Implicit Definability and Infinitary Logic in Finite Model Theory

(Extended Abstract)

Anuj Dawar*[1], Lauri Hella[2], Phokion G. Kolaitis**[3]

[1] Dept. of Comp. Science, Univ. of Wales, Swansea, Swansea SA2 8PP, U.K.
[2] Dept. of Mathematics, P.O. Box 4, 00014 Univ. of Helsinki, Finland.
[3] Comp. and Info. Sciences, University of California, Santa Cruz, CA 95064, U.S.A.

Abstract. We study the relationship between the infinitary logic $L^{\omega}_{\infty\omega}$ with finitely many variables and implicit definability in effective fragments of $L^{\omega}_{\infty\omega}$ on finite structures. We show that fixpoint logic has strictly less expressive power than first-order implicit definability. We also establish that the separation of fixpoint logic from a certain restriction of first-order implicit definability to $L^{\omega}_{\infty\omega}$ is equivalent to the separation of PTIME from UP $\cap$ co-UP. Finally, we delineate the relationship between partial fixpoint logic and implicit definability in partial fixpoint logic on finite structures.

1 Introduction and Summary of Results

Finite model theory is the study of first-order logic and extensions of first-order logic on finite structures. Finite model theory has enjoyed steady growth through a continuous interaction with computational complexity, database theory, and combinatorics. In particular, the interaction between finite model theory and computational complexity has resulted in the development of descriptive complexity, an area of research whose main aim is to classify algorithmic problems according to the resources needed to express such problems in various logical formalisms. One of the most notable successes of descriptive complexity is the discovery that essentially all major complexity classes have natural characterizations in terms of logical expressibility on finite structures (cf. [Gur88,Imm89]). The prototypical result in this vein is Fagin's theorem [Fag74] which asserts that a class of finite structures is in NP if and only if it is definable by a formula of existential second-order logic. Quite often, certain logical characterizations of other major complexity classes are valid only on classes of ordered structures. Thus, Immerman [Imm86] and, independently, Vardi [Var82] showed that PTIME coincides with fixpoint logic LFP on the class of all ordered finite structures. Similarly, Vardi [Var82] established that partial fixpoint logic PFP

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captures PSPACE on ordered structures (cf. also [AV89]). However, on the class of all finite structures the collection of queries expressible in fixpoint logic LFP is properly contained in PTIME, and the same is true of partial fixpoint PFP and PSPACE (cf. [CH82]). In spite of these limitations, both LFP and PFP constitute powerful extensions of first-order logic on the class of all finite structures, as they incorporate recursion and iteration mechanisms in the form of fixpoint operators. For this reason, they have been studied extensively in database theory and have become a standard for comparing and calibrating the expressive power of other database query languages (cf. [Cha88]).

A subsequent turning point in the study of the connections between logic and complexity was the paper by Abiteboul and Vianu [AV91], which established that the problem of separating fixpoint logic LFP from partial fixpoint logic PFP on the class of all finite structures is equivalent to the separation of PTIME from PSPACE. This result was further extended by Abiteboul, Vardi, and Vianu [AVV92] to results concerning a whole range of complexity classes between PTIME and EXPTIME. For this, they introduced a variety of logics equipped with suitable fixpoint operators and showed that inclusions and separations between these fixpoint logics on the class of all finite structures mirror inclusions and separations between corresponding complexity classes. In other words, each of these fixpoint logics gives rise to what is called a *relational* complexity class on all finite structures; such a class embodies the salient properties of a corresponding complexity class, but it is defined in a machine-independent and order-invariant way.

The aforementioned results in [AV91,AVV92] exploit the fact that logics with fixpoint operators can be viewed as effective fragments of the infinitary logic $L^{\omega}_{\infty\omega}$ with finitely many variables. During the past several years, the study of $L^{\omega}_{\infty\omega}$ became a focal point of research in finite model theory. One of the reasons for the interest in $L^{\omega}_{\infty\omega}$ is that its expressive power has an elegant characterization in terms of two-player pebble games [Bar77,Imm82]. This has made it possible to apply game-theoretic techniques in order to derive lower bounds for expressibility in $L^{\omega}_{\infty\omega}$, as well as to obtain positive structural results, such as 0-1 laws (cf. [KV92c]). Moreover, as demonstrated in [AV91,DLW92], there is a deeper connection between $L^{\omega}_{\infty\omega}$ and LFP, a connection which is behind the phenomenon that outstanding open problems in complexity theory turn out to be equivalent to separation questions between fragments of $L^{\omega}_{\infty\omega}$ on the class of all finite structures.

Research in finite model theory has also focused on the study of concepts from the classical model theory of first-order logic in the context of finite structures. While early work in this direction centered on the failure of many classical theorems when restricted to finite models (cf. [Gur84]), in recent years some effort has been devoted to exploring the positive aspects of this restriction. By taking as a point of departure Gurevich's [Gur84] result that Beth's definability theorem fails in the finite case, Kolaitis [Kol90] embarked on a systematic investigation of first-order implicit definability on finite structures. This investigation was carried out by studying the expressive power and computational content of the class IMP(FO) of queries Q that are members of a sequence $(Q, Q_1, \ldots, Q_n)$ of queries implicitly definable by some first-order sentence $\varphi(S, S_1, \ldots, S_n)$. In [Kol90], it was shown that LFP $\subseteq$ IMP(FO), which means that on the class of all finite structures first-order implicit definability is at least as expressive as fixpoint logic. The question of whether the inclusion LFP $\subseteq$ IMP(FO) is proper on the class of all finite structures was left as an open problem. On the other hand, it was also shown in [Kol90] that on the class of all ordered finite structures IMP(FO) coincides with UP $\cap$ co-UP, where UP is the class of NP problems computable by unambiguous Turing machines.

In the present paper, we study the interaction between infinitary logic $L^{\omega}_{\infty\omega}$ and implicit definability in first-order logic, fixpoint logic, and partial fixpoint logic. This investigation touches upon all directions of research in finite model theory mentioned above. Thus, we view our work as a continuation of the study of implicit definability on finite structures and as a contribution to both descriptive complexity and the model theory of $L^{\omega}_{\infty\omega}$. Building on recent work of Gurevich and Shelah [GS94], we show that there is a query in IMP(FO) which is not definable in $L^{\omega}_{\infty\omega}$. This result implies that on the class of all finite structures the inclusion LFP $\subseteq$ IMP(FO) is indeed a proper one. After this, we introduce the class IMP(FO)$|L^{\omega}_{\infty\omega}$ of queries that are obtained by restricting first-order implicit definition to sequences of queries all of which are in $L^{\omega}_{\infty\omega}$. We investigate the computational strength of the class IMP(FO)$|L^{\omega}_{\infty\omega}$ and demonstrate that it is a relational complexity class corresponding to UP $\cap$ co-UP. In particular, we establish that the separation of LFP from IMP(FO)$|L^{\omega}_{\infty\omega}$ on the class of all finite structures is equivalent to the separation of PTIME from UP $\cap$ co-UP. This result provides a new instance of the phenomenon that a complexity theoretic problem is equivalent to a problem about fragments of $L^{\omega}_{\infty\omega}$ on the class of all finite structures. Finally, we study implicit definability in partial fixpoint logic and determine the exact relationship between the classes PFP, IMP(PFP)$|L^{\omega}_{\infty\omega}$, and IMP(PFP) $\cap$ $L^{\omega}_{\infty\omega}$.

2 Background Information

A *vocabulary* σ is a finite sequence $(R_1, \ldots, R_m)$ of relation symbols of arities $r_1, \ldots, r_m$, respectively. As usual, a *structure* $\mathbf{A}$ *over* σ is a tuple $\mathbf{A} = (A, R_1^{\mathbf{A}}, \ldots, R_m^{\mathbf{A}})$, where A is a non-empty set, called the *universe* of $\mathbf{A}$, and $R_i^{\mathbf{A}} \subseteq A^{r_i}$ is a relation on A of arity r_i, $1 \leq i \leq m$. From now on, we assume that all structures studied are finite.

Let k be positive integer. A *k-ary query* Q is a function which for every structure $\mathbf{A}$ returns as value a k-ary relation $Q(\mathbf{A})$ on the universe of $\mathbf{A}$ such that $Q(\mathbf{A})$ is preserved under isomorphisms, i.e., if $\mathbf{A}$ and $\mathbf{B}$ are structures and g is an isomorphism from $\mathbf{A}$ to $\mathbf{B}$, then g is also an isomorphism from $(A, Q(\mathbf{A}))$ to $(B, Q(\mathbf{B}))$.

We say that a k-ary query Q is *definable in a logic* L if there is a formula $\varphi(x_1, \ldots, x_k)$ of L such that for every structure $\mathbf{A}$ we have $Q(\mathbf{A}) = \{(a_1, \ldots, a_k) \in A^k : \mathbf{A} \models \varphi(a_1, \ldots, a_k)\}$.

If L is a logic, then the notation L will also be used to denote the class of queries definable in L. If L_1 and L_2 are two logics, then $L_1 \subseteq L_2$ means that the class of queries definable in L_1 is contained in the class of queries definable in L_2, while $L_1 \subset L_2$ means that the above containment is a *proper* one.

The preceding definitions relativize naturally to a class $\mathcal{C}$ of structures that is closed under isomorphisms, so that we have the concept of a *query on* $\mathcal{C}$ and the notation $L(\mathcal{C})$ for the class of queries on $\mathcal{C}$ definable in L.

2.1 Fixpoint Logic and Partial Fixpoint Logic

Let $\varphi(x_1, \ldots, x_n, S)$ be a first-order formula over a vocabulary $\sigma \cup \{S\}$, where S is an n-ary relation symbol that is not in σ, and let $\mathbf{A}$ be a structure over σ. The formula φ gives rise to an operator $\Phi(S)$ from n-ary relations on the universe A of $\mathbf{A}$ to n-ary relations on A, where

$$\Phi(S) = \{(a_1, \ldots, a_n) : \mathbf{A} \models \varphi(a_1, \ldots, a_n, S)\}$$

for every n-ary relation S on A. Every such operator $\Phi(S)$ can be iterated and, thus, it gives rise to the sequence of *stages* Φ^m, $m \geq 1$, defined by the induction: $\Phi^1 = \Phi(\emptyset)$, $\Phi^{m+1} = \Phi(\Phi^m)$.

If the operator $\Phi(S)$ is *monotone* (which means that $\Phi(S_1) \subseteq \Phi(S_2)$, whenever $S_1 \subseteq S_2$), then by the Knaster-Tarski Theorem the operator $\Phi(S)$ has a *least fixpoint*, which is denoted by φ^∞. Moreover, the least fixpoint φ^∞ of $\Phi(S)$ can be obtained as the "limit" of the stages Φ^m, $m \geq 1$, that is to say for every structure $\mathbf{A}$ there is an integer m_0 such that $\varphi^\infty = \Phi^{m_0} = \Phi^m$ on $\mathbf{A}$ for every $m \geq m_0$. Monotonicity is a semantic property which follows from the syntactic property of positivity. More precisely, we say that a formula $\varphi(x_1, \ldots, x_n, S)$ is *positive in* S if every occurrence of S in φ is within an even number of negations. For such formulas, the associated operator $\Phi(S)$ is monotone and, thus, the least fixpoint φ^∞ exists.

Definition 1. *Fixpoint Logic* LFP *is the extension of* FO *that is closed under the operations of first-order logic and the least fixpoint rule: if* $\varphi(x_1, \ldots, x_n, S)$ *is a positive formula, then* $\varphi^\infty(x_1, \ldots, x_n)$ *is also a formula of* LFP.

If the operator Φ is not monotone, then on a given structure $\mathbf{A}$ either there is a positive integer m such that $\Phi^m = \Phi^{m+1}$ or the sequence Φ^m, $m \geq 1$, of stages cycles without ever yielding a fixpoint of Φ. If $\varphi(x_1, \ldots, x_n, S)$ is a first-order formula (not necessarily a positive one), then we define the *partial fixpoint* φ^∞ of φ to be a stage Φ^m such that $\Phi^m = \Phi^{m+1}$, if such a stage exists, or the empty set $\emptyset$, otherwise.

Definition 2. *Partial Fixpoint Logic* PFP *is the extension of* FO *that is closed under the operations of first-order logic and the partial fixpoint rule:*
if $\varphi(x_1, \ldots, x_n, S)$ *is a formula, then* $\varphi^\infty(x_1, \ldots, x_n)$ *is also a formula of* PFP.

Fixpoint logic on finite structures was first studied by Chandra and Harel [CH82]. Partial fixpoint logic was introduced by Abiteboul and Vianu [AV89], who showed that the class of PFP queries coincides with the class of queries that are expressible in first-order logic augmented with WHILE-*looping*. The latter is a powerful iteration construct investigated by Chandra and Harel [CH82].

It is easy to verify that on the class of all finite structures LFP $\subseteq$ PTIME and PFP $\subseteq$ PSPACE. The above containments are proper, since counting properties, such as "even cardinality", are not expressible in LFP or in PFP (cf. [CH82]). In contrast, on classes of *ordered* structures LFP captures PTIME and PFP captures PSPACE, where $\mathcal{C}$ is a class of *ordered* structures over a vocabulary σ if σ contains a binary relation symbol $<$ and for every $\mathbf{A}$ in $\mathcal{C}$ the relation $<^{\mathbf{A}}$ is a linear order on A. Immerman [Imm86] and Vardi [Var82] showed independently that if $\mathcal{C}$ is a class of ordered structures, then LFP$(\mathcal{C})$ = PTIME on $\mathcal{C}$; moreover, Vardi [Var82] showed that PFP$(\mathcal{C})$ = PSPACE on $\mathcal{C}$ (cf. also [AV89]).

Since the partial fixpoint of a positive formula coincides with its least fixpoint, it is clear that LFP $\subseteq$ PFP. Chandra and Harel [CH82] raised the problem of showing that LFP is properly contained in PFP on the class of all finite structures. Abiteboul and Vianu [AV91] established that this separation amounts to separating PTIME from PSPACE, i.e., LFP = PFP if and only if PTIME = PSPACE.

2.2 Infinitary Logics with Finitely Many Variables

Powerful extensions of first-order logic result by augmenting the syntax with infinitary connectives. In particular, $\mathrm{L}_{\infty\omega}$ is the extension of first-order logic obtained by allowing infinite disjunctions and conjunctions, while keeping the quantifier strings finite. This logic, however, is too powerful to be of any use in finite model theory, since every class of finite structures that is closed under isomorphisms is definable by a sentence of $\mathrm{L}_{\infty\omega}$. On the other hand, the fragments of $\mathrm{L}_{\infty\omega}$ obtained by restricting the number of variables have turned out to be of importance in finite model theory (cf. [KV92c]).

Definition 3. *Let k be a positive integer.*

- *The infinitary logic $\mathrm{L}^k_{\infty\omega}$ with k variables consists of all formulas of $\mathrm{L}_{\infty\omega}$ with at most k distinct variables.*
- *The infinitary logic $\mathrm{L}^\omega_{\infty\omega}$ consists of all formulas of $\mathrm{L}_{\infty\omega}$ with a finite number of distinct variables, i.e., $\mathrm{L}^\omega_{\infty\omega} = \bigcup_{k=1}^{\infty} \mathrm{L}^k_{\infty\omega}$.*
- *We write $\mathrm{L}^k_{\omega\omega}$ for the collection of all first-order formulas with at most k variables. Thus, $\mathrm{FO} = \bigcup_{k=1}^{\infty} \mathrm{L}^k_{\omega\omega}$.*

The infinitary logic $\mathrm{L}^\omega_{\infty\omega}$ subsumes both fixpoint logic and partial fixpoint logic on finite structures, i.e., LFP $\subseteq$ PFP $\subseteq \mathrm{L}^\omega_{\infty\omega}$ (cf. [KV92b]). Moreover, on the class of all finite structures both LFP and PFP are properly contained in $\mathrm{L}^\omega_{\infty\omega}$, since $\mathrm{L}^\omega_{\infty\omega}$ can express non-recursive queries. Although LFP constitutes a small effective fragment of $\mathrm{L}^\omega_{\infty\omega}$, it plays a key role in the study of the model theory of $\mathrm{L}^\omega_{\infty\omega}$.

Definition 4. *Let k be a positive integer and let $\mathbf{A}$ be a structure.*

- *If $(a_1, \ldots, a_k)$ is a tuple from the universe of $\mathbf{A}$, then the k-type of $(a_1, \ldots, a_k)$ is the collection of all formulas ψ of $\mathrm{L}^k_{\infty\omega}$ such that $\mathbf{A} \models \psi(a_1, \ldots, a_k)$.*
- *If $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are two tuples having the same k-type, then we say that $(a_1, \ldots, a_k)$ is $\mathrm{L}^k_{\infty\omega}$-equivalent to $(b_1, \ldots, b_k)$ on $\mathbf{A}$ and write $(\mathbf{A}, a_1, \ldots, a_k) \equiv^k_{\infty\omega} (\mathbf{A}, b_1, \ldots, b_k)$.*

It can be shown that each k-type is equivalent to a formula of $\mathrm{L}^k_{\infty\omega}$. Dawar et al. [DLW92] established that each k-type is actually definable by a formula of $\mathrm{L}^k_{\omega\omega}$. Moreover, it was shown in [DLW92,KV92a] (and implicitly in [AV91]) that for each $k \geq 1$ there is a formula of fixpoint logic LFP that defines $\mathrm{L}^k_{\infty\omega}$-equivalence. These results make use of the fact that $\mathrm{L}^k_{\infty\omega}$-equivalence has a combinatorial characterization in terms of k-pebble games (cf. [Bar77,Imm82]).

Abiteboul and Vianu [AV91] showed that for every $k \geq 1$ there is a formula of fixpoint logic that defines a linear order $<_k$ on (the representatives of) the equivalence classes of $\mathrm{L}^k_{\infty\omega}$-equivalence (cf. also [DLW92]). This is the main technical tool used by Abiteboul and Vianu [AV91] in establishing that the separation of LFP from PFP is equivalent to the separation of PTIME from PSPACE.

2.3 Implicit Definability

So far, we have been considering *explicit definability* of queries in a logic L. The concept of *implicit definability* in a logic L arises when the value of a query is the unique relation that satisfies a sentence of L.

Definition 5. *Let L be a logic, and $(Q_1, \ldots, Q_n)$ a sequence of queries. We say that the sequence $(Q_1, \ldots, Q_n)$ is implicitly definable in L if there is a sentence $\varphi(P_1, \ldots, P_n)$ of L over the vocabulary $\sigma \cup \{P_1, \ldots, P_n\}$ such that for every structure $\mathbf{A}$ the sequence $(Q_1(\mathbf{A}), \ldots, Q_n(\mathbf{A}))$ is the only sequence of relations $(S_1, \ldots, S_n)$ on A such that $\mathbf{A} \models \varphi(S_1, \ldots, S_n)$.*

We write $\mathrm{IMP}(L)$ *to denote the collection of all queries Q such that $Q = Q_1$ for some sequence $(Q_1, \ldots, Q_n)$ of queries which is implicitly definable in L.*

The classical Beth definability theorem (cf. [CK90]) asserts that if a query is implicitly definable in first-order logic on all *finite and infinite* structures over a vocabulary σ, then it is also first-order definable on all finite and infinite structures over σ. Gurevich [Gur84] pointed out that Beth's theorem fails on finite structures, which means that the class FO of first-order definable queries is properly contained in IMP(FO). The expressive power and computational strength of first-order implicit definability on finite structures was investigated by Kolaitis [Kol90], where it was shown that LFP $\subseteq$ IMP(FO) and IMP(FO) $=$ IMP(LFP). Moreover, in [Kol90] it was established that if $\mathcal{C}$ is a class of ordered structures, then $\mathrm{IMP(FO)}(\mathcal{C}) = \mathrm{UP} \cap \text{co-UP}$ on $\mathcal{C}$, where UP is the class of *unambiguous* NP problems introduced by Valiant [Val76]. Thus, separating LFP from IMP(FO) on ordered structures is equivalent to separating PTIME from UP $\cap$ co-UP.

In view of the above, in [Kol90] the question was raised of whether LFP is properly contained in IMP(FO) on the class of all finite structures. This question is settled in the next section.

3 Fixpoint Logic vs. First-Order Implicit Definability

Recently, Gurevich and Shelah [GS94] constructed certain finite structures, called *multipedes*, and used them to refute a conjecture of Dawar [Daw93] stating that if $\mathcal{C}$ is a first-order definable class of rigid finite structures, then there is a formula of fixpoint logic that defines a linear order on every member of $\mathcal{C}$. In the next definition, we present a somewhat different, though essentially equivalent, variant of multipedes.

Definition 6. *A multipede is a structure $(U, \preceq, L, H_1, H_2, \varepsilon)$ such that the following hold.*

- *The universe U is the disjoint union of three sets S, F, and P.*
- *$\preceq$ is a linear order on S.*
- *$L \subseteq F \times S$ is a binary relation with the following properties: for every $f \in F$ there is exactly one s such that $(f, s) \in L$; for every $s \in S$ there are exactly two elements f_1 and f_2 in F such that $(f_i, s) \in L,\ i = 1, 2$.*
  If $(f_i, s) \in L,\ i = 1, 2$, then we say that f_i is a foot of s; we also say that s is the segment corresponding to $f_i,\ i = 1, 2$.
- *There is a one-to-one correspondence between P and the powerset of S; moreover, ε is the corresponding membership relation between elements of S and elements of P.*
- *H_1 is a ternary relation on S such that if $(s_1, s_2, s_3) \in H_1$, then the three elements are distinct and for every permutation π of the set $\{1, 2, 3\}$ we have that $(s_{\pi(1)}, s_{\pi(2)}, s_{\pi(3)}) \in H_1$.*
  We treat H_1 as a collection of three-element subsets of S.
- *H_2 is a ternary relation on F satisfying the following conditions: for every triple in H_2, the triple of corresponding segments is in H_1; for every $h \in H_1$, exactly four triples are in H_2 out of the eight triples that can be formed by choosing exactly one foot of each element of h; moreover, these four triples are such that any one of them can be obtained from any other by exchanging the feet of exactly two elements of h.*

A multipede is *odd* if for every non-empty subset X of S there is an $h \in H_1$ such that either $h \subseteq X$ or $|X \cap h| = 1$. Gurevich and Shelah [GS94] showed that the class of odd multipedes is first-order definable and that every odd multipede is a *rigid* structure, which means that the identity function is its only automorphism. Moreover, using a difficult probabilistic construction, Gurevich and Shelah established the following result.

Theorem 1. **[GS94]** *There is no formula of $\mathrm{L}^{\omega}_{\infty\omega}$ that defines a linear order on the class of odd multipedes.*

In contrast, we can show the following:

Lemma 1. *There is a first-order formula $\varphi(\leq)$ which implicitly defines a linear order on all odd multipedes.*

Proof: (*Sketch*) We can write a first-order formula stating that:

1. all elements of S occur before any element of P and all elements of P are before any element of F in the linear order $\leq$;
2. $\leq$ coincides on S with the linear order $\preceq$, while on P it is the linear order generated from $\preceq$ by the lexicographic order on the powerset of S;
3. for every two elements f and f' of F we have that $f \leq f'$ if the segment corresponding to f in the order $\preceq$ occurs before the segment corresponding to f' in $\preceq$.

Any two linear orders that satisfy the above conditions are distinguished only by the choice of ordering the feet of individual segments. That is, for any two such orders there is a set $X \subseteq S$ such that one order is obtained from the other by exchanging the feet of exactly the segments in X.

We associate with each order $\leq$ satisfying the above conditions a subset G of H_1 defined as follows. For each $h = \{a, b, c\} \in H_1$, let (a_1, a_2), (b_1, b_2), and (c_1, c_2) be the pairs of feet of a, b, and c respectively, where $a_1 \leq a_2$, $b_1 \leq b_2$ and $c_1 \leq c_2$. We then put $h \in G$ if and only if $\{a_1, b_1, c_1\} \in H_2$.

It can be verified that two distinct choices of order cannot yield the same set G, since otherwise the set X as defined above would witness that the multipede is not odd. Now, we can write a first-order formula $\varphi(\leq)$ asserting that the set G obtained from the order $\leq$ is lexicographically minimal among all linear orders satisfying conditions 1–3 above. ∎

From Theorem 1 and Lemma 1 it follows immediately that $\mathrm{IMP(FO)} \not\subseteq \mathrm{L}^{\omega}_{\infty\omega}$. Thus, we have established the following separations.

Theorem 2. *The class of fixpoint queries is properly contained in the class of first-order implicitly definable queries on finite structures, i.e.,* $\mathrm{LFP} \subset \mathrm{IMP(FO)}$. *Moreover,* $\mathrm{IMP(FO)} \cap \mathrm{L}^{\omega}_{\infty\omega} \subset \mathrm{IMP(FO)}$.

Theorem 2 settles the question raised in [Kol90] about the relationship between fixpoint logic and first-order implicit definability on finite structures. On the other hand, an examination of the proof that $\mathrm{LFP} \subseteq \mathrm{IMP(FO)}$ in [Kol90] reveals that a seemingly stronger containment holds, namely for every fixpoint query Q there is another fixpoint query Q' such that the pair (Q, Q') is first-order implicitly definable on finite structures. In other words, not only is every fixpoint query a member of a sequence which is first-order implicitly definable, but actually the other members of this sequence can be taken to be fixpoint queries (and, hence, $\mathrm{L}^{\omega}_{\infty\omega}$ queries) as well. This observation motivates the following definition.

Definition 7. *If L is one of the logics* FO, LFP, PFP, *then* $\mathrm{IMP}(L)|\mathrm{L}^{\omega}_{\infty\omega}$ *is the class of queries Q that are definable in $\mathrm{L}^{\omega}_{\infty\omega}$ and such that there are $\mathrm{L}^{\omega}_{\infty\omega}$-definable queries $Q_1, \ldots, Q_n$ and a formula φ of L which implicitly defines the sequence of queries $(Q, Q_1, \ldots, Q_n)$.*

It is important to differentiate between $\mathrm{IMP}(L)|\mathrm{L}^{\omega}_{\infty\omega}$ and $\mathrm{IMP}(L) \cap \mathrm{L}^{\omega}_{\infty\omega}$. It is clear that $\mathrm{IMP}(L)|\mathrm{L}^{\omega}_{\infty\omega} \subseteq \mathrm{IMP}(L) \cap \mathrm{L}^{\omega}_{\infty\omega}$, but, as will be shown later on, this containment may be a proper one. Thus, a query Q may be both in $\mathrm{L}^{\omega}_{\infty\omega}$ and in $\mathrm{IMP}(L)$, but there may be no L-implicitly definable sequence $(Q, Q_1, \ldots, Q_n)$ of queries such that $Q_1, \ldots, Q_n$ are *all* $\mathrm{L}^{\omega}_{\infty\omega}$-definable. It should also be noted that if $\mathcal{C}$ is a class of ordered structures, then $\mathrm{IMP}(L)|\mathrm{L}^{\omega}_{\infty\omega}(\mathcal{C})$ is the same as $\mathrm{IMP}(L)(\mathcal{C})$, since in this case every query on $\mathcal{C}$ is definable in $\mathrm{L}^{\omega}_{\infty\omega}$.

By combining Theorem 2 with the remarks preceding Definition 7, we obtain the following picture

$$\mathrm{LFP} \subseteq \mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega} \subseteq \mathrm{IMP(FO)} \cap \mathrm{L}^{\omega}_{\infty\omega} \subset \mathrm{IMP(FO)}.$$

A natural question arises whether the containment of LFP in $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega}$ is proper. In view of the above, this can be construed as the "right" way or the "fair" way to ask if the containment of fixpoint logic in first-order implicit definability is a proper one. It should be pointed out that an answer to this new question cannot be obtained from the proof of Theorem 2, because the IMP(FO) query that defines a linear order on odd multipedes is not definable in $\mathrm{L}^{\omega}_{\infty\omega}$. Our next result shows that the problem of separating LFP from $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega}$ is equivalent to a major open problem in computational complexity that we encountered earlier.

Theorem 3. $\mathrm{LFP} = \mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega}$ *if and only if* $\mathrm{PTIME} = \mathrm{UP} \cap \text{co-UP}$.

Proof: (*Hint*) The proof uses the methods developed by Abiteboul and Vianu [AV91] to show that LFP = PFP if and only if PTIME = PSPACE. More specifically, by restricting first-order implicit definability to $\mathrm{L}^{\omega}_{\infty\omega}$, we can work with $\mathrm{L}^{k}_{\infty\omega}$-equivalence classes and take advantage of the existence of a fixpoint definable linear order on k-types. In addition, we apply the result in [Kol90] asserting that $\mathrm{IMP(FO)} = \mathrm{UP} \cap \text{co-UP}$ on classes of ordered structures. ∎

Several remarks are in order now. First, it should be pointed out that results similar to Theorem 3 can be established about the relationship of $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega}$ to other effective fragments of $\mathrm{L}^{\omega}_{\infty\omega}$. In particular, using the same technique, we can show that $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega} = \mathrm{PFP}$ if and only if $\mathrm{UP} \cap \text{co-UP} = \mathrm{PSPACE}$. Thus, Theorem 3 and its extensions provide a new manifestation that separation problems between complexity classes are equivalent to separations of appropriate fragments of $\mathrm{L}^{\omega}_{\infty\omega}$ on the class of all finite structures, even if the fragments of $\mathrm{L}^{\omega}_{\infty\omega}$ do not capture the corresponding complexity classes. Moreover, these results indicate that the class $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega}$ should be regarded as the "relational" version of the complexity class $\mathrm{UP} \cap \text{co-UP}$.

Although the separation of LFP from $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega}$ does not appear to be within reach, it turns out that we can separate LFP from $\mathrm{IMP(FO)} \cap \mathrm{L}^{\omega}_{\infty\omega}$. In order to achieve this, we need to find a query Q which is both in $\mathrm{L}^{\omega}_{\infty\omega}$ and in IMP(FO), but whose implicit definition appears to require in a crucial way an auxiliary query that is not definable in $\mathrm{L}^{\omega}_{\infty\omega}$.

Observe that if $\mathcal{C}$ is a class of structures in which there is a linear order definable in IMP(FO), then every query on $\mathcal{C}$ that is computable in $\mathrm{UP} \cap \text{co-UP}$

is also definable in IMP(FO), using the linear order as an "auxiliary query". Consequently, on such a class $\mathcal{C}$ we have the containment $\mathrm{L}^{\omega}_{\infty\omega} \cap \mathrm{PTIME} \subseteq \mathrm{IMP(FO)} \cap \mathrm{L}^{\omega}_{\infty\omega}$. In [DLW92], it is proved that LFP is properly contained in $\mathrm{L}^{\omega}_{\infty\omega} \cap \mathrm{PTIME}$ by constructing an appropriate query on complete binary trees. The size of each complete binary tree is exponential in its height, but, by a result of Lindell [Lin91], the number of distinct automorphism types, and, a fortiori, the number of distinct k-types is only polynomial in the height of the tree. However, it is obvious that there is no definable linear order on complete binary trees, as they are not rigid. Nevertheless, we are able to combine the necessary properties of binary trees with those of multipedes in order to obtain structures that are rigid, but where the number of distinct k-types is logarithmic in the size of the structure. In these structures, called *tree multipedes*, each segment, instead of having two feet, is the root of a complete binary tree. Moreover, we replicate the relations H_1 and H_2 at all levels of the resulting forest.

As with multipedes, we can show that every odd tree multipede is rigid, that the class of odd tree multipedes is definable in LFP and that there is an FO formula that implicitly defines a linear order in every odd tree multipede. We can also show that for every k and n, there is an odd tree multipede of height n in which no two siblings are distinguished by a formula of $\mathrm{L}^{\omega}_{\infty\omega}$. Together these facts enable us to establish the following theorem.

Theorem 4. $\mathrm{LFP} \subset \mathrm{IMP(FO)} \cap \mathrm{L}^{\omega}_{\infty\omega}$.

We conjecture that the above Theorem 4 can be strengthened to show that $\mathrm{IMP(FO)}|\mathrm{L}^{\omega}_{\infty\omega} \subset \mathrm{IMP(FO)} \cap \mathrm{L}^{\omega}_{\infty\omega}$. At present, we can obtain this separation only modulo the complexity-theoretic assumption that $\mathrm{UP} \cap \text{co-UP}$ is properly contained in $\mathrm{UEXP} \cap \text{co-UEXP}$.

4 Implicit Definability in Partial Fixpoint Logic

Next, we focus on partial fixpoint logic and implicit definability in this logic. It should be clear from the definitions that the following sequence of inclusions holds:

$$\mathrm{PFP} \subseteq \mathrm{IMP(PFP)}|\mathrm{L}^{\omega}_{\infty\omega} \subseteq \mathrm{IMP(PFP)} \cap \mathrm{L}^{\omega}_{\infty\omega} \subseteq \mathrm{IMP(PFP)}.$$

As it turns out, it is possible to determine which of these inclusions are proper.

Theorem 5. $\mathrm{PFP} = \mathrm{IMP(PFP)}|\mathrm{L}^{\omega}_{\infty\omega}$.

Proof: (*Hint*) Abiteboul and Vianu [AV91] showed that there is a formula of LFP that defines a linear order $<_k$ on the representatives of $\equiv^k_{\infty\omega}$-equivalence classes. We use this formula to construct a formula φ of PFP such that on every structure the stages of φ cycle through all $\mathrm{L}^{k}_{\infty\omega}$-definable relations. We can then check at each stage whether the relation satisfies the condition specified by the implicit definition. ∎

Theorem 5 can be viewed as a Beth definability theorem for partial fixpoint logic on finite structures. As such, it should be compared and contrasted with the earlier Theorem 3, which reveals that the analogous theorem for fixpoint logic is true if and only if UP $\cap$ co-UP collapses to PTIME, an unlikely complexity-theoretic eventuality. Thus, Theorems 3 and 5 indicate yet another difference between fixpoint logic and partial fixpoint logic on finite structures.

Finally, we complete the picture for partial fixpoint logic by establishing the following proper containments.

Theorem 6. PFP $\subset$ IMP(PFP)$\cap \mathrm{L}^{\omega}_{\infty\omega}$ *and* IMP(PFP)$\cap \mathrm{L}^{\omega}_{\infty\omega} \subset$ IMP(PFP).

Proof: (*Hint*) The first separation is established along the lines of Theorem 4 by showing that the inclusion PFP $\subseteq \mathrm{L}^{\omega}_{\infty\omega} \cap$ PSPACE is proper on the class of odd tree multipedes. The second separation follows from Theorem 2. ∎

5 Conclusions

In this paper, we investigate the relationship between the infinitary logic $\mathrm{L}^{\omega}_{\infty\omega}$ and implicit definability in effective fragments of this logic on finite structures.

We were able to obtain the following sharp picture that delineates the relationship between partial fixpoint logic and implicit definability in partial fixpoint logic on finite structures:

$$\mathrm{PFP} = \mathrm{IMP(PFP)}|\mathrm{L}^{\omega}_{\infty\omega} \subset \mathrm{IMP(PFP)} \cap \mathrm{L}^{\omega}_{\infty\omega} \subset \mathrm{IMP(PFP)}.$$

Concerning the relationship between fixpoint logic and implicit definability, we establish that LFP $\subseteq$ IMP(FO)$|\mathrm{L}^{\omega}_{\infty\omega} \subseteq$ IMP(FO)$\cap\mathrm{L}^{\omega}_{\infty\omega} \subset$ IMP(FO) and LFP $\subset$ IMP(FO) $\cap\, \mathrm{L}^{\omega}_{\infty\omega}$.

This picture is not as sharp as the one for partial fixpoint logic. However, we exhibited good reasons that prevent us from making it sharper, since we also demonstrated that further separations are intimately related to outstanding open problems in complexity theory, such as the separation of PTIME from UP $\cap$ co-UP.

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