6 Computation Theory (amp12)

A set A equipped with a binary operation $@ : A \times A \to A$ is a *combinatory algebra* if there are elements $K, S \in A$ satisfying for all $a, b, c \in A$

$$@(@(K, a), b) = a \tag{1}$$
$$@(@(@(S, a), b), c) = @(@(a, c), @(b, c)) \tag{2}$$

(*a*) Show that there is a binary operation on the set of equivalence classes of closed λ-terms for the equivalence relation of β-conversion that makes it a combinatory algebra. [5 marks]

(*b*) Show that every combinatory algebra A contains an element I satisfying

$$@(I, a) = a \tag{3}$$

for all $a \in A$. [Hint: what does (2) tell us when $a = b = K$?] [2 marks]

(*c*) For an arbitrary combinatory algebra A, let $A[x]$ denote the set of expressions given by the grammar

$$e ::= x \mid \ulcorner a \urcorner \mid (ee)$$

where x is some fixed symbol not in A and a ranges over the elements of A. Given $e \in A[x]$ and $a \in A$, let $e[x := a]$ denote the element of A resulting from interpreting occurrences of x in e by a, interpreting the expressions of the form $\ulcorner a' \urcorner$ by a' and interpreting expressions of the form (ee') using @.

(*i*) Give the clauses in a definition of $e[x := a]$ by recursion on the structure of e. [2 marks]

(*ii*) For each $e \in A[x]$ show how to define an element $\Lambda_x e \in A$ with the property that

$$@(\Lambda_x e, a) = e[x := a] \tag{7}$$

for all $a \in A$. [6 marks]

(*d*) Recall the usual encoding of Booleans in λ-calculus. Using Part (*c*)(*ii*), show that in any combinatory algebra A there are elements $\mathit{True}, \mathit{False} \in A$ and a function $\mathit{If} : A \times A \to A$ satisfying

$$@(\mathit{If}(a, b), \mathit{True}) = a \tag{11}$$
$$@(\mathit{If}(a, b), \mathit{False}) = b \tag{12}$$

for all $a, b \in A$ [5 marks]