5 Denotational Semantics (mpf23)

You may use standard results provided that you state them clearly.

(*a*) For a domain D, let *fix* be the function mapping a continuous function $f \in (D \to D)$ to its least pre-fixed point $\mathit{fix}(f) \in D$.

Prove that $\mathit{fix} : (D \to D) \to D$ is continuous. [4 marks]

(*b*) For a PCF type τ, let $\Omega_\tau = \mathbf{fix}(\mathbf{fn}\ x : \tau.\ x)$ and consider the following closed PCF terms of type $(\tau \to \tau) \to (\mathit{nat} \to \tau)$.

$$\mathrm{M}_\tau = \mathbf{fn}\ f : \tau \to \tau.\ \mathbf{fn}\ n : \mathit{nat}.\ \mathbf{fix}(f)$$

$$\begin{aligned}\mathrm{N}_\tau = {} & \mathbf{fn}\ f : \tau \to \tau. \\ & \quad \mathbf{fix}\Big(\ \mathbf{fn}\ h : \mathit{nat} \to \tau.\ \mathbf{fn}\ n : \mathit{nat}. \\ & \qquad\qquad f\big(\mathbf{if}\ \mathbf{zero}(n)\ \mathbf{then}\ \Omega_\tau\ \mathbf{else}\ h(\mathbf{pred}(n))\big)\ \Big)\end{aligned}$$

Give an explicit description of the denotations $[\![\mathrm{M}_\tau]\!]$ and $[\![\mathrm{N}_\tau]\!]$ in the domain $([\![\tau]\!] \to [\![\tau]\!]) \to (\mathbb{N}_\perp \to [\![\tau]\!])$. [4 marks]

(*c*) Recall that the contextual preorder $\vdash M \leq_{\mathrm{ctx}} N : \tau$ holds whenever M and N are closed PCF terms of type τ and for all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M]$ and $\mathcal{C}[N]$ are closed PCF terms of type $\gamma \in \{\mathit{nat}, \mathit{bool}\}$ and for all values V of type γ, if $\mathcal{C}[M] \Downarrow_\gamma V$ then $\mathcal{C}[N] \Downarrow_\gamma V$.

Say whether the following statements concerning the PCF terms in Part (*b*) are true or false and, respectively, either prove or disprove them:

(*i*) For all PCF types τ, $\vdash \mathrm{M}_\tau \leq_{\mathrm{ctx}} \mathrm{N}_\tau : (\tau \to \tau) \to (\mathit{nat} \to \tau)$.

(Hint: Consider the case $\tau = \mathit{nat} \to \mathit{nat}$.) [6 marks]

(*ii*) For all PCF types τ, $\vdash \mathrm{N}_\tau \leq_{\mathrm{ctx}} \mathrm{M}_\tau : (\tau \to \tau) \to (\mathit{nat} \to \tau)$.

(Hint: Recall that every PCF type is of the form $\tau_1 \to (\cdots(\tau_\ell \to \gamma)\cdots)$ where $\ell \in \mathbb{N}$, τ_i $(1 \leq i \leq \ell)$ are types, and $\gamma \in \{\mathit{nat}, \mathit{bool}\}$.) [6 marks]