6 Numerical Analysis (abr28)

(*a*) You are given a system of real equations in matrix form $Ax = b$ where A is non-singular. Give three factorization techniques to solve this system, depending on the shape and structure of A: tall, square, symmetric. For each technique, give the relevant matrix equations to obtain the solution x, and point out the properties of the matrices involved. Highlight one potential problem from an implementation (computer representation) standpoint. [*Note:* You do not need to detail the factorization steps that give the matrix entries.] [5 marks]

(*b*) We want to estimate travel times between stops in a bus network, using ticketing data. The network is represented as a directed graph, with a vertex for each bus stop, and edges between adjacent stops along a route. For each edge $j \in \{1, \ldots, p\}$ let the travel time be d_j. The following ticketing data is available: for each trip $i \in \{1, \ldots, n\}$, we know its start time s_i, its end time f_i, and also the list of edges it traverses. The total trip duration is the sum of travel times along its edges.

We shall estimate the d_j using linear least squares estimation, i.e. solve $\arg\min_\beta \|y - X\beta\|^2$ for a suitable matrix X and vectors β and y.

(*i*) Give an example of ticket data for a trip traversing 5 edges, and write the corresponding equation of its residual. [1 mark]

(*ii*) Give the dimensions and contents of X, β, and y for this problem. State a condition on X that ensures we can solve for β. [3 marks]

(*iii*) Give an example with $p = 2$ and $n = 3$ for which it is *not* possible to estimate the d_j. Compute $X^T X$ for your example. [2 marks]

(*c*) Let A be an $n \times n$ matrix with real entries.

(*i*) We say that A is *diagonalisable* if there exists an invertible $n \times n$ matrix P such that the matrix $D = P^{-1}AP$ is diagonal. Show that if A is diagonalisable and has only one eigenvalue then A is a constant multiple of the identity matrix. [3 marks]

(*ii*) Let A be such that when acting on vectors $x = [x_1, x_2, \ldots, x_n]^T$ it gives $Ax = [x_1, x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]^T$. Write out the contents of A and find its eigenvalues and eigenvectors. Scale the eigenvectors so they have unit length (i.e. so their magnitude is equal to 1). [6 marks]