text{-}qf} (Q', \kappa')$ *if there is a reduction from (Q, κ) to (Q', κ') that is* slicewise Θ-definable*, where Θ is the collection of* quantifier-free *formulas.*

It is clear from the definition that $(Q, \kappa) \leq^{\texttt{s-bfo}} (Q', \kappa')$ implies $(Q, \kappa) \leq^{\texttt{s-fo}} (Q', \kappa')$. Furthermore, since the definition of slicewise reductions requires the interpretations to be of fixed width, and the only variables that occur in a quantifier-free formula are the free variables, it can be easily seen that a $\leq^{\texttt{s-qf}}$ reduction is defined with a bounded number of variables. Thus, $(Q, \kappa) \leq^{\texttt{s-qf}} (Q', \kappa')$ implies $(Q, \kappa) \leq^{\texttt{s-bfo}} (Q', \kappa')$ and the reductions in Definition 1 are increasingly weak as we go down the list. The last two of them are also weaker than *fpt*-reductions, in the sense that, since FO^s formulas are also LFP^s formulas, we have that $(Q, \kappa) \leq^{\texttt{s-bfo}} (Q', \kappa')$ implies $(Q, \kappa) \leq^{\texttt{fpt}} (Q', \kappa')$. As we show in Section 5, it is unlikely that $(Q, \kappa) \leq^{\texttt{s-fo}} (Q', \kappa')$ implies $(Q, \kappa) \leq^{\texttt{fpt}} (Q', \kappa')$ as this would entail the collapse of the W-hierarchy.

4 Bounded-Variable Reductions

In this section, we construct problems that are complete for the class $W[t]$ under $\leq^{\texttt{s-bfo}}$ reductions.

We first consider the decision problem of *alternating reachability*, also known as *game*. We are given a directed graph $G = (V, E)$ along with a bipartition of the vertices $V = V_\exists \uplus V_\forall$ and two distinguished vertices a and b. We are asked to decide whether the pair (a, b) is in the *alternating transitive closure* defined by $(V_\exists, V_\forall, E)$. This is equivalent to asking whether the existential player has a winning strategy in the following two-player token pushing game played on G as follows. The token is initially on a. At each turn, if the token is on an element of $V_\exists$, it is the existential player that moves and, if it is on an element of $V_\forall$, it is the universal player that moves. Each move consists of the player whose turn it is moving the token from a vertex u to a vertex v such that $(u, v) \in E$. If the token reaches b, the existential player has won. In general, we call a directed graph $G = (V, E)$ along with a bipartition $V = V_\exists \uplus V_\forall$ an *alternating graph*; we call the vertices in $V_\exists$ the existential vertices of G and those in $V_\forall$ the universal vertices; and we call a the source vertex and b the target vertex. We can assume without loss of generality that the target vertex has no outgoing edges.

An *alternating path* from a to b is an *acyclic* subgraph $V' \subseteq V, E' \subseteq E$ with $a, b \in V'$ such that for every $u \in V' \cap V_\exists$, $u \neq b$, there is a $v \in V'$ with $(u, v) \in E'$; for every $u \in V' \cap V_\forall$, $u \neq b$ and every $v \in V$ with $(u, v) \in E$ we have that $v \in V'$ and $(u, v) \in E'$; and for every $u \in V'$, there is a path from u to b in (V', E'). It is easily checked that (a, b) is in the alternating transitive closure of $(V_\exists, V_\forall, E)$ if, and only if, there is an alternating path from a to b.

It is known that the alternating reachability problem is complete for P under first-order reductions, in the presence of order (see [10]). Indeed, it is also known that in the absence of order, the problem is still complete for the class of problems that are definable in LFP [3]. Moreover, it is easily shown from the reductions constructed by Dahlhaus in [3] that every problem definable in LFP^s is reducible to alternating reachability by means of a first-order reduction whose width depends only on s, giving us the following lemma.

Lemma 1. *For any s there is an r such that for any formula $\varphi(\mathbf{z})$ of LFP^s in the signature σ, we can find an FO^r-interpretation with parameters $\mathbf{z}$ that takes each $(\mathbb{A}, \mathbf{c})$, where $\mathbb{A}$ is a σ-structure and $\mathbf{c}$ an interpretation of the parameters, to an alternating graph $(V, V_\exists, V_\forall, E, a, b)$ so that $\mathbb{A} \models \varphi[\mathbf{c}]$ if, and only if, there is an alternating path from a to b in $(V_\exists, V_\forall, E)$.*

It is easily checked that alternating reachability is defined by the following formula of LFP^2.

$$[\mathbf{lfp}_{X,x}\big(x = b \vee (V_\exists(x) \wedge \exists y(E(x, y) \wedge X(y))) \vee \\ (V_\forall(x) \wedge \exists y E(x, y) \wedge \forall y(E(x, y) \rightarrow X(y)))\big)](a)$$

For an alternating graph $G = (V, V_\exists, V_\forall, E)$ and a subset $U \subseteq V$, we say that there is a U*-avoiding alternating path* from a to b if there is an alternating path from a to b which does not include any vertex of U. Note, that this is *not* the same as saying there is an alternating path from a to b in the subgraph of G induced by $V \setminus U$. In particular, a U*-avoiding alternating path* may not include any universal vertex which has an outgoing edge to a vertex in U, though such vertices may appear in an alternating path in the graph $G[V \setminus U]$.

We will now define a series of variants of the alternating reachability problem, which will lead us to the $W[t]$-complete problems we seek to define. In the following definitions, k is a fixed positive integer.

k-conjunctive restricted alternating reachability. Given an alternating graph $G = (V, V_\exists, V_\forall, E)$, along with sets of vertices $C \subseteq U \subseteq V$ and distinguished vertices a and b, where a has at most k outgoing edges, decide whether for every v such that $(a, v) \in E$, there is an $s_v \in C$ and a $(U \setminus \{s_v\})$-avoiding alternating path from v to b. If the answer is yes, we say there is a *k-conjunctive restricted alternating path* from a to b.

In other words, the problem asks whether there is an alternating path from a to b, where $a \in V_\forall$, of a particular restricted kind. The path is not to use the vertices in U apart from C, and these may be used only in a limited way. That is, each outgoing edge from a leads to a path which may use only one vertex of C, though this vertex may be different for the different edges leaving a.

We define a dual to the above for starting vertices a that are existential.

k-disjunctive restricted alternating reachability. Given an alternating graph $G = (V, V_\exists, V_\forall, E)$, along with sets of vertices $C \subseteq U \subseteq V$ and distinguished vertices a and b, where a has at most k outgoing edges, decide whether there is a vertex v with $(a, v) \in E$ and an $s_v \in C$ such that there is a $(U \setminus \{s_v\})$-avoiding alternating path from v to b. If the answer is yes, we say there is a *k-disjunctive restricted alternating path* from a to b.

We next define, by induction on t, the problems of conjunctive and disjunctive k, t-*restricted alternating reachability*, for which the above two problems serve as base cases.

Definition 2 (k, t-restricted alternating reachability). *The* conjunctive $k, 0$-restricted alternating reachability *problem is just the k-conjunctive restricted alternating reachability problem defined above, and similarly, the* disjunctive $k, 0$-restricted alternating reachability *problem is the k-disjunctive restricted alternating reachability problem.*

The conjunctive $k, t+1$-restricted alternating reachability *is the problem of deciding, given an alternating graph $G = (V, V_\exists, V_\forall, E)$, along with sets of vertices $C \subseteq U \subseteq V$ and distinguished vertices a and b, whether for every v such that $(a, v) \in E$, there is a disjunctive k, t-restricted alternating path from v to b.*

Dually, the disjunctive $k, t+1$-restricted alternating reachability *is the problem of deciding whether there is a v such that $(a, v) \in E$ and there is a conjunctive k, t-restricted alternating path from v to b.*

Roughly speaking, the conjunctive k, t-restricted alternating reachability problem asks for an alternating path from a to b, with a a universal node, where we are allowed t alternations before the restrictions on the use of vertices in the sets U and C kick in. The disjunctive version is dual.

We are ready to define the parameterized problems we need. By a *clique* in a directed graph, we just mean a set of vertices such that for each pair of distinct vertices in the set, there are edges in both directions.

Definition 3 (clique-restricted alternating reachability). *For any fixed t, the* parameterized t-clique-restricted alternating reachability problem *is:*

p-t-CLIQUE RESTRICTED ALTERNATING REACHABILITY	
Input:	$G = (V = V_\exists \uplus V_\forall, E)$, $U \subseteq V$, $a, b \in V$ and $k \in \mathbb{N}$.
Parameter:	k
Problem:	Is there a clique $C \subseteq U$ with $\lvert C \rvert = k$ such that $(V, V_\exists, V_\forall, E, U, C)$ admits a conjunctive k, t-restricted alternating path from a to b?

Theorem 4. *For each $t \geq 1$, p-t-clique-restricted alternating reachability is in $W[t]$.*

Proof. This is easily established, using Theorem 2, by writing a formula of $\Sigma_{t,1}$-Bool(LFP^4) that defines the problem for each fixed value of k. This is obtained by taking the prenex normal form of the formula

$$\exists x_1 \cdots x_k (\bigwedge_i U(x_i) \wedge \bigwedge_{i \neq j} E(x_i, x_j) \wedge x_i \neq x_j) \wedge \forall y_1 (E(a, y_1) \rightarrow (\exists y_2 (E(y_1, y_2) \wedge \cdots \Gamma \cdots)$$

where Γ is the formula $\exists y_{t-1}(E(y_{t-2}, y_{t-1}) \wedge \bigwedge_{1 \leq i \leq k}(\bigvee_{1 \leq j \leq k} \theta_i(y_{t-1}, x_j)))$ if t is odd and the formula $\forall y_{t-1}(E(y_{t-2}, y_{t-1}) \rightarrow \bigvee_{1 \leq i,j \leq k} \theta_i(y_{t-1}, x_j))$ if t is even; and $\theta_i(y_{t-1}, x_j)$ is the formula of LFP^4 which states that there is a $U \setminus \{x_j\}$-avoiding alternating path from z to b, where z is the ith (in the linear ordering $\leq$) vertex such that there is an edge from y_{t-1} to z. This formula is obtained as an easy modification of the LFP^2 formula above defining alternating reachability. □

Theorem 5. *For each $t \geq 1$, p-t-clique-restricted alternating reachability problem is $W[t]$-hard.*

Proof. (sketch): Suppose (Q, κ) is in $W[t]$. Assume that t is odd (the case for even t is dual). By Theorem 3, there is a u and an s so that (Q, κ) is slicewise-$\Sigma_{t,u}$-Conj(LFP^s)-definable. Thus, for each k, there is a formula

$$\varphi \equiv \exists x_{11} \cdots \exists x_{1l_1} \forall x_{21} \cdots \forall x_{2l_2} \ \cdots \ \exists x_{t1} \cdots \exists x_{tl_t} \bigwedge_{j \in S} \theta_j$$

where each θ_j is in LFP^s, which defines the structures $\mathbb{A}$ such that $\mathbb{A} \in Q$ and $\kappa(\mathbb{A}) = k$. We give an informal description of the reduction that takes $\mathbb{A}$ to an instance $\mathbb{G}$ of t-clique-restricted alternating reachability. The reduction is definable by an FO-interpretation using a number of variables that is a function of s, t and u but independent of k. In what follows, we assume that $\lvert S \rvert \leq \binom{l_1}{s}$. If this is not the case, we can add dummy variables to the first quantifier block without changing the meaning of the formula.

By Lemma 1, we know that each θ_j gives rise to an FO^r-interpretation (for a fixed value of r) that maps $\mathbb{A}$ to an instance of alternating reachability. Note that, as the width of the Since θ_j has (as many as s) free variables, the interpretation will have up to s parameters from among the variables $x_{11}, \ldots, x_{tl_t}$. For notational purposes, we will distinguish between those parameters that are in the variables quantified in the first existential block (i.e. $x_{11} \ldots x_{1l_1}$) and the others. Thus, we write $\mathrm{AR}_j^{\alpha,\beta}$ for the instance of alternating reachability obtained from θ_j, with α the assignment of values to the parameters among $x_{11} \dots x_{1l_1}$ and β the assignment of values to the other parameters. $\mathbb{G}$ will contain the disjoint union of all of these instances (slightly modified as explained below). Note that, since the interpretation taking $\mathbb{A}$ to $\mathrm{AR}_j^{\alpha,\beta}$ has width at most r, the size of $\mathrm{AR}_j^{\alpha,\beta}$ is at most n^r (where n is the size of $\mathbb{A}$). Furthermore, there are at most $|S| \cdot n^s$ such instances. $\mathbb{G}$ also contains a target vertex b with an incoming edge from the target vertex in each $\mathrm{AR}_j^{\alpha,\beta}$.

In addition, for each initial segment $\mathbf{x}$ of the sequence of variables $x_{21} \dots x_{tl_t}$ that ends at a quantifier alternation (i.e. $\mathbf{x} = x_{21} \dots x_{t',l_{t'}}$ for some $t' \leq t$), and each assignment ρ of values from $\mathbb{A}$ to the variables in $\mathbf{x}$, $\mathbb{G}$ contains a new element. Note that the number of such elements is less than $2n^{\sum_{2 \leq i \leq t} l_i}$, which is at most $2n^{u(t-1)}$. For each ρ and ρ', we include an edge from ρ to ρ' just in case ρ' extends the assignment ρ by exactly one quantifier block. If that block is existential, ρ is in $V_\exists$ and if it is universal, ρ is in $V_\forall$. Those ρ which assign a value to every variable in $x_{21} \dots x_{tl_t}$ are in $V_\forall$ and have outgoing edges to a vertex ρ_j, one for each $j \in S$. These vertices are existential and have outgoing edges to the source vertex of each $\mathrm{AR}_j^{\alpha,\beta}$ where the assignment β is consistent with ρ. That is, if a variable x among $x_{21} \dots x_{tl_t}$ occurs among the parameters of θ_j we should have $\beta(x) = \rho(x)$. The unique empty assignment ϵ is the source vertex of $\mathbb{G}$.

Let ψ be the part of φ after the first existential block, i.e. $\varphi \equiv \exists x_{11} \cdots \exists x_{1l_1} \psi$. The structure so far codes the interpretation of ψ with each θ_j replaced by equivalent alternating reachability conditions. If there is an assignment of values to the variables $x_{11} \dots x_{1l_1}$ that makes ψ true, there is an alternating path from source to target in $\mathbb{G}$. However, the converse is not true, as distinct θ_j may share free variables from among $x_{11} \dots x_{1l_1}$ and there is nothing in the alternating path that ensures consistency in the values they assign to these variables. In fact, it can be shown that, in the structure described so far, there is an alternating path from source to target if, and only if, we can assign values to the free variables $x_{11} \dots x_{1l_1}$, *independently for each* θ_j in a way that makes ψ true. We now add a gadget to $\mathbb{G}$ that ensures consistency of the assignment of these values.

$\mathbb{G}$ also contains a set U of vertices, disjoint from those constructed so far. There is one vertex in U for each assignment of values from $\mathbb{A}$ to a subset of the variables $x_{11} \dots x_{1l_1}$ *of size* s. Thus, U contains a total of $\binom{l_1}{s} \cdot n^s$ vertices. All vertices in U are existential (i.e. in $V_\exists$) and for $\alpha, \beta \in U$, there is an edge from α to β if the two assignments agree on all variables they have in common. It is easily checked that the maximal cliques in U are of size $\binom{l_1}{s}$. There is one such clique for each assignment of values to all l_1 variables.

To connect the gadget U with the rest of the construction, we replace every vertex v in each $\mathrm{AR}_j^{\alpha,\beta}$ with two vertices v_{in} and v_{out}. All edges into v now lead to v_{in} and all edges out of v are replaced by edges out of v_{out}. v_{out} is existential if v is existential and universal if v is universal. v_{in} is universal for all v and has exactly two outgoing edges, one to v_{out} and the other to $\alpha \in U$ (i.e. the element of U giving the assignment to the parameters among $x_{11} \dots x_{1l_1}$ that corresponds to the instance $\mathrm{AR}_j^{\alpha,\beta}$). Finally, there is also an edge from $\alpha \in U$ to v_{out} for each vertex v in any instance of the form $\mathrm{AR}_j^{\alpha,\beta}$.

This completes the description of $\mathbb{G}$. It is not difficult to argue that $\mathbb{G}$ contains a clique $C \subseteq U$ of size $\binom{l_1}{s}$ such that $(\mathbb{G}, C)$ admits a conjunctive $\binom{l_1}{s}$, t-restricted alternating path from ϵ to b if, and only if, $\mathbb{A} \models \varphi$. Indeed, if $\mathbb{A} \models \varphi$ and γ is an assignment to the variables $x_{11} \dots x_{1l_1}$ that witnesses this, we can choose C to consist of all nodes α that are consistent with γ. As we have argued above, this will yield the required alternating path through $\mathbb{G}$. Conversely, any clique $C \subseteq U$ of size $\binom{l_1}{s}$ must correspond to such an assignment γ and thus any alternating path in $\mathbb{G}$ that uses only C will provide a witness that $\mathbb{A} \models \varphi$.

We omit from this sketch the construction of the formulas that show that the interpretation that takes $\mathbb{A}$ to $\mathbb{G}$ can be given by first-order formulas where the number of variables is independent of k (i.e. independent of l_1), but we will make two points in this connection. One is that the total number of vertices in $\mathbb{G}$ is bounded by $\binom{l_1}{s}n^s + |S|n^{s+r} + 2n^{(t-1)u}$. This is a polynomial in n whose degree depends on s, t and u (recall, by Lemma 1, that r is a function of s) but not on l_1. We use this to establish that the width of the interpretation is bounded. One subtle point is that in defining the set U, we need to define not only all s-tuples of elements of $\mathbb{A}$ but to pair them with s-element sets of variables. We do this by identifying the variables with the first l_1 elements of the linear order $\leq$. We then use the fact that any fixed element of a linear order can be identified with a formula of FO^3 (see, for instance, [4]). $\square$

5 Other First-Order Reductions

As we pointed out in Section 2, one standard definition of the class $W[t]$ is as $[p\text{-WD-}\Pi_t]^{\leq^{\text{fpt}}}$, i.e. the class of problems *fpt*-reducible to p-WD$_\varphi$ for some $\varphi \in \Pi_t$. Here we consider the class $[p\text{-WD-}\Pi_t]^{\leq^{\text{s-qf}}}$, i.e. the problems reducible by *slicewise quantifier-free reductions* to p-WD$_\varphi$ for some $\varphi \in \Pi_t$ and show this class is most likely weaker, in the sense that there are problems in $W[t+1]$ that are *provably* not in this class.

It is known that the alternation of quantifiers in a first-order formula yields a strict hierarchy of increasing expressive power, even on ordered structures, in the presence of arithmetic relations [14]. We use this to establish our result.

Say that an alternating graph G is *strictly alternating* if each vertex in $V_\exists$ only has edges to vertices in $V_\forall$ and vice versa. We can write, for each $t \geq 1$, a formula $\varphi_t(X) \in \Pi_t$ with a free set variable X that is satisfied by a strictly alternating graph G with a set S interpreting X if, and only if, the source vertex of G is universal and G contains an alternating path from a to all $b \in S$ (or for some $b \in S$, when t is even) with exactly t alternations. We are able to show, by a reduction from the problems that Sipser [14] uses to establish the strictness of the first-order quantifier alternation hierarchy, that $\varphi_{t+1}(X)$ is not equivalent to any formula of Π_t, even on ordered structures with arithmetic. On the other hand, it is not difficult to show that if p-WD$_{\varphi_{t+1}} \leq^{\text{s-qf}} p$-WD$_\psi$ for a formula $\psi \in \Pi_t$ then, composing ψ with the interpretation, we would obtain a formula of Π_t equivalent to φ_{t+1}. This leads to the following theorem.

Theorem 6. *For each $t \geq 1$, there is a $\varphi \in \Pi_{t+1}$ such that for any $\psi \in \Pi_t$, p-WD$_\varphi \not\leq^{s\text{-}qf} p$-WD$_\psi$.*

On the other hand, for any formula $\varphi(X)$, it is easy to construct a sequence of first-order formulas $\varphi_k (k \in \mathbb{N})$, without the variable X, that define the slices of p-WD$_\varphi$. These can be used to construct a slicewise first-order reduction of p-WD$_\varphi$ to a trivial problem, giving us the following observation.

Theorem 7. *For any $\varphi \in$ FO, there is an FPT problem Q such that p-WD$_\varphi \leq^{s\text{-}fo} Q$.*

6 Concluding Remarks

We have considered varying the notion of reductions used in the definition of the classes of the W-hierarchy. The results of Section 5 show that slicewise quantifier-free reductions are too weak, and slicewise first-order reductions are too strong for the purpose. The intermediate case of slicewise bounded-variable first-order reductions is considered in Section 4 and though these reductions are considerably weaker than *fpt*-reductions, we are able to show the existence of complete problems for the classes of the W-hierarchy. It would be interesting to investigate whether other, natural $W[t]$-complete problems remain complete under these reductions. In particular, is it the case that the closure of p-WD-Π_t under such reductions is all of $W[t]$?

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