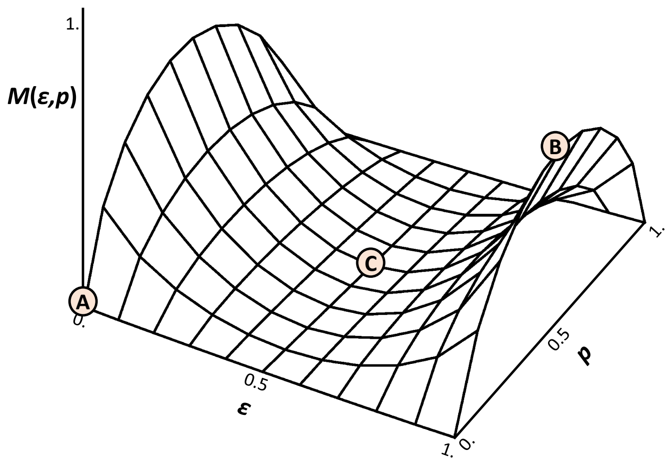
9 Information Theory (jgd1000)

(*a*) A binary symmetric channel receives as input a bit whose values $\{0, 1\}$ have probabilities $\{p, 1-p\}$, but in either case, a transmission error can occur with probability ϵ which flips the bit. The surface plot below describes the mutual information of this channel as a function $M(\epsilon, p)$ of these probabilities:



(*i*) At the point marked A, the error probability is $\epsilon = 0$. Why then is the channel mutual information minimal in this case: $M(\epsilon, p) = 0$? [2 marks]

(*ii*) At the point marked B, an error always occurs ($\epsilon = 1$). Why then is the channel mutual information maximal in this case: $M(\epsilon, p) = 1$? [2 marks]

(*iii*) At the point marked C, the input bit values are equiprobable ($p = 0.5$), so the symbol source has maximal entropy. Why then is the channel mutual information in this case $M(\epsilon, p) = 0$? [2 marks]

(*iv*) Define mathematically the function $M(\epsilon, p)$ in terms of ϵ and p. [4 marks]

(*b*) An important operation in pattern recognition is convolution. If f and g are two functions $\mathbb{R} \to \mathbb{C}$ then their convolution is $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$.

If their respective Fourier transforms are $\mathcal{F}_{[f]}(\omega)$ and $\mathcal{F}_{[g]}(\omega)$, prove that the convolution $(f*g)(x)$ has a Fourier transform $\mathcal{F}_{[f*g]}(\omega)$ that is the simple product

$$\mathcal{F}_{[f*g]}(\omega) = 2\pi \mathcal{F}_{[f]}(\omega) \cdot \mathcal{F}_{[g]}(\omega)\,.$$

[6 marks]

(*c*) Show how a generating (or "mother") wavelet $\Psi(x)$ can spawn a self-similar family of "daughter" wavelets $\Psi_{jk}(x)$ by simple scaling and shifting operations. Explain the advantages of analysing data in terms of such a self-similar family of dilates and translates of a mother wavelet. [4 marks]