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Properties of Almost All Graphs and Generalized Quantifiers*

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Abstract. We study 0-1 laws for extensions of first-order logic by Lindström quantifiers. We state sufficient conditions on a quantifier Q expressing a graph property, for the logic FO$[Q]$ – the extension of first-order logic by means of the quantifier Q – to have a 0-1 law. We use these conditions to show, in particular, that FO[Rig], where Rig is the quantifier expressing rigidity, has a 0-1 law. We also show that extensions of first-order logic with quantifiers for Hamiltonicity, regularity and self-complementarity of graphs do not have a 0-1 law. Blass and Harary pose the question whether there is a logic which is powerful enough to express Hamiltonicity or rigidity and which has a 0-1 law. It is a consequence of our results that there is no such regular logic (in the sense of abstract model theory) in the case of Hamiltonicity, but there is one in the case of rigidity. We also consider sequences of vectorized quantifiers, and show that the extensions of first-order logic obtained by adding such sequences generated by quantifiers that are closed under substructures have 0-1 laws. The positive results also extend to the infinitary logic with finitely many variables.

1. Introduction

The study of random graphs in combinatorics has focused attention on the asymptotic probabilities of graph properties. Informally, the asymptotic probability $\mu(P)$ of a graph property P is the limit, as n goes to infinity, of the proportion of graphs of cardinality n that satisfy P, if this limit exists. It turns out that many interesting properties of graphs have asymptotic probability 0 or 1. Intuitively, this means that such properties either hold in almost all (finite) graphs, or they hold in almost none of them. For instance, almost all graphs are connected, rigid, and Hamiltonian, whereas almost no graph is planar or

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k-colourable, for any fixed k (see [4]). In contrast, it is clear that evenness – the property of the order of a graph being even – does not have an asymptotic probability.

The study of the *logical* properties of random structures has focused on the existence of 0-1 laws, and other limit laws, for a variety of logics. We say that a logic L has a 0-1 law if, for every property that is expressible by a sentence of L, the asymptotic probability is defined and is either 0 or 1. Glebskiĭ *et al.* [12] and Fagin [10] independently showed that first-order logic has a 0-1 law. Such laws have also been established for fragments of second-order logic [17], extensions of first-order logic by inductive operators [1, 21, 22] and the infinitary logic with finitely many variables [18] (see [5] for a survey of results on 0-1 and limit laws).

Most of the known 0-1 laws in logic are proved by means of *extension axioms*. For atomic types s, t where $s \subseteq t$, the s-t-extension axiom is a first-order sentence stating that every tuple realizing the type s can be extended to a tuple realizing t. It is an elementary exercise in probability theory to show that each extension axiom has asymptotic probability 1. For graphs this amounts to saying that for all $k \leq m$ and all collections $v_1, \dots, v_m$ of m distinct nodes there almost surely exists a node w with an edge to each of $v_1, \dots, v_k$ but to none of $v_{k+1}, \dots, v_m$. Since every extension axiom holds in almost all graphs, the same is true for any property that is a consequence of a finite collection of extension axioms.

A number of properties that hold for almost all graphs can be explained by means of extension axioms, even for properties that are not definable in first-order logic. For instance, the property of having diameter two is implied by the conjunction of two extension axioms. As a consequence, we obtain that $\mu(\text{connectivity}) = 1$, even though connectivity is not a first-order property. Similarly, given any graph H, there is a finite collection of extension axioms implying that almost all graphs contain H as an induced subgraph. Thus, every non-trivial property which is closed under taking subgraphs has asymptotic probability 0; in particular this proves that $\mu(\text{planarity}) = 0$ and $\mu(k\text{-colourability}) = 0$.

However, there are important graph properties which have asymptotic probability 0 or 1 and for which this does not follow from the extension axioms, the most notable being Hamiltonicity and rigidity. Blass and Harary [3] prove that there is no first-order sentence with asymptotic probability 1 which implies either Hamiltonicity or rigidity.

We can thus divide the properties of almost all graphs, or more generally, properties of almost all finite structures, into three classes:

(1) Properties expressible in first-order logic. This includes only rather simple properties, such as having diameter $\leq k$, for any fixed k, or containing a copy of H as an induced subgraph, for any fixed graph H.

(2) Properties that are not expressible in first-order logic, but are a consequence of a finite collection of extension axioms. It has turned out that many, perhaps most, interesting properties of almost all graphs fall into this class.

(3) Properties that hold for almost all finite graphs, but are not implied by any finite collection of extension axioms (or equivalently, by any first-order sentence with asymptotic probability 1). This class includes Hamiltonicity and rigidity [3], but also other properties such as not being self-complementary, having two nodes of different degrees (non-regularity), and not having an Euler tour [2].

We focus here on the third class. To prove that a property of almost all graphs (such as Hamiltonicity or rigidity) is not implied by extension axioms, one typically proceeds as follows:

Construct a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs that do *not* have the property, and prove that every finite collection of extension axioms is satisfied by some graph G_n. A variation of this technique is based on probabilistic arguments: Construct a sequence $\Gamma = (\Gamma_n)$ of probability spaces such that the property in question has asymptotic probability 0 on Γ, and prove that the extensions axioms have asymptotic probability 1 on Γ.

This second technique has been used by Blass and Harary for Hamiltonicity and rigidity. We briefly sketch the argument for rigidity. Let Γ_n be the space of random graphs with vertex set $V = \{1, \dots, n\} \cup \{-1, \dots, -n\}$ such that whenever i, j are adjacent, then so are $-i, -j$. For every potential pair of edges (i, j), $(-i, -j)$ decide whether to include none or both in G by tossing a fair coin. No graph in $G \in \Gamma_n$ is rigid, since $\pi(i) = -i$ is a non-trivial automorphism of G. However, all extension axioms have asymptotic probability 1 on Γ.

The first technique works, in particular, for properties of Paley graphs. Let p be a prime with $p \equiv 1 \pmod 4$; then the Paley graph of order p has vertices $0, \dots, p-1$, so that x and y are connected by an edge if, and only if, $x - y$ is a quadratic residue modulo p. Notice that, since $p \equiv 1 \pmod 4$, the field $\mathbb{F}_p$ has a square root of -1, so $x - y$ is a quadratic residue if, and only if, $y - x$ is one. Hence Paley graphs are indeed undirected graphs. The simplest Paley graph is C_5, the cycle of length five.

It is known that Paley graphs are regular, self-complementary (by taking x to $xa \pmod p$ where a is any non-square in $\mathbb{F}_p$) and Eulerian. Notice that all of these are properties that hold for almost no graph. However, it has been shown in [2] that the Paley graph of order p satisfies the extension axioms with $\leq n$ vertices on each side provided that $p > n^2 2^{4n}$. This proves that having vertices of distinct degrees, not being self-complementary and not having an Euler tour are properties of almost all graphs that are not implied by finitely many extension axioms.

We study the question, posed by Blass and Harary [3] of whether there is any natural logic that can express properties in this class (such as Hamiltonicity or rigidity) and for which a 0-1 law holds. This problem is also commented on in an informal way and reported as "still wide open" in [15]. Of course, we need to clarify what is meant by a "natural logic". A trivial solution to the problem posed by Blass and Harary would be to add Hamiltonicity or rigidity as a sentence to first-order logic. However, such a "logic" would lack the most basic closure properties that one usually requires from a logical system. In model theory, the notion of a regular logic has been introduced, in order to make precise ideas of what constitutes a natural extension of first-order logic. A regular logic can be described as a logic that can express all atoms, and is closed under negation, conjunction, particularization (or existential quantification), relativization and substitution (see [8]).

Here substitution means the replacement of basic relations by defined ones: if the logic can express a property P of the structures $\mathfrak{A}$ on which it is evaluated, then it should also be able to express that the property P is satisfied in the modified structure that is obtained from $\mathfrak{A}$ by replacing a basic relation R by a defined relation $\varphi^{\mathfrak{A}} := \{\bar{a} : \mathfrak{A} \models \varphi(\bar{a})\}$, for any appropriate formula $\varphi(\bar{x})$ of the logic. This is intimately related to the notion of a *generalized quantifier* or *Lindström quantifier*. For any class K of structures $\mathfrak{A} = (A, R_1, \dots, R_k)$, which is closed under isomorphism, we may add a quantifier Q_K to first-order logic, to obtain the extension FO$[Q_k]$ that allows for the construction of formulae $Q_K\bar{x}(\psi_1, \dots, \psi_k)$ saying that the structure defined by $\psi_1, \dots, \psi_k$ belongs to the class K. For a more detailed definition, see Sect. 2.2 below. It is easy to see that, for any class K of σ-structures, FO$[Q_K]$ is the minimal logic closed under negation, conjunction, particularization and substitution that can express K.

However, notice that FO$[Q_K]$ is not necessarily regular since it need not be closed under relativization. Here relativization means the restriction to definable substructures. To obtain a regular logic one needs to modify the notion of a generalized quantifier, adding to the scope of the quantifier a formula $\delta(x)$ that defines the universe. Then $Q_K\bar{x}(\delta, \psi_1, \dots, \psi_k)$ expresses that the structure whose universe is defined by δ and whose relations are defined by $\psi_1, \dots, \psi_k$ lies in the class K.

To address the question posed by Blass and Harary, we investigate 0-1 laws for extensions of first-order logic by Lindström quantifiers. Such extensions were also considered, from the point of view of 0-1 laws by Fayolle *et al.* [11], where a sufficient condition was established on a quantifier Q, for a restricted fragment of the logic FO$[Q]$ to have a 0-1 law. We extend such results and formulate other sufficient conditions on quantifiers Q associated with graph properties which guarantee that the logic FO$[Q]$ has a 0-1 law in the language of graphs.

We use our conditions to establish, in particular, that FO[Rig] has a 0-1 law, where Rig is the quantifier associated with the class of rigid graphs. By contrast, for all the other mentioned properties K that hold for almost all graphs but are not implied by extension axioms, we find examples showing that FO$[Q_K]$ does not have a 0-1 law. We also extend the result for FO[Rig] to its closure under relativizations. This enables us to establish that there is no regular logic (in the sense in which this term is used in abstract model theory) which can express Hamiltonicity, regularity, self-complementarity, or existence of Euler tours and which has a 0-1 law, but there is one in the case of rigidity.

We also consider, in Section 6 extensions of first-order logic by means of vectorized quantifiers. In particular, we show that for any quantifier that is closed under substructures, the corresponding extension of first-order logic by means of a vectorized sequence of quantifiers has a 0-1 law, greatly generalizing a result of [11]. This establishes 0-1 laws for the extensions of first-order logic by the sequences of quantifiers obtained by vectorizing the graph quantifiers for 3-colourability and planarity.

Finally, we show in Section 7 that our positive results on the existence of 0-1 laws can be extended from logics FO$[Q]$ to $L^{\omega}_{\infty\omega}[Q]$, i.e. the extension by means of the quantifier Q of the infinitary logic with finitely many variables. This applies both to the results where Q is a single quantifier and when it is vectorized.

2. Preliminaries

Let σ, τ be finite relational signatures. We denote structures by $\mathfrak{A}, \mathfrak{B}, \dots$ and their universes by $A, B, \dots$. Let Str(σ) and Str$_n(\sigma)$ denote, respectively, the set of all finite σ-structures and the set of all σ-structures with universe $[n] = \{0, \dots, n-1\}$. For a σ-structure $\mathfrak{A}$ and a formula $\psi(x_1, \dots, x_k)$, we write $\psi^{\mathfrak{A}}$ to denote $\{\bar{a} \in A^k : \mathfrak{A} \models \psi(\bar{a})\}$, i.e. the relation that ψ defines on $\mathfrak{A}$. Similarly, if ψ has additional free variables $\bar{y}$, then for any valuation $\bar{b}$ of those variables, we define $\psi^{\mathfrak{A},\bar{b}}$ as $\{\bar{a} \in A^k : \mathfrak{A} \models \psi(\bar{a}, \bar{b})\}$, i.e. the relation defined by ψ on $\mathfrak{A}$ by fixing the interpretation of the parameters $\bar{y}$ to be $\bar{b}$. For a sentence φ, we write Mod(φ) to denote the set of all (finite) models of φ. A structure $\mathfrak{B}$ is a substructure of $\mathfrak{A}$, if $B \subseteq A$, and the relations on $\mathfrak{B}$ are the restrictions of the corresponding relations on $\mathfrak{A}$ to the universe B.

Definition 2.1. An *atomic type in* $x_1, \dots, x_k$ *over* σ is a maximal consistent set of σ-atoms and negated σ-atoms in the variables $x_1, \dots, x_k$. Often, we call an atomic type in k variables a k-type. We denote atomic types by $t, t', s, \dots$ or by $t(x_1, \dots, x_k), \dots$ to display the variables. By abuse of notation, we do not distinguish between an atomic type and the conjunction over all formulae in it.

The following lemma is immediate.

Lemma 2.1. Every quantifier-free formula is equivalent to a disjunction of atomic types.

Proof:
Let $\varphi(x_1, \ldots, x_k)$ be a quantifier-free formula over σ. Then

$$\varphi(x_1, \ldots, x_k) \equiv \bigvee_{t \models \varphi} t(x_1, \ldots, x_k),$$

where t ranges over the atomic types in $x_1, \ldots, x_k$ over σ. □

2.1. Asymptotic Probabilities

Let $0 \leq p \leq 1$. A *Bernoulli trial with mean* p is a random variable X that takes only the values 0 and 1 and such that $P[X = 1] = p$.

Let $\Gamma(\sigma) = (\Gamma_n(\sigma))_{n \in \mathbb{N}}$ be a sequence of probability spaces over σ-structures, where $\Gamma_n(\sigma)$ is obtained by assigning a probability distribution μ_n to $\mathrm{Str}_n(\sigma)$. Some important examples are:

- $\Omega_n(\sigma, 1/2)$ denotes the probability space with the uniform probability distribution, i.e. every structure $\mathfrak{A} \in \mathrm{Str}_n(\sigma)$ has the same probability $\mu(\mathfrak{A}) = 1/|\mathrm{Str}_n(\sigma)|$.

- For arbitrary functions $p : \mathbb{N} \to [0, 1]$ we define the probability spaces $\Omega_n(\sigma, p)$ as follows: the truth of all instances $R(i_1, \ldots, i_r)$ of σ-atoms over universe $[n]$ are determined by independent Bernoulli trials with mean $p(n)$.

  It is clear that when p is the constant function $1/2$, this gives the uniform probability distribution. Where σ is understood, we simply write $\Omega(p)$ to denote the sequence of distributions $(\Omega_n(\sigma, p))_{n \in \mathbb{N}}$.

- $\mathcal{G}(n, p)$ is the probability space of random graphs with edge probability p (again p may depend on n). We write $\mathcal{G}(p)$ for the sequence $(\mathcal{G}(n, p))_{n \in \mathbb{N}}$. Note that $\mathcal{G}(n, p)$ is not the same space as $\Omega_n(\{E\}, p)$, since a graph is assumed to be undirected and loop–free.

For a fixed sequence $\Gamma(\sigma) = (\Gamma_n(\sigma))_{n \in \mathbb{N}}$ of probability spaces, define the probability $\mu_n(P)$ of a class P of σ-structures as the probability that a structure $\mathfrak{A}$ with universe $\{0, \ldots, n-1\}$ is in the class P. Define the asymptotic probability of P as $\mu(P) = \lim_{n \to \infty} \mu_n(P)$, if this limit exists. If the limit does not exist, we say that P has no asymptotic probability for $\Gamma(\sigma)$.

For any logic L, we define the asymptotic probability $\mu(\varphi)$ of a sentence of L to be $\mu(\mathrm{Mod}(\varphi))$. If every sentence of L in the vocabulary σ has an asymptotic probability for $\Gamma(\sigma)$, we say that L has a limit law for $\Gamma(\sigma)$. Furthermore, if $\mu(\varphi)$ is 0 or 1 for every σ-sentence φ of L, we say that L has a 0-1 law for $\Gamma(\sigma)$.

The following theorem is at the core of the proof due to Glebskiĭ *et al.* [12] that first-order logic has a 0-1 law (see also [13]). It can be seen as establishing an "almost sure" quantifier elimination property for the theory of finite structures.

Theorem 2.1. For every formula $\psi(\bar{x})$ of first-order logic, there is a quantifier free formula $\theta(\bar{x})$ such that the sentence $\forall \bar{x}(\theta \leftrightarrow \psi)$ has asymptotic probability 1 in $\Omega(p)$ for any constant p.

Furthermore, it is an easy adaptation to see that the same holds for the distributions $\mathcal{G}(p)$, for constant p. For formulae ψ and θ as in Theorem 2.1 above, we will say that ψ and θ are equivalent *almost everywhere*.

2.2. Interpretations and Quantifiers

Let the signature τ be $\{R_1, \ldots, R_m\}$ where R_i is a relation symbol of arity r_i. A sequence $\Psi = \psi_1(\bar{x}_1), \ldots, \psi_m(\bar{x}_m)$ of formulae of signature σ, where $\psi_i(\bar{x}_i)$ has the free variables $x_1, \ldots, x_{r_i}$ defines an interpretation

$$\begin{array}{rcl} \Psi : \mathrm{Str}(\sigma) & \to & \mathrm{Str}(\tau) \\ \mathfrak{A} & \mapsto & \Psi\mathfrak{A} = (A, \psi_1^{\mathfrak{A}}, \ldots, \psi_m^{\mathfrak{A}}). \end{array}$$

An *interpretation with parameters* is given by a sequence $\Psi(\bar{y}) = \psi_1(\bar{x}_1, \bar{y}), \ldots, \psi_m(\bar{x}_m, \bar{y})$ of σ-formulae ψ_i which may contain, besides $\bar{x}_i$, additional free variables $\bar{y}$. For any σ-structure $\mathfrak{A}$ and any valuation $\bar{a}$ for $\bar{y}$ we obtain an interpreted structure

$$\Psi(\mathfrak{A}, \bar{a}) = (A, \psi_1^{\mathfrak{A},\bar{a}}, \ldots, \psi_m^{\mathfrak{A},\bar{a}}).$$

The following definition of a generalized quantifier is essentially due to Lindström [20].

Definition 2.2. Let K be a collection of structures of some fixed signature τ, which is closed under isomorphisms, i.e. if $\mathfrak{A} \in K$ and $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{B} \in K$. With K we associate the *generalized quantifier* Q_K, which can be adjoined to first-order logic to form an extension FO$[Q_K]$, which is defined by closing FO under the following rule for building formulae:

If $\Psi(\bar{y}) = (\psi_1, \ldots, \psi_k)$ is an interpretation with parameters $\bar{y}$ from σ to τ then $Q_K\bar{x}(\psi_1, \ldots, \psi_k)$ is a formula of FO$[Q_K]$ of signature σ with free variables $\bar{y}$. Here $\bar{x}$ denotes the list of free variables of $\psi_1, \ldots, \psi_k$, not including the parameters $\bar{y}$.

The semantics of Q_K is given by the following rule: for a σ-structure $\mathfrak{A}$ and a valuation $\bar{a}$ for $\bar{y}$,

$$(\mathfrak{A}, \bar{a}) \models Q_K\bar{x}(\psi_1, \ldots, \psi_k) \iff \Psi(\mathfrak{A}, \bar{a}) \in K.$$

An interpretation Ψ from σ-structures to τ-structures also maps a probability space $\Gamma_n(\sigma)$ to a new probability space $\Psi\Gamma_n(\sigma)$ of τ-structures, defined by assigning to $\mathfrak{B} \in \mathrm{Str}_n(\tau)$ the probability

$$\nu(\mathfrak{B}) = \sum_{\Psi\mathfrak{A}=\mathfrak{B}} \mu(\mathfrak{A}),$$

where $\mu(\mathfrak{A})$ is the probability of $\mathfrak{A}$ in $\Gamma_n(\sigma)$.

On the other hand, if we are given $\Psi(\bar{y})$, an interpretation with parameters, it does not define a map from σ-structures to τ-structures. Rather, it defines a map from pairs $(\mathfrak{A}, \bar{a})$, where $\mathfrak{A}$ is a σ-structure and $\bar{a}$ is a valuation of the parameters $\bar{y}$ in $\mathfrak{A}$, to τ-structures. Thus, we will assume we are given a probability space $\Gamma_n(\sigma, \bar{y})$ that assigns a probability to $(\mathfrak{A}, \bar{a})$ for each $\mathfrak{A} \in \mathrm{Str}_n(\sigma)$ and each valuation $\bar{a}$ of the parameters $\bar{y}$ in $\mathfrak{A}$. We then define the probability space $\Psi\Gamma_n(\sigma, \bar{y})$ by assigning to $\mathfrak{B} \in \mathrm{Str}_n(\tau)$ the probability

$$\nu(\mathfrak{B}) = \sum_{\Psi(\mathfrak{A},\bar{a})=\mathfrak{B}} \mu(\mathfrak{A}, \bar{a}).$$

One of the goals of this paper is to elucidate the structure of $\Psi\Gamma_n(\sigma, \bar{y})$.

2.3. Graph quantifiers

In this paper, a graph always means a loop-free undirected graph $G = (V, E)$. A graph quantifier is a generalized quantifier given by an isomorphism closed class $\mathcal{H}$ of graphs. It is applied to interpretations that map graphs to graphs. Thus, a graph quantifier binds two variables, say x and y, and is applied to a single formula $\varphi(x, y, \bar{z})$ of signature $\{E\}$. A little complication arises because we have to make sure that the interpreted structure $\Phi(G, \bar{a})$ is indeed a graph. To avoid the necessity of verifying the semantic condition that a formula does indeed define an irreflexive and symmetric relation (a condition that has to be met for all valuations of the parameters), we impose no restriction on the formulae, but modify the interpretation of formulae.

Definition 2.3. For any class L of formulae over $\{E\}$ and any isomorphism closed class $\mathcal{H}$ of graphs, we define the logic $L[Q^G_{\mathcal{H}}]$ by closing L under the following rule: given any formula $\varphi(x, y, \bar{z})$, we can also build the formula $Q^G_{\mathcal{H}}x, y\ \varphi$, with free variables $\bar{z}$.

The semantics is given by the equivalence

$$Q^G_{\mathcal{H}}x, y\ \varphi \equiv Q_{\mathcal{H}}x, y\ (x \neq y \wedge (\varphi(x, y) \vee \varphi(y, x))).$$

(where $\varphi(y, x)$ is $\varphi(x, y)$ with variables x and y interchanged.)

For any formula $\varphi(x, y, \bar{z})$ we will refer to the interpretation with parameters $\Phi(\bar{z})$ defined by the formula $x \neq y \wedge (\varphi(x, y, \bar{z}) \vee \varphi(y, x, \bar{z}))$ as the *graph interpretation* associated with φ. We also call $Q^G_{\mathcal{H}}$ the graph quantifier associated with $\mathcal{H}$.

3. Properties of Almost All Graphs Not Implied by Extension Axioms

We now proceed to investigate conditions that can be imposed on a class of graphs $\mathcal{H}$ in order for the logic FO$[Q^G_{\mathcal{H}}]$ to have a 0-1 law. We begin, in this section, by formulating some necessary conditions and showing that they are not sufficient. We also obtain a necessary and sufficient condition for a certain fragment of FO$[Q^G_{\mathcal{H}}]$ to have a 0-1 law.

Further for the examples described above of properties of almost all graphs that are not implied by extension axioms, with the exception of rigidity, we construct sentences in the associated logic FO$[Q]$ that do not have an asymptotic probability, thus giving, for all these cases, a negative answer to the question of Blass and Harary.

For any property K, let FO$^r[Q_K]$ denote those sentences of FO$[Q_K]$ of the form $Q_K\bar{x}\varphi$, where φ is first-order, i.e. FO$^r[Q_K]$ can express exactly those properties that are reducible to K by means of a first-order interpretation *without* parameters. This fragment was considered by Fayolle *et al.* [11], who showed for any generalized quantifier Q_K, and any signature σ, a sufficient condition for the logic FO$^r[Q_K]$ to have a 0-1 law on the class of σ-structures is that K is monotone and closed under extensions. Clearly a necessary condition is that K itself has asymptotic probability 0 or 1, because the property K can be trivially expressed by a sentence of FO$^r[Q_K]$. We first show that this latter condition is not sufficient, by means of an example.

Example 3.1. Let $\mathcal{E}$ be the class of Eulerian graphs. It is well known that a connected graph G is Eulerian if, and only if, every vertex in G has even degree. Furthermore it follows from known results about degrees in random graphs (see [4], chapter 3) that $\mathcal{E}$ has asymptotic probability 0 for $\mathcal{G}(p)$ for any constant $0 \leq p < 1$. However, consider the following sentence of FO^r[Eul]:

$$\varphi \equiv (\text{Eul}\, x, y\ (x \neq y))$$

It is clear that a graph G on n satisfies φ if, and only if, K_n, the complete graph on n vertices is Eulerian. This is true if, and only if, n is odd. It follows that φ does not have an asymptotic probability for $\mathcal{G}(p)$ for any p.

The above example shows that a graph property might have asymptotic probability 0 or 1 for $\mathcal{G}(p)$ for a fixed p, without the logic FO$^r[Q^G_\mathcal{H}]$, let alone FO$[Q^G_\mathcal{H}]$, having a 0-1 law. The next result establishes a necessary *and* sufficient condition for FO$^r[Q^G_\mathcal{H}]$ to have a 0-1 law for $\mathcal{G}(p)$.

Theorem 3.1. For any graph property $\mathcal{H}$, FO$^r[Q^G_\mathcal{H}]$ has a 0-1 law for $\mathcal{G}(p)$ if, and only if, $\mathcal{H}$ has asymptotic probability 0 or 1 for each of $\mathcal{G}(0)$, $\mathcal{G}(1)$, $\mathcal{G}(p)$ and $\mathcal{G}(1-p)$.

Proof:
Let $\varphi \equiv Q^G_\mathcal{H} x, y\ \psi$ be a sentence of FO$^r[Q^G_\mathcal{H}]$. By Theorem 2.1, there is a quantifier free formula θ that is equivalent to ψ almost everywhere. The asymptotic probability of φ is given by the asymptotic probability of $\mathcal{H}$ on $\Theta\mathcal{G}(p)$, where Θ is the graph interpretation defined by the formula θ.

Up to equivalence, there are only four quantifier-free formulae in two variables that define an irreflexive and symmetric relation:

True, False, Exy and $\neg Exy$.

Thus, $\Theta\mathcal{G}(p)$ is one of $\mathcal{G}(1)$, $\mathcal{G}(0)$, $\mathcal{G}(p)$ or $\mathcal{G}(1-p)$. □

In Example 3.1 above, the class of Eulerian graphs does not have an asymptotic probability defined for $\mathcal{G}(1)$. To take another example, recall that the class of Hamiltonian graphs has asymptotic probability 1 for $\mathcal{G}(p)$ for any constant $p > 0$ (see [4]). Since it is clear that this class has asymptotic probability 0 for $\mathcal{G}(0)$, it follows from Theorem 3.1 that FO^r[Ham] has a 0-1 law. We will show next that there is no such law for FO[Ham], which implies that the condition in Theorem 3.1 is, in general, not sufficient to establish a 0-1 law for the unrestricted logic FO$[Q^G_\mathcal{H}]$.

Example 3.2. Consider the sentence:

$$\varphi \equiv \exists z(\text{Ham}\, x, y\ \psi), \text{ where } \psi \equiv Exz \wedge \neg Eyz$$

The interpretation $\Psi(z)$ defined by ψ maps a pair (G, v) (where $G = (V, E)$ is a graph and v a distinguished vertex of G) to the complete bipartite graph $H = (V, E')$, where $E' = \{(a, b) \mid (v, a) \in E \text{ and } (v, b) \notin E\}$. Letting $D(v)$ denote the set $\{a \in V \mid (v, a) \in E\}$, it can be verified that the graph H has a Hamiltonian cycle if and only if $|D(v)| = |V - D(v)|$. In particular, if G is a graph of odd size, then it cannot satisfy the sentence φ. On the other hand, φ is true in a graph of cardinality $2n$ just in case the graph contains a vertex of degree n. But, as n goes to infinity this happens almost surely in $\mathcal{G}(p)$ for any constant $p > 0$ (see [4, Chapter 3]). We conclude that φ does not have an asymptotic probability for $\mathcal{G}(p)$.

Notice that precisely the same example applies to (non-)regularity. Indeed the bipartite graph $H = (V, E')$, defined by the interpretation $\Psi(z)$ from $G = (V, E)$, is regular if, and only if, it is Hamiltonian.

We next consider the case of self-complementary graphs.

Example 3.3. Recall that a graph $G = (V, E)$ is *self-complementary* if it is isomorphic to its complement $\bar{G} = (V, \bar{E})$ where $\bar{E}$ is the set of edges (u, v) such that $u \neq v$ and $(u, v) \notin E$.

A simple example of a self-complementary graph is P_4, the simple path on four vertices $v_0 - v_1 - v_2 - v_3$, as its complement is the path $v_1 - v_3 - v_0 - v_2$. We can generalize this example by considering four disjoint sets V_0, V_1, V_2 and V_3 each with n vertices and the graph G on the set of vertices $V_0 \cup V_1 \cup V_2 \cup V_3$ which induces a clique on each of V_1 and V_2 and an independent set on each of V_0 and V_3 and in addition has the edges $V_0 \times V_1$, $V_1 \times V_2$ and $V_2 \times V_3$. Then, this graph is self-complementary by any bijection that maps V_0 to V_1, V_1 to V_3, V_2 to V_0 and V_3 to V_2.

Let SC be the quantifier for self-complementarity and consider the sentence

$$\varphi \equiv \exists z_1 \exists z_2 (\mathrm{SC}\, x, y\ \psi)$$

where

$$\psi \equiv \begin{array}{l} \big(E(x, z_1) \wedge \neg E(x, z_2) \wedge E(y, z_1) \wedge \neg E(y, z_2)\big) \vee \\ \big(\neg E(x, z_1) \wedge E(x, z_2) \wedge \neg E(y, z_1) \wedge E(y, z_2)\big) \vee \\ \big(\neg E(x, z_1) \wedge \neg E(x, z_2) \wedge E(y, z_1) \wedge \neg E(y, z_2)\big) \vee \\ \big(E(x, z_1) \wedge \neg E(x, z_2) \wedge \neg E(y, z_1) \wedge E(y, z_2)\big) \vee \\ \big(\neg E(x, z_1) \wedge E(x, z_2) \wedge E(y, z_1) \wedge E(y, z_2)\big) \end{array}$$

Given a graph $G = (V, E)$ and two vertices u and v interpreting the parameters z_1 and z_2 respectively, the interpretation Ψ defined by ψ gives a graph which consists of a clique on each of V_1 and V_2 and an independent set on each of V_0 and V_3 and in addition has the edges $V_0 \times V_1$; $V_1 \times V_2$ and $V_2 \times V_3$, where V_0 is the set of vertices that are adjacent to neither of u and v; V_1 contains the vertices adjacent to u but not v; V_2 contains the vertices adjacent to v but not u; and V_3 contains the vertices adjacent to both u and v. Thus, in particular, if the four sets of vertices have the same size, the graph $\Psi(G, u, v)$ is self-complementary. Now, it follows from a simple probability calculation (such as in [4, Chapter 3]) that in any graph with $4n$ vertices we can find, with probability tending to 1, two vertices u and v, each of which has $2n$ neighbours and so that there are exactly n vertices that are neighbours of both. Hence, the sentence φ is true almost surely on any graph with $4n$ vertices.

On the other hand, note that for any graph G and pair of vertices u, v, in $\Psi(G, u, v)$, the set $V_1 \cup V_2$ induces a clique and $V_0 \cup V_3$ induces an independent set. If $\Psi(G, u, v)$ is self-complementary by a bijection π, the image of $V_1 \cup V_2$ under π must be an independent set in $\Psi(G, u, v)$ and similarly the image of $V_0 \cup V_3$ under π must be a clique. Thus, we must have $|V_1 \cup V_2| = |V_0 \cup V_3|$. We conclude that $\Psi(G, u, v)$ cannot be self-complementary if G has odd order. Thus, φ does not have an asymptotic probability for $\mathcal{G}(p)$ for any constant p.

The following result follows immediately from the above examples, since any regular logic that can express that a graph has an Eulerian tour will have a sentence equivalent to the one in Example 3.1 and similarly any regular logic that can express Hamiltonicity will have one equivalent to the sentence in Example 3.2, etc.

Theorem 3.2. There are no regular logics that can express any of the following properties

- the existence of an Eulerian tour;
- the existence of a Hamiltonian cycle;
- regularity of a graph; and
- self-complementarity of a graph

and which have a 0-1 law for $\mathcal{G}(1/2)$.

Indeed, we can strengthen the theorem by replacing "regular logics" with weaker closure conditions on the logics which can be extracted from the examples. In particular, none of these properties can be expressed in a logic that has a 0-1 law and that is closed under *substitutions by quantifier-free formulas* and *existential quantification*. Thus, in particular, it follows that none of these properties is expressible in the fragments of second-order logic for which 0-1 laws have been proved in [17].

The contrast between the examples for Hamilton cycles and Euler tours also reveals that quantifier-free interpretations with parameters can be much more complex than those without parameters. We take up the analysis of the case with parameters in the next section.

4. Probability Spaces Defined by Graph Interpretations

In order to formulate a condition on the graph quantifiers Q which guarantees that the logic FO$[Q]$ has a 0-1 law, we will construct an argument by quantifier elimination. That is, we will state sufficient conditions on Q so that, for every quantifier free formula ψ, $Qx, y\,\psi$ is itself equivalent, almost everywhere, to a quantifier free formula. This, along with Theorem 2.1 will then enable us to derive the required result.

To establish this quantifier elimination, we consider the action of a quantifier-free interpretaion with parameters $\Psi(\bar{y})$ on a probability distribution that assigns probabilities to structures $(G, \bar{a})$. For this, we consider each atomic type of the tuple $\bar{a}$ separately. That is, for each atomic type t, we define a distribution $\mathcal{G}_{t,n}(p)$ which assigns a probability to each pair $(G, \bar{a})$, where $\bar{a}$ is a tuple of elements of G. This probability is 0 if $\bar{a}$ is not of type t in G, and otherwise it is the same for all tuples of type t in G.

More formally, let t be an atomic type in the variables $\bar{z} = z_1, \dots, z_m$ such that for each $1 \leq i < j \leq m$, $t \models z_i \neq z_j$. We denote by $\mathcal{G}_{t,n}(p)$ the probability space obtained from $\mathcal{G}_n(p)$ as follows: for each graph $G \in \mathcal{G}_n$, and each m-tuple $\bar{a}$ of elements in G, let the probability $\mu_{t,n}(G, \bar{a}) = 0$ if $G \not\models t[\bar{a}]$ and $\mu_{t,n}(G, \bar{a}) = \mu_n(G)/k$ otherwise, where k is the number of distinct tuples in G of type t. In other words, $\mu_{t,n}(G, \bar{a})$ is the probability of obtaining the pair $G, \bar{a}$ subject to the condition $G \models t(\bar{a})$ for edge probability p and equal probability for each tuple of type t. We write $\mathcal{G}_t(p)$ for any sequence of probability spaces $(\Gamma_n)_{n\in\mathbb{N}}$ such that Γ_n is $\mathcal{G}_{t,n}(p)$ for $n \geq m$.

Definition 4.1. Let Ψ be an interpretation in m parameters and t a type in m variables, as above. A class of graphs $\mathcal{H}$ *converges quickly* to 1 (resp. 0) for $\Psi\mathcal{G}_t(p)$ if $\mu_{t,n}(\mathcal{H}) = 1 - o(n^{-m})$ (resp. $o(n^{-m})$).

This definition enables us to formulate the following lemma.

Lemma 4.1. If $\psi(x, y, \bar{z})$ is a first-order formula, $\Psi(\bar{z})$ is the associated graph interpretation, and $\mathcal{H}$ is a class of graphs which converges quickly to 1 for $\Psi\mathcal{G}_t(p)$, then the sentence $\forall\bar{z}(t(\bar{z}) \rightarrow (Q^G_{\mathcal{H}} x, y\, \psi))$ has asymptotic probability 1 for $\mathcal{G}(p)$. Similarly, if $\mathcal{H}$ converges quickly to 0 for $\Psi\mathcal{G}_t(p)$, then the sentence $\exists\bar{z}(t(\bar{z}) \wedge (Q^G_{\mathcal{H}} x, y\, \psi))$ has asymptotic probability 0 for $\mathcal{G}(p)$.

Proof:
We sketch the proof for the universal case, the existential case being dual. The number of tuples of type t in a graph G of cardinality n tends to n^m/c for some constant c as n goes to infinity. Thus, if $\mathcal{H}$ converges quickly to 1 on $\Psi\mathcal{G}_t(p)$, then in almost all graphs, for all tuples $\bar{a}$ of type t, $\Psi(G, \bar{a}) \in \mathcal{H}$. Hence $\forall\bar{z}(t(\bar{z}) \rightarrow (Q^G_{\mathcal{H}} x, y\, \psi))$ has asymptotic probability 1 for $\mathcal{G}(p)$. □

Lemma 4.1 is used in proving the next result, which defines the conditions for one step of our quantifier elimination.

Lemma 4.2. If ψ is a formula defining a graph interpretation $\Psi(\bar{z})$ in m parameters, and $\mathcal{H}$ is a class of graphs which converges quickly to 0 or 1 for $\Psi\mathcal{G}_t(p)$, for every m-type t, then there is a quantifier free formula $\theta(\bar{z})$ such that the sentence $\forall\bar{z}(\theta \leftrightarrow Q^G_{\mathcal{H}} x, y\, \psi)$ has asymptotic probability 1 for $\mathcal{G}(p)$.

Proof:
Let $\theta \equiv \bigvee\{t(\bar{z}) : \mathcal{H}$ converges quickly to 1 for $\Psi\mathcal{G}_t(p)\}$. □

We now proceed to study the structure of the spaces $\Psi\mathcal{G}_t(p)$. By Theorem 2.1, it suffices to consider the case where the interpretation $\Psi(\bar{z})$ is given by a quantifier-free formula. For the remainder of this section, we will also confine ourselves to the case where p is the constant function $1/2$.

Let $\psi(x, y, \bar{z})$ be a quantifier free formula defining a graph interpretation $\Psi(\bar{z})$ with m parameters. By Lemma 2.1, ψ is equivalent to a disjunction of $(m+2)$-types. Let S be the collection of the types in this disjunction that extend the m-type t. Clearly, $\Psi\mathcal{G}_t(1/2)$ is completely determined by which types are in S.

Furthermore, if s is a type in the variables $x, y, \bar{z}$ extending $t(\bar{z})$, then s is determined by its subtypes $s_1(x, \bar{z})$, $s_2(y, \bar{z})$ and whether or not $s \models Exy$. Moreover, in the case where either $s \models x = z_i$ or $s \models y = z_i$ for some i, the last of these is already determined by the two $(m+1)$-types s_1 and s_2. Thus, given two $(m+1)$-types s_1 and s_2 extending t, there may be one or two $(m+2)$-types consistent with s_1 and s_2. For our purposes, we can identify a set S of $(m+2)$-types extending t with a function f that maps pairs of $(m+1)$-types extending t into the set $\{0, 1, 1/2\}$. Thus, $f(s_1, s_2) = 0$ if there is no type in S that extends s_1 and s_2; $f(s_1, s_2) = 1/2$ if there are two $(m+2)$-types that extend s_1 and s_2 and exactly one of them is in S; and $f(s_1, s_2) = 1$ if all the $(m+2)$-types that extend s_1 and s_2 (whether there are one or two of them) are in S.

Now, there are $m + 2^m$ distinct types in the variables $x, \bar{z}$, extending t. These are obtained by taking $x = z_i$ for some i, yielding m distinct types, and for the case when $x \neq z_i$ for all i, by taking the 2^m ways in which x can be connected by edges to $z_1, \ldots, z_m$.

Thus, given a random graph G and a tuple $\bar{a}$ such that $G \models t[\bar{a}]$, we can divide the vertices $b \in G$ into $m + 2^m$ sets according to the $(m+1)$-type of $(b, \bar{a})$. Of these sets, m are singletons (containing the vertices that are in the tuple $\bar{a}$) and the rest of the vertices are distributed randomly among the other 2^m sets. The probability that a pair (b_1, b_2) satisfies $G \models \psi[b_1, b_2, \bar{a}]$, and therefore that there is an edge (b_1, b_2) in $\Psi(G, \bar{a})$, is then given by $f(s_1, s_2)$ where s_i is the $(m+1)$-type of $(b_i, \bar{a})$. This discussion motivates the following definitions.

Definition 4.2. A pair (m, f) is an *interpretive measure* for $\mathcal{G}(1/2)$ if and only if $m \in \mathbb{N}$ and there are disjoint sets $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_{2^m}\}$ such that f is a function from $(P \cup Q)^2$ to $\{0, 1, 1/2\}$ subject to the following conditions: $f(x, y) = f(y, x)$ and if either $x \in P$ or $y \in P$, then $f(x, y) \in \{0, 1\}$.

Definition 4.3. For any interpretive measure (m, f), and any $n \geq m$, let $\mathcal{T}_n$ be the collection of all functions $T : \{0, \ldots, n-1\} \to (P \cup Q)$, for which there are m distinguished points $0 \leq a_1, \ldots, a_m < n$ such that $T(x) = p_i$ if and only if $x = a_i$.

For each $T \in \mathcal{T}_n$, the probability space Γ_T is obtained by determining for each pair of points $a, b \in \{0, \ldots, n-1\}$ whether there is an edge between them by means of independent Bernoulli trials with mean $f(T(a), T(b))$.

Finally, the probability space $\Gamma_n(m, f)$ is defined by assigning to each graph G with n vertices the probability $(\sum_{T \in \mathcal{T}_n} \mu_T(G))/\text{card}(\mathcal{T}_n)$, where $\mu_T(G)$ is the probability assigned to G in the probability space Γ_T.

We write $\Gamma(m, f)$ for the sequence $(\Gamma_n(m, f))_{n \in \mathbb{N}}$, where for $n < m$, Γ_n is chosen arbitrarily.

The relevance of the above definition to $\Psi\mathcal{G}_t(1/2)$ emerges in the following lemma.

Lemma 4.3. If $\psi(x, y, \bar{z})$ is a quantifier-free first-order formula, $\Psi(\bar{z})$ is the associated graph interpretation, and t is a type in the variables $\bar{z}$, then $\Psi\mathcal{G}_t(1/2)$ is $\Gamma(m, f)$ for some interpretive measure (m, f).

Proof:
Let ψ^* be the formula $x \neq y \wedge (\psi(x, y) \vee \psi(y, x))$, i.e. the formula that defines the interpretation Ψ. By Lemma 2.1, we can assume that ψ^* is presented as a disjunction over a set R of atomic types in the variables $x, y, \bar{z}$. Let S be the set of all types s in the variables $x, y, \bar{z}$ such that:

$s \models x \neq y$; and

$s \models t$, i.e. s extends t.

Clearly, $\Psi\mathcal{G}_t(1/2)$ is completely determined by the set $R \cap S$.

We proceed to define the measure (m, f). Let m be the number of *distinct* parameters in $\bar{z}$, i.e. it is the cardinality of a maximal set $P = \{z_{i_1}, \ldots, z_{i_m}\}$ of variables from $\bar{z}$ such that $t \models z_{i_j} \neq z_{i_k}$. We will assume without loss of generality, by renaming variables if necessary, that P consists of the variables $\{z_1, \ldots, z_m\}$. Let $Q = \{q_1, \ldots, q_{2^m}\}$ be the power set of P.

Intuitively, $P \cup Q$ represents the $m + 2^m$ sets of vertices as mentioned in the discussion preceding Definition 4.2. Therefore, each pair $(x, y) \in (P \cup Q)^2$ either uniquely determines a type $s \in S$ (if either x or y is in P), or it determines two types $s_0, s_1 \in S$. Thus, for each pair $(x, y) \in (P \cup Q)^2$, we will determine the value of the function f based on whether or not the corresponding types are in R.

We formally define f as follows:

1. $f(z, z) = 0$ for all $z \in P$;

2. For $z_i, z_j \in P, i < j$, let s be the unique type in S such that $s \models x = z_i \wedge y = z_j$. We let $f(z_i, z_j) = f(z_j, z_i) = 1$ if $s \models \psi$, and $f(z_i, z_j) = f(z_j, z_i) = 0$ otherwise.

3. For $z_i \in P$ and $q \in Q$, let s be the unique type in S satisfying:

$s \models x = z_i$;
$s \models y \neq z_j$, for $z_j \in P$;
$s \models Eyz_j$, for $z_j \in q$; and
$s \models \neg Eyz_j$, for $z_j \notin q$.

We let $f(z_i, q) = f(q, z_i) = 1$ if $s \models \psi$ and $f(z_i, q) = f(q, z_i) = 0$ otherwise.

4. for $q_i, q_j \in Q, i \leq j$, let s_0 and s_1 be the two types in S satisfying:

   $s_c \models x \neq z_k$, for $z_k \in P$ and $c = 0, 1$;
   $s_c \models y \neq z_k$, for $z_k \in P$ and $c = 0, 1$;
   $s_c \models Exz_k$, for $z_k \in q_i$ and $c = 0, 1$;
   $s_c \models \neg Exz_k$, for $z_k \notin q_i$ and $c = 0, 1$;
   $s_c \models Eyz_k$, for $z_k \in q_j$ and $c = 0, 1$;
   $s_c \models \neg Eyz_k$, for $z_k \notin q_j$ and $c = 0, 1$;
   $s_0 \models Exy$; and
   $s_1 \models \neg Exy$.

   We let:
   $$f(q_i, q_j) = f(q_j, q_i) = p,$$
   where,
   $$p = \begin{cases} 0 & \text{if } s_0 \not\models \psi \text{ and } s_1 \not\models \psi \\ 1 & \text{if } s_0 \models \psi \text{ and } s_1 \models \psi \\ 1/2 & \text{if } s_0 \models \psi \text{ and } s_1 \not\models \psi \\ 1/2 & \text{if } s_0 \not\models \psi \text{ and } s_1 \models \psi \end{cases}$$

It then follows from the discussion preceeding Definition 4.2 that $\Psi\mathcal{G}_t(1/2)$ is $\Gamma(m, f)$. □

Lemma 4.3 tells us the structure of the probability spaces on which $\mathcal{H}$ must converge quickly in order for us to be able to apply Lemma 4.2 to eliminate an occurrence of a quantifier. If this can be done for every $\Gamma(m, f)$, then starting with an arbitrary sentence φ of FO[$Q^G_{\mathcal{H}}$], by repeated application of this procedure, we can obtain a quantifier free sentence that is equivalent to φ almost everywhere. Furthermore, suppose $\mathcal{Q}$ is a collection of quantifiers, each of which is of the form $Q^G_{\mathcal{H}}$ for a class of graphs $\mathcal{H}$ that converges quickly on every $\Gamma(m, f)$. Let FO[$\mathcal{Q}$] denote the extension of first-order logic with all the quantifiers in $\mathcal{Q}$. Then, again for any sentence φ of FO[$\mathcal{Q}$], we can apply the above procedure to eliminate all quantifiers and obtain a quantifier free sentence that is equivalent to φ almost everywhere. This then yields the main theorem of this section:

Theorem 4.1. If $\mathcal{Q}$ is a collection of quantifiers, each of the form $Q^G_{\mathcal{H}}$ for a graph property $\mathcal{H}$ that converges quickly to 0 or 1 on $\Gamma(m, f)$ for every interpretive measure (m, f), then FO[$\mathcal{Q}$] has a 0-1 law for $\mathcal{G}(1/2)$.

Proof:
Let φ be a sentence of FO$[\mathcal{Q}]$. We prove, by induction on the total number of quantifiers in φ, that φ is equivalent almost everywhere to a quantifier free sentence, i.e. to True or False. This is trivially true when this number is 0. Let φ contain $q+1$ quantifiers. There is a subformula χ of φ which is either of the form $\exists x\psi$, or of the form $Q^{G}_{\mathcal{H}}x, y\psi$, where ψ is quantifier free. In either case, χ is equivalent almost everywhere to a quantifier free formula θ. In the first case this is true by Theorem 2.1 while in the second it follows from Lemma 4.2 and Lemma 4.3. Thus, by replacing χ by θ in φ, we obtain a sentence φ' that is equivalent to φ over a class with asymptotic probability 1, and that has only q quantifiers. But then, by the induction hypothesis, φ' is equivalent to a quantifier free sentence, on a class of asymptotic probability 1. Since the intersection of two classes that have asymptotic probability 1 must itself have asymptotic probability 1, we conclude that φ is equivalent almost everywhere to a quantifier free sentence. □

We have assumed throughout this paper that we are working with purely relational signatures. It is well known that when we have constants in our signature, then even the 0-1 law for first order logic fails. However, one can still show that every sentence is equivalent almost everywhere to a quantifier free sentence (*cf.* Theorem 2.1). This extends also to the above Theorem 4.1. Thus, for any $\mathcal{Q}$ that satisfies the hypotheses of the theorem, any sentence of FO$[\mathcal{Q}]$, perhaps including constants, is equivalent almost everywhere to a quantifier free sentence.

5. Rigidity

We now use the characterization provided by Theorem 4.1 to show that FO[Rig] has a 0-1 law, where Rig is the graph quantifier formed from the class of rigid graphs.

Theorem 5.1. For every interpretive measure (m, f), the probability that a graph is rigid converges exponentially fast to either 0 or 1 for $\Gamma(m, f)$.

Proof:
We distinguish three cases for interpretive measures (m, f). Recall that $f : (P \cup Q)^2 \to \{0, 1, 1/2\}$.

(i) There exists a non-trivial permutation π on P such that $f(p, p') = f(\pi p, \pi p')$ for all $p, p' \in P$, and $f(p, q) = f(\pi p, q)$ for all $p \in P$, $q \in Q$.

(ii) There exists a $q \in Q$ such that $f(q, q') \in \{0, 1\}$ for all $q' \in Q$.

(iii) All other cases.

In case *(i)* the permutation π defines a non-trivial automorphism on all $G \in \Gamma_n(m, f)$. In case *(ii)* we have a non-trivial automorphism for $G \in \Gamma_n(m, f)$ provided G contains at least two nodes in the class defined by q. But, this holds with probability tending to 1 exponentially fast. We now aim to prove that in case *(iii)*, the graphs $G \in \Gamma_n(m, f)$ are almost surely rigid.

The random process of constructing $G \in \Gamma_n(m, f)$ can be split into two subprocesses. In the first stage, the nodes from $[n] = \{0, \dots, n-1\}$ are distributed over the $m + 2^m$ classes $P \cup Q$. In the second stage, edges are determined according to the probabilities given by f.

Recall that the first subprocess randomly selects m points to form the singleton sets $p \in P$, and then distributes the remaining $n-m$ nodes over the sets $q \in Q$. For every $q \in Q$, the probability that q gets precisely k points is described by a binomial distribution $b(k; n-m, 2^{-m})$, where $b(k;n,p)$ is the usual abbreviation for

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

Obviously, the expected number of elements in every class q is $2^{-m}(n-m)$. More precisely, basic facts on binomial distributions (see e.g. [4, pp. 10-14]) imply that for every $\delta > 0$, the probability that some class q contains less than $(1-\delta)2^{-m}(n-m)$ or more than $(1+\delta)2^{-m}(n-m)$ elements, is bounded by $2^{-\varepsilon n}$ for some $\varepsilon > 0$.

It is convenient to exclude from further consideration those rare events, where the nodes are not 'evenly' distributed over Q. Fix a constant $d > 0$, and let $\Gamma_n^d(m,f)$ be a new probability space, obtained from $\Gamma_n(m,f)$ by throwing away all those graphs where the first stage of the construction produces any class $q \in Q$ with less than $(1-d)2^{-m}n$ elements. Since these graphs form a set whose measure goes to 0 exponentially fast, it suffices to prove our result for the sequence of probability spaces $\Gamma_n^d(m,f)$.

Let $X(G)$ be the number of non-trivial automorphisms of G. We will prove that the expectation $E(X)$ on $\Gamma_n^d(m,f)$ tends exponentially fast to 0 as n goes to infinity. Since, by Markov's inequality, $P[X \geq 1] \leq E(X)$, this immediately implies the desired result. For $\pi \in S_n$, let X_π be the indicator random variable, defined by

$$X_\pi(G) = \begin{cases} 1 & \text{if } \pi \in \mathrm{Aut}(G) \\ 0 & \text{otherwise} \end{cases}$$

By linearity of expectation we have that

$$E(X) = \sum_{\pi \in S_n - \{\mathrm{id}\}} E(X_\pi).$$

The *support* of a permutation π, denoted $\mathrm{supp}(\pi)$ is the set of points moved by π. Let $h = |\mathrm{supp}(\pi)|$ and $T_{n,h} = \{\pi \in S_n : |\mathrm{supp}(\pi)| = h\}$.

It is sufficient to prove the following claim.

Claim. *There exists a $\delta > 0$ such that*

$$E(X_\pi) \leq 2^{-\delta h n}$$

for all h and all $\pi \in T_{n,h}$.

Indeed, the claim implies that

$$\begin{aligned} E(X) &\leq \sum_{h=1}^{n} |T_{n,h}| 2^{-\delta h n} \\ &\leq \sum_{h=1}^{n} 2^{h \log n} 2^{-\delta h n} \leq 2^{-\varepsilon n} \end{aligned}$$

for some $\varepsilon > 0$.

We first prove a bound on $E(X_\pi)$ that holds for arbitrary size of the support.

Lemma 5.1. If $\pi \neq \mathrm{id}$, then $E(X_\pi) \leq 2^{-\varepsilon n}$ for some $\varepsilon > 0$ and sufficiently large n.

Proof:
π moves at least one point, say $\pi(i) = j$. Assume that the first subprocess produces $p = \{i\}$ and $p' = \{j\}$ for $p, p' \in P$. Then, since condition *(i)* does not hold, there exists a class $q \in Q$ such that $f(p,q) \neq f(p',q)$. Thus, to be an automorphism of G, π has have to move the whole class q. But this means that π must preserve $\Omega(n^2)$ non-trivial edge events.

Otherwise, at least one of the nodes i and j is put into a class $q \in Q$. But then, there exists an entire class $q' \in Q$ such that the edge-probabilities from this node to q' are 1/2. Since q' has $\Omega(n)$ elements, the result follows. □

Note that Lemma 5.1 proves the claim for $h < k$ where k is fixed (independent of n).

Before we prove the claim for permutations that move more points, we make some general observations that hold for arbitrary probability spaces of graphs.

Let $\pi \in S_n$ and K be the set of potential edges, i.e. the set of unordered pairs of elements of $[n]$. We call $R \subseteq K$ a *witness set* for π if $K - R$ intersects every orbit of the operation of π on K; in other words, for every pair $(i,j) \in R$ there exists $k \in \mathbb{N}$ such that $(\pi^k(i), \pi^k(j)) \notin R$. If R is a witness set for $\pi \in \mathrm{Aut}(G)$ and we fix the edges and non-edges of G outside of R, then those inside R are determined as well.

The following is a possible way to construct witness sets: Let $B, C \subseteq \mathrm{supp}(\pi)$ such that $B \cap C = \emptyset$ and C contains, for every $b \in B$, precisely one element of the orbit of b under π. Further, let $D = [n] - (B \cup C)$. Then $B \times D$ is a witness set.

Thus, given a permutation $\pi \in S_n$ we can establish an upper bound for $E(X_\pi)$ as follows: We choose suitable sets B, C and prove that the first stage of the construction of a random graph must assign edge probability 1/2 to at least r pairs in the associated witness set $B \times D$. Then $E(X_\pi) \leq 2^{-r}$.

Lemma 5.2. Let $c < (1-d)2^{-m}$ and let $2m < h \leq cn$. Then there exists an $\varepsilon > 0$ such that $E(X_\pi) \leq 2^{-\varepsilon(h/2-m)n}$ for $\pi \in T_{n,h}$.

Proof:
Let $C \subset \mathrm{supp}(\pi)$ be any set obtained by picking precisely one element out of every nontrivial cycle of π, and let $B = \mathrm{supp}(\pi) - C$. Thus, $D = [n] - (B \cup C)$ coincides with the set of fixed points of π and therefore contains at least $(1-c)n$ elements. B contains at least $h/2$ nodes, since the support of π is decomposed into cycles of length ≥ 2 and C contains only one element of each cycle. Thus, at least $h/2 - m$ of the nodes of B are put into some $q \in Q$ so that each of these has nontrivial edge-probabilities to at least one entire class $q' \in Q$. Since $|q'| \geq (1-d)2^{-m}n$, it follows that $|D \cap q'| \geq \varepsilon n$ where $\varepsilon = (1-d)2^{-m} - c$. Thus $B \times D$ contains at least $\varepsilon(h/2-m)n$ pairs with edge probability 1/2. Thus the probability that a random graph $G \in \Gamma_n^d(m,f)$ is fixed by π is bounded by $2^{-\varepsilon(h/2-m)n}$. □

Lemma 5.3. For the same constant c as in the previous lemma and $h > cn$, there exists a $\delta > 0$ such that $E(X_\pi) \leq 2^{-\delta n^2}$.

Proof:
Let B, C be disjoint subsets of $\mathrm{supp}(\pi)$ such that $|B| = cn/2$ and C contains precisely one element

of each cycle of π that intersects with B. Again $D = [n] - (B \cup C)$ has at least $(1-c)n$ elements. With precisely the same reasoning as in the previous lemma, we infer that $B \times D$ contains at least $(cn/2 - m)\varepsilon n$ pairs with edge probability 1/2. By choosing $\delta > 0$ such that $\delta n^2 \leq (cn/2 - m)\varepsilon n$, the result follows. □

Together, the three lemmata prove the claim, and therefore the theorem. □

Theorem 5.1, together with Theorem 4.1 yields:

Corollary 5.1. FO[Rig] has a 0-1 law on $\mathcal{G}(1/2)$.

In order to show that there is a regular logic that can express rigidity on graphs and has a 0-1 law, we need to consider the closure of FO[Rig] under relativization. This can be obtained by considering a relativized version of the rigidity quantifier, denoted Rig′, which binds two formulae $\delta(x, \bar{z})$ and $\varphi(x, y, \bar{z})$. Let $\psi = (x \neq y) \wedge (\varphi(x, y, \bar{z}) \vee \varphi(y, x, \bar{z}))$ be the irreflexive and symmetric formula associated with φ. Then the meaning of a formula Rig′ $x, y(\delta, \varphi) \in$ FO[Rig′] in a structure $\mathfrak{A}$ with valuation $\bar{b}$ for $\bar{z}$, is that the graph $(\delta^{\mathfrak{A},\bar{b}}, \psi^{\mathfrak{A},\bar{b}})$ is rigid.

A simple modification of the proof of Theorem 5.1 extends the result to the logic FO[Rig′]. We outline the modification here. Define a *relativized interpretive measure* as a triple (m, d, f) such that $m \in \mathbb{N}$ and there are disjoint sets $P = \{p_1, \dots, p_m\}$ and $Q = \{q_1, \dots, q_{2^m}\}$ such that $d \subseteq P \cup Q$ and $f : d \times d \to \{0, 1, 1/2\}$ is a function satisfying the restrictions in Definition 4.2. We let the class of functions $\mathcal{T}_n$ be as in Definition 4.3, but now define the probability distribution Γ_T over the collection of graphs with vertex set $T^{-1}(d)$. In particular, edge probabilities for $a, b \in T^{-1}(d)$ are determined by Bernoulli trials with mean $f(T(a), T(b))$.

We can then define the probability distribution $\Gamma_n(m, d, f)$ as before as $\sum_{T \in \mathcal{T}_n} \mu_T(G)/\text{card}(\mathcal{T}_n)$. Note that $\Gamma_n(m, d, f)$ is a probability distribution over all graphs whose vertex set is a subset of $\{0, \dots, n-1\}$. Then, the following lemma can be established along the lines of Lemma 4.3.

Lemma 5.4. If Ψ is a graph interpretation defined by quantifier-free formulae $\delta(x, \bar{z})$ and $\varphi(x, y, \bar{z})$, and t is a type in $\bar{z}$, then $\Psi\mathcal{G}_t(1/2)$ is $\Gamma(m, d, f)$ for some relativized interpretive measure (m, d, f).

Similarly, in the case distinction at the beginning of the proof of Theorem 5.1, we need to relativize the cases to the set d. This leads us to the result.

Theorem 5.2. FO[Rig′] is a regular logic that has a 0-1 law for $\mathcal{G}(1/2)$.

6. Vectorized Quantifiers

In this section, we consider extensions of first-order logic formed by adding vectorized quantifiers. A single Lindström quantifier can be seen as giving rise to an infinite sequence of quantifiers formed by vectorization. This allows us to consider interpretations that are not bound by the universe of a given structure and can map it to potentially larger structures. Vectorized interpretations and quantifiers capture a natural notion of logical reduction. For a discussion of this and its significance for descriptive complexity, see [9, Chapter12] and [6].

We begin with some definitions. Let $\tau = \{R_1, \ldots, R_m\}$ be a signature where R_i has arity r_i. A vectorized interpretation of τ in σ of width k is given by a sequence of σ-formulas, $\psi_1(\bar{x}_1, \bar{y}), \ldots, \psi_m(\bar{x}_m, \bar{y})$, where the length of $\bar{x}_i$ is $k \cdot r_i$. The variables in $\bar{y}$ are parameters. The interpretation maps a σ-structure $\mathfrak{A}$ along with an interpretation $\bar{a}$ of the parameters in $\mathfrak{A}$ to a τ-structure $\mathfrak{B}$, whose universe is A^k, with the relation $R_i^{\mathfrak{B}}$ given by $\psi_i^{\mathfrak{A},\bar{a}}$.

For any graph quantifier $Q_{\mathcal{H}}$, we define its kth vectorization $Q^k_{\mathcal{H}}$ as a quantifier that binds $2k$ variables and whose semantics is given by the following rule: if $\psi(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are, respectively, a $2k$-tuple and an m-tuple of variables, defines a vectorized interpretation $\Psi(\bar{y})$ of width k, then $(G, \bar{a}) \models Q^k_{\mathcal{H}}\bar{x}\psi$ if and only if $\Psi(G, \bar{a}) \in \mathcal{H}$. We define FO$[Q^\star_{\mathcal{H}}]$ to be the extension of first-order logic by the infinite sequence of quantifiers $\{Q^k_{\mathcal{H}} | k \in \mathbb{N}\}$.

Let Φ be a vectorized interpretation of width k given by a quantifier-free formula φ with m parameters. Let G and H be graphs and $\bar{a}$, $\bar{b}$ be m-tuples of vertices from G and H respectively, such that there is an isomorphic embedding $f : H \to G$ with $f(\bar{b}) = \bar{a}$. Let Φf denote the map from $\Phi(H, \bar{b})$ to $\Phi(G, \bar{a})$ given by the natural extension of f to k-tuples. The following lemma is based on the observation that quantifier free formulas are preserved under isomorphic embeddings.

Lemma 6.1. Φf is an isomorphic embedding of $\Phi(H, \bar{b})$ in $\Phi(G, \bar{a})$.

Proof:
If $\bar{h}_1$ and $\bar{h}_2$ are two k-tuples of vertices in H, then whether or not there is an edge between them in $\Phi(H, \bar{b})$ is determined by the quantifier-free formula φ. However, quantifier-free formulae are clearly preserved under the isomorphic embedding f, and therefore $\Phi f(\bar{h}_1)$ and $\Phi f(\bar{h}_2)$ have an edge if and only if $\bar{h}_1$ and $\bar{h}_2$ do. □

Let H be any fixed graph, $\bar{b}$ an m-tuple of vertices in H and t the atomic type of $\bar{b}$ in H. Recall that $\mathcal{G}_{t,n}(p)$ is a probability space on structures $(G, \bar{a})$, for graphs G of cardinality n and m-tuples $\bar{a}$ of vertices of G, such that the probability $\mu_{t,n}(G, \bar{a})$ is non zero only if $\bar{a}$ has type t in G. Let $F_{(H,\bar{b})}$ denote those structures $(G, \bar{a})$ for which there is an isomorphic embedding $f : (H, \bar{b}) \to (G, \bar{a})$.

Lemma 6.2. For any graph H, and any m-tuple $\bar{b}$ of vertices of atomic type t in H,

$$\mu_{t,n}(F_{(H,\bar{b})}) = 1 - o(n^{-m}).$$

Proof:
The proof is immediate from the fact that the probability of each of the extension axioms converges exponentially quickly to 1 [10]. □

Let $\mathcal{H}$ be a collection of graphs that is closed under taking extensions. The following lemma, which is analogous to Lemma 4.2, is derived from Lemmas 6.1 and 6.2.

Lemma 6.3. For any quantifier free formula ψ defining a vectorized interpretation $\Psi(\bar{y})$ of width k, with parameters $\bar{y}$, there is a quantifier free formula θ such that the sentence $\forall \bar{y}(\theta \leftrightarrow Q^k_{\mathcal{H}}\bar{x}\,\psi)$ has asymptotic probability 1 for $\mathcal{G}(p)$, for any constant p.

Proof:
We show that for any m-type t, either there is no pair $(H, \bar{b})$, such that $\bar{b}$ has type t in H, and $(H, \bar{b}) \models Q^k_{\mathcal{H}} \bar{x}\, \psi$, and therefore $\mathcal{H}$ converges quickly to 0 for $\mathcal{G}_t(p)$; or $\mathcal{H}$ converges quickly to 1 for $\mathcal{G}_t(p)$. It then follows that we can take θ to be the disjunction of types t such that there is such a pair $(H, \bar{b})$.

Suppose now, that for a given t, there is a graph H and a tuple $\bar{b}$ of type t in H such that $(H, \bar{b}) \models Q^k_{\mathcal{H}} \bar{x}\, \psi$, i.e., $\Psi(H, \bar{b}) \in \mathcal{H}$. Then, by Lemma 6.1, there is an isomrphic embedding of $\Psi(H, \bar{b})$ in $\Psi(G, \bar{a})$. Since $\mathcal{H}$ is closed under extensions, this implies that $\Psi(G, \bar{a}) \in \mathcal{H}$. In other words, for every $(G, \bar{a}) \in F_{(H,\bar{b})}$, $\Psi(G, \bar{a}) \in \mathcal{H}$. Therefore, by Lemma 6.2 $\mathcal{H}$ converges quickly to 1 for $\mathcal{G}_t(p)$. □

Observe that, by duality, the argument outlined above also works for classes of graphs that are closed under substructures rather than extensions. This enables us to prove the following theorem, by an elimination of quantifiers along the lines of Theorem 4.1.

Theorem 6.1. For any class of graphs $\mathcal{H}$ closed under taking extensions (or, dually, substructures), the logic FO$[Q^\star_{\mathcal{H}}]$ has a 0-1 law for $\mathcal{G}(p)$, for any constant p.

Theorem 6.1 should be compared with a result in [11] which shows that the logic FO$^r[Q_{\mathcal{H}}]$ has a 0-1 law if $\mathcal{H}$ is monotone and closed under extensions. We have weakened the hypothesis by dropping the requirement of monotonicity and greatly strengthened the theorem by allowing both vectorization and nesting of quantifiers.

Writing 3-Col for the graph quantifier defined by the class of 3-colourable graphs, and Plan for the graph quantifier corresponding to the class of planar graphs, the following two corollaries of Theorem 6.1 are immediate.

Corollary 6.1. FO[3-Col*] has a 0-1 law for $\mathcal{G}(p)$, for any constant p.

Corollary 6.2. FO[Plan*] has a 0-1 law for $\mathcal{G}(p)$, for any constant p.

Moreover, Corollaries 6.1 and 6.2 are easily extended to the closure of these logics under relativizations by arguments entirely analogous to those used to establish Theorem 5.2. Indeed, it is easily shown that these classes converge quickly even under relativized interpretive measures. These results, first established in [7] show that 3-colourability and planarity are expressible in a regular logic that is closed under vectorization and has a 0-1 law. Corollary 6.1, in particular, answered a question posed by Iain Stewart. Grohe has since shown that graph planarity is definable in least-fixed-point logic [14]. This provides an alternative proof of Corollary 6.2, since this logic has a 0-1 law.

It should also be pointed out that the various quantifiers can be combined in a single logic, as in Theorem 4.1. Thus, there is a 0-1 law for the logic FO[3-Col,Plan] which extends first-order logic with quantifiers for both 3-colourability and planarity. This tells us that there is, for example, a regular logic that can express the property of a graph being planar and not 3-colourable, and that still has a 0-1 law.

7. Infinitary Logic

We write $L_{\infty\omega}$ for the infinitary logic that is formed by adding to first-order logic the ability to take conjunctions and disjunctions over infinite sets of formulae. To be precise, if S is an arbitrary set of formulae of $L_{\infty\omega}$, then so is $\bigwedge_{\varphi \in S} \varphi$ and $\bigvee_{\varphi \in S} \varphi$. We write $L^k_{\infty\omega}$ for the collection of formulae of $L_{\infty\omega}$

which use only the variables $x_1, \dots, x_k$, free or bound, and $L^\omega_{\infty\omega}$ for the collection of all formulae that are in $L^k_{\infty\omega}$ for some k. That is, $L^\omega_{\infty\omega}$ consists of those formulae of $L_{\infty\omega}$ which contain only finitely many distinct variables. The logic $L^\omega_{\infty\omega}$ has been much studied in finite model theory as it provides an upper bound on the expressive power of interesting fixed-point logics (see [9, 19]) and indeed provides a powerful tool for showing inexpressibility in such logics. It was shown by Kolaitis and Vardi [18] that the logic admits a 0-1 law. It is a natural question to ask whether the extensions of $L^\omega_{\infty\omega}$ by means of the generalized quantifiers we have considered here also admit 0-1 laws. Of course, this question makes sense only for those quantifiers Q where we have shown that the 0-1 law holds for FO$[Q]$. In this section we briefly outline how Theorems 4.1 and 6.1 can be extended to the case of infinitary logic.

For a generalized quantifier Q, let $L^k_{\infty\omega}[Q]$ denote the logic $L^k_{\infty\omega}$ extended with the formula formation rule for the quantifier Q as in Definition 2.2, and let $L^\omega_{\infty\omega}[Q]$ denote the collection of all formulae that are in $L^k_{\infty\omega}[Q]$ for some k.

The proof by Kolaitis and Vardi that $L^\omega_{\infty\omega}$ has a 0-1 law relies on the fact that for each k, there are only finitely many extension axioms using the variables $x_1, \dots, x_k$. They show that for any sentence φ of $L^\omega_{\infty\omega}$, either φ or $\neg\varphi$ is a logical consequence of the conjunction of all such axioms. Since each axiom individually has asymptotic probability 1, so does the finite conjunction, and the result follows. At first sight, such an argument cannot be extended to the quantifier-elimination technique in the proof of Theorem 4.1 since, while a sentence of $L^\omega_{\infty\omega}[Q]$ only has finitely many variables, it may have infinitely many occurrences of quantifiers. If the induction on quantifier rank in the proof of Theorem 4.1 is carried into the transfinite, we may need to take infinite intersections of classes of graphs of asymptotic probability 1, and these may have smaller probability. Nevertheless, we are able to prove that Theorem 4.1 does extend to the infinitary logic as we show next, provided that the collection $\mathcal{Q}$ of quantifiers is itself finite.

Theorem 7.1. If $\mathcal{Q}$ is a finite collection of quantifiers, each of the form $Q^G_{\mathcal{H}}$ for a graph property $\mathcal{H}$ that converges quickly to 0 or 1 on $\Gamma(m, f)$ for every interpretive measure (m, f), then $L^\omega_{\infty\omega}[\mathcal{Q}]$ has a 0-1 law for $\mathcal{G}(1/2)$.

Proof:
Suppose $\varphi(\bar{x})$ is a formula of $L^k_{\infty\omega}[\mathcal{Q}]$ for some k. We wish to show that there is a quantifier-free formula θ such that $\mu(\forall\bar{x}(\varphi \leftrightarrow \theta)) = 1$. Note that there are, up to logical equivalence, only finitely many distinct quantifier-free formulae in the variables $x_1, \dots, x_k$ and each of these is equivalent to a *finitary* formula, i.e. one without infinitary connectives. Let Θ be a finite collection of such formulae including representatives of all up to equivalence. Let $\psi_1, \dots, \psi_l$ be an enumeration of all formulae of the form $\exists x_i \alpha$ or $Q^G_{\mathcal{H}} x_i x_j \, \alpha$ for $i, j \in \{1, \dots, k\}$, $Q^G_{\mathcal{H}} \in \mathcal{Q}$ and $\alpha \in \Theta$. Then, each ψ_i is a formula of FO$[\mathcal{Q}]$ and by the proof of Theorem 4.1, there is a class C_i and a quantifier-free formula θ_i such that $\mu(C_i) = 1$ and ψ_i is equivalent on C_i to θ_i. Let $C = \bigcap_{1 \le i \le l} C_i$. As C is a *finite* intersection of classes of asymptotic probability 1, we have that $\mu(C) = 1$. We claim that there is a quantifier-free formula θ such that all graphs in C satisfy $\varphi \leftrightarrow \theta$. This is proved by a transfinite induction on the structure of φ.

If φ is quantifier-free, there is nothing to prove. The case where φ is $\neg\psi$ is equally easy. If φ is $\exists x \psi$ or $Q^G_{\mathcal{H}} xy \, \psi$ then, by induction hypothesis, ψ is equivalent on C to a quantifier-free formula and therefore φ is equivalent, on C, to one of the formulas ψ_i. But since $C \subseteq C_i$ and ψ_i is equivalent on C_i to θ_i, it follows that φ is equivalent, on C, to θ_i. Finally, if φ is $\bigwedge S$ and for each $\psi \in S$, there is a quantifier-free formula θ_ψ that is equivalent to ψ on C, then φ is equivalent, on C, to $\bigwedge_{\psi \in S} \theta_\psi$. □

Combining this with Theorem 5.1 immediately yields the following corollary.

Corollary 7.1. $L^{\omega}_{\infty\omega}[\mathrm{Rig}]$ has a 0-1 law on $\mathcal{G}(1/2)$.

A related result, more general than Corollary 7.1, has been obtained by Kaila [16]. This establishes a 0-1 law on $\mathcal{G}(1/2)$ for the extension of $L^{\omega}_{\infty\omega}$ with rigidity quantifiers for all signatures, rather than just graphs. Corollary 7.1 could further be extended to $\mathcal{G}(p)$ for any constant probability p. This would require a more complicated definition of probability spaces associated with interpretive measures. In Definition 4.3, the probability assigned to a graph G would not be obtained by a simple sum over $\mathcal{T}_n$ but a suitably weighted sum. We have omitted this generalization in the interests of clarity.

An entirely analogous argument applies to extensions of the results in Section 6 to infinitary logics. To be precise, suppose $\mathcal{H}$ is a graph property and let $L^{k}_{\infty\omega}[Q^{\star}_{\mathcal{H}}]$ denote the extension of $L^{k}_{\infty\omega}$ with the vectorized quantifiers generated by the class $\mathcal{H}$. Then, since any formula φ of $L^{k}_{\infty\omega}[Q^{\star}_{\mathcal{H}}]$ contains only k distinct variables, it is easy to see that any occurrence of a generalized quantifier in φ can be replaced by a quantifier $Q^{l}_{\mathcal{H}}$ for $l \leq k$. Thus, the logic is equivalent to an extension of $L^{k}_{\infty\omega}$ with finitely many generalized quantifiers and an argument like in the proof of Theorem 7.1 above shows that there are only finitely many formulas with one quantifier. If each of the quantifier-free interpretations yields a measure on which H converges quickly, we can construct a class C with $\mu(C) = 1$ on which each of the formulas with one quantifier is equivalent to a quantifier-free formula. This yields, in particular, the following generalization of Theorem 6.1.

Theorem 7.2. For any class of graphs $\mathcal{H}$ closed under taking extensions (or, dually, substructures), the logic $L^{\omega}_{\infty\omega}[Q^{\star}_{\mathcal{H}}]$ has a 0-1 law for $\mathcal{G}(p)$, for any constant p.

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